Pricing Synthetic CDOs based on Exponential Approximations to the Payoff Function*

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Abstract

Correlation-dependent derivatives, such as Asset-Backed Securities (ABS) and Collateralized Debt Obligations (CDO), are common tools for offsetting credit risk. Factor models in the conditional independence framework are widely used in practice to capture the correlated default events of the underlying obligors. An essential part of these models is the accurate and efficient evaluation of the expected loss of the specified tranche, conditional on a given value of a systematic factor (or values of a set of systematic factors). Unlike other papers that focus on how to evaluate the loss distribution of the underlying pool, in this paper we focus on the tranche loss function itself. It is approximated by a sum of exponentials so that the conditional expectation can be evaluated in closed form without having to evaluate the pool loss distribution. As an example, we apply this approach to synthetic CDO pricing. Numerical results show that it is efficient.

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1 Introduction

Correlation-dependent derivatives, such as Asset-Backed Securities (ABS) and Collateralized Debt Obligations (CDO), are common tools for offsetting credit risk.[1] Factor models in the conditional independence framework are widely used in practice for pricing because they are tractable, analytic or semi-analytic formulas being available. An essential part of these models is the accurate and efficient evaluation of the expected loss of the specified tranche, conditional on a given value of a systematic factor (or correspondingly values of a set of systematic factors). To be specific, the problem is how to evaluate the conditional expectation of the piecewise linear payoff function of the loss Z

$$f(Z) = \min\left(u - \ell, (Z - \ell)^+\right), \tag{1}$$

where $x^+ = \max(x, 0)$, $Z = \sum_{k=1}^{K} Z_k$, Z_k are conditionally mutually independent, but not necessarily identically distributed, nonnegative random variables in a conditional independence framework (see Section 2 for explanations), and ℓ and u are the attachment and the detachment points of the tranche, respectively, satisfying $u > \ell \geq 0$. Generally Z_k, for obligor k, is the product of the two components: a random variable directly related to its credit rating and a loss-given-default or mark-to-market related value. The payoff function, f, is also known as the stop-loss function in actuarial science [4], [23].

Note that the expectation of a function of a random variable depends on two things: the distribution of the underlying random variable and the function itself. A standard approach to compute the expectation of a function of a random variable is to compute firstly the distribution of the underlying random variable, Z in our case, and then to compute the expectation of the given function, possibly using its special properties. Almost all research in finance [2], [9], [10], [14], [22], [25] and in actuarial science [7], [28], to name a few, has focused on the first part due to the piecewise linearity of the payoff function.

In this paper, we propose a new approach to solving the problem. We approximate the

[1]For background on, and a description of, these instruments, see for example [5]. We restrict attention to synthetic CDOs in this paper.

non-smooth function f by a sum of exponentials over $[0, \infty)$. Based on this approximation, the conditional expectation can be computed from a series of simple expectations. Consequently, we do not need to compute the distribution of Z. Due to the central role played by the exponential approximation to the payoff function in our method, we refer to it as the EAP method.

The remainder of this paper is organized as follows. The details of our approach outlined above are described in Section 2. As an example, this method is applied to synthetic CDO pricing in Section 3. The paper ends with some conclusions in Section 4, in which we summarize the advantages of our method over others, and indicate its scope of applicability.

2 Conditional expectation based on an exponential approximation

In the conditional independence framework, a central problem is how to evaluate the expectation

$$\mathbb{E}[f(Z)] = \int_M \mathbb{E}_M \left[f\left(Z\right)\right] \mathrm{d}\Phi(M),$$

where $\Phi(M)$ is the distribution of an auxiliary factor $\mathscr{M}$ (which can be a scalar or a vector),

$$\mathbb{E}_M \left[f\left(Z\right)\right] \equiv \mathbb{E}\left[f\left(Z\right)|\mathscr{M} = M\right]$$

and

$$Z = \sum_{k=1}^{K} Z_k, \tag{2}$$

where $Z_k \geq 0$ are mutually independent random variables, conditional on $\mathscr{M}$. It is obvious that Z is nonnegative. We denote by Ψ_M the distribution of Z conditional on $\mathscr{M} = M$, so that

$$\mathbb{E}_M \left[f\left(Z\right)\right] = \int_z f\left(z\right) \mathrm{d}\Psi_M(z). \tag{3}$$

Due to the piecewise linearity of the function f defined by (1), it is clear that once the distribution Ψ_M is obtained, the conditional expectation $\int_z f\left(z\right) \mathrm{d}\Psi_M(z)$ can be readily computed. Most research has focused on how to evaluate the conditional distribution of Z given the conditional distributions of Z_k. A fundamental result about a sum of independent random variables states that Z's conditional distribution can be computed as the convolution of Z_k's conditional distributions. Numerically, this idea is realized through forward and inverse fast Fourier transformations (FFT). A disadvantage of this approach is that it may be very slow when there are many obligors due to the number of convolutions to be calculated. For pools with special structures, it might be much slower than methods that are specially designed for those pools, such as recursive methods proposed by De Pril [7] and Panjer [28] and their extensions discussed in [4] and [23], and the one proposed by Jackson, Kreinin, and Ma [22] for portfolios where the Z_k sit on a properly chosen common lattice. To avoid computing too many convolutions, the target conditional distribution can be approximated by parametric distributions matching the first few conditional moments of the true conditional loss distribution. For a large pool, a normal approximation is a natural choice as a consequence of the central limit theorem and due to its simplicity, although it may not capture some important properties, such as skewness and fat tails.

To capture these important properties for medium to large portfolios, compound approximations, such as the compound Poisson [20], improved compound Poisson [13], compound binomial and compound Bernoulli [29] distributions have been used. They have proved to be very successful, since they match not only the first few conditional moments, but, most importantly, they match the tails much better than either normal or normal power distributions do. A key step in a method based on these compound approximations is the computation of convolutions of the conditionally independent distributions of the Z_k, by FFTs. As a result, the computational complexity of such an algorithm is superlinear in K, the number of terms in the sum (2).

As an alternative, in this paper, we propose an algorithm for which the computational complexity is linear in K. We focus on the stop-loss function f, instead of the conditional distribution Ψ_M of Z. To emphasis the role of the attachment and the detachment points ℓ and u, we denote $f(x)$ by $f(x; \ell, u)$ and introduce an auxiliary function $h(x)$ defined on $[0, \infty)$:

$h(x) = 1 - x$ if $x \leq 1$, 0 otherwise. Then we have

$$f(x) = f(x;\ell,u) = u\left[1 - h\left(\frac{x}{u}\right)\right] - \ell\left[1 - h\left(\frac{x}{\ell}\right)\right]. \tag{4}$$

In particular, if $\ell = 0$, we have

$$f(x;0,u) = \min\left(u, x^{+}\right) = \min\left(u, x\right) = u\left[1 - h\left(\frac{x}{u}\right)\right].$$

Note that $h(x)$ is independent of the constants ℓ and u. Therefore, if it can be approximated by a sum of exponentials over $[0,\infty)$, it is clear that $f(x;\ell,u)$ can be approximated by a sum of exponentials. Let

$$h(x) \approx \sum_{n=1}^{N} \omega_n \exp(\gamma_n x), \tag{5}$$

where ω_n and γ_n are complex numbers. Then from (4) we can see that $f(x;\ell,u)$ can be approximated by a sum of exponentials:

$$f(x;\ell,u) \approx u\left[1 - \sum_{n=1}^{N} \omega_n \exp\left(\gamma_n \frac{x}{u}\right)\right] - \ell\left[1 - \sum_{n=1}^{N} \omega_n \exp\left(\gamma_n \frac{x}{\ell}\right)\right] \tag{6}$$

$$\approx (u - \ell) - u\sum_{n=1}^{N} \omega_n \exp\left(\frac{\gamma_n}{u} x\right) + \ell\sum_{n=1}^{N} \omega_n \exp\left(\frac{\gamma_n}{\ell} x\right). \tag{7}$$

Based on this expression the conditional expectation $\mathbb{E}_M\left[f\left(Z\right)\right]$ defined in (3) can be computed as follows:

$$
\begin{aligned}
&\mathbb{E}_M\left[f\left(Z\right)\right] \\
&= \int_z f\left(z\right) \mathrm{d}\Psi_M(z) \\
&\approx \int_z \left[(u-\ell) - u\sum_{n=1}^{N}\omega_n \exp\left(\frac{\gamma_n}{u}z\right) + \ell\sum_{n=1}^{N}\omega_n \exp\left(\frac{\gamma_n}{\ell}z\right)\right] \mathrm{d}\Psi_M(z) \\
&= (u-\ell) - u\sum_{n=1}^{N}\omega_n \int_z \exp\left(\frac{\gamma_n}{u}z\right) \mathrm{d}\Psi_M(z) \\
&\quad +\ell\sum_{n=1}^{N}\omega_n \int_z \exp\left(\frac{\gamma_n}{\ell}z\right) \mathrm{d}\Psi_M(z) \\
&= (u-\ell) \\
&\quad -u\sum_{n=1}^{N}\omega_n \int_{z_1,\ldots,z_K} \prod_{k=1}^{K} \exp\left(\frac{\gamma_n}{u}z_k\right) \mathrm{d}\Psi_{M,1}(z_1)\cdots \mathrm{d}\Psi_{M,K}(z_K) \\
&\quad +\ell\sum_{n=1}^{N}\omega_n \int_{z_1,\ldots,z_K} \prod_{k=1}^{K} \exp\left(\frac{\gamma_n}{\ell}z_k\right) \mathrm{d}\Psi_{M,1}(z_1)\cdots \mathrm{d}\Psi_{M,K}(z_K) \\
&= (u-\ell) - u\sum_{n=1}^{N}\omega_n \prod_{k=1}^{K} \mathbb{E}_M\left[\exp\left(\frac{\gamma_n}{u}Z_k\right)\right] \\
&\quad +\ell\sum_{n=1}^{N}\omega_n \prod_{k=1}^{K} \mathbb{E}_M\left[\exp\left(\frac{\gamma_n}{\ell}Z_k\right)\right], \qquad (8)
\end{aligned}
$$

where $\Psi_{M,k}$ is the conditional distribution of Z_k, $\mathbb{E}_M\left[\exp\left(cZ_k\right)\right]$ is the conditional expectation of $\exp\left(cZ_k\right)$, for $c = \frac{\gamma_n}{\ell}$ or $\frac{\gamma_n}{u}$, respectively. The last equality holds by noting that Z_k, thus cZ_k, are mutually independent conditional on a given value of $\mathscr{M}$. In this way we can see that, to compute the conditional expectation $\mathbb{E}_M\left[f\left(Z\right)\right]$, we only need to compute the conditional expectations $\mathbb{E}_M\left[\exp\left(cZ_k\right)\right]$ of individual obligors.

The approximation in (5) is understood in the uniform sense and the uniformity is inherited in (7). Denoting the function on the right-hand side of (7) by $\hat{f}$, then the right-hand side of (8) is $\mathbb{E}_M[\hat{f}(Z)]$. If $\sup_{z\geq 0}|h(z)-\hat{h}(z)| \leq \varepsilon/(u+\ell)$ then, by (6), $\sup_{z\geq 0}|f(z)-\hat{f}(z)| \leq \varepsilon$, and hence

$$
|\mathbb{E}_M[f(z)] - \mathbb{E}_M[\hat{f}(z)]| \leq \varepsilon \qquad (9)
$$

and similarly for the unconditional expectations.

Since $h(z)$ is independent of the constants ℓ and u, for a given approximation accuracy the coefficients ω_n and γ_n for (5) need to be computed only once and the number of terms required can be determined a priori. As shown in a separate paper by the authors [17], the maximum absolute error in the approximation (5) is roughly equal to $0.16/N$:

Table 1: The maximum absolute error in the approximation (5) for several values of N

N	25	50	100	200	400
Max absolute error	6.4×10^{-3}	3.2×10^{-3}	1.6×10^{-3}	8×10^{-4}	4×10^{-4}

This numerical result leads to the following rule of thumb for a practical choice of N. To achieve a desired accuracy ε in the calculation of an expected tranche loss, with attachment and detachment points ℓ and u, respectively, one can choose $N > 0.16(u + \ell)/\varepsilon$. As such, from Table 1, $\sup_{z\geq 0} |h(z) - \hat{h}(z)| \leq \varepsilon/(u + \ell)$ and then (9) holds. If ℓ and u are expressed as percentages of the total pool notional, then $u + \ell \leq 2$ and we can simply take $N > 0.32/\varepsilon$ to obtain a maximum relative error of size ε in the calculation of *any* expected tranche loss, relative to the total pool notional.

As an example, the parameters γ_n and ω_n for a 25-term approximation are shown in Table 2. It is proved in [17] that, if γ_n is real, then ω_n is also real, and if γ_i and γ_j are a complex conjugate pair, then the corresponding ω_i and ω_j are also a complex conjugate pair, and vice versa. The data in the table exhibit this property. Exploiting this property, we can simplify the summations in (8) by noting that the sum of the complex conjugated i-th and j-th terms equals twice the real part of either one of these two terms. The table also shows that the real part of each γ_n is strictly negative. This property guarantees that the exponential approximation of (5) converges to zero as $x \to \infty$, and thus guarantees the existence of the conditional expectation $\mathbb{E}_M [\exp(cZ_k)]$. For a more complete discussion of the exponential approximation (5), see [17]. In particular, it is proved therein, that each γ_n has a strictly negative real part for any N.

Table 2: The real and imaginary parts of ω_n and γ_n (weights and exponents) for the 25-term exponential approximation

n	$\Re(\omega_n)$	$\Im(\omega_n)$	$\Re(\gamma_n)$	$\Im(\gamma_n)$
1	1.68E-4	-3.16E-5	-5.68E-2	1.45E+2
2	1.68E-4	3.16E-5	-5.68E-2	-1.45E+2
3	2.04E-4	-5.98E-5	-1.72E-1	1.32E+2
4	2.04E-4	5.98E-5	-1.72E-1	-1.32E+2
5	2.69E-4	-1.02E-4	-3.51E-1	1.20E+2
6	2.69E-4	1.02E-4	-3.51E-1	-1.20E+2
7	3.87E-4	-1.70E-4	-5.98E-1	1.07E+2
8	3.87E-4	1.70E-4	-5.98E-1	-1.07E+2
9	6.02E-4	-2.94E-4	-9.25E-1	9.49E+1
10	6.02E-4	2.94E-4	-9.25E-1	-9.49E+1
11	1.01E-3	-5.39E-4	-1.35E+0	8.24E+1
12	1.01E-3	5.39E-4	-1.35E+0	-8.24E+1
13	1.87E-3	-1.09E-3	-1.89E+0	6.99E+1
14	1.87E-3	1.09E-3	-1.89E+0	-6.99E+1
15	3.86E-3	-2.53E-3	-2.58E+0	5.74E+1
16	3.86E-3	2.53E-3	-2.58E+0	-5.74E+1
17	9.17E-3	-7.44E-3	-3.50E+0	4.49E+1
18	9.17E-3	7.44E-3	-3.50E+0	-4.49E+1
19	2.45E-2	-3.19E-2	-4.73E+0	3.24E+1
20	2.45E-2	3.19E-2	-4.73E+0	-3.24E+1
21	7.57E-3	-2.10E-1	-6.44E+0	2.04E+1
22	7.57E-3	2.10E-1	-6.44E+0	-2.04E+1
23	3.81E+0	0.00E+0	-9.65E+0	0.00E+0
24	-1.46E+0	1.03E-1	-8.54E+0	9.38E+0
25	-1.46E+0	-1.03E-1	-8.54E+0	-9.38E+0

3 Pricing synthetic CDOs

We illustrate the EAP method by applying it to synthetic CDO pricing. The underlying collateral of a synthetic CDO is a set of credit default swaps (CDSs).

3.1 Overview of models and pricing methods

Factor models, such as the reduced-form model proposed by Laurent and Gregory [25] or the one proposed by Duffie and Gârleanu [10] and extended successively by Mortensen [27] and Eckner [11], and the structural model proposed by Vasicek [31] and Li [26], are widely used in practice to obtain analytic or semi-analytic formulas to price synthetic CDOs efficiently. For a comparative analysis of different copula models, see the paper by Burtschell, Gregory and Laurent [6]. A generalization of the reduced-form model which, along with the structural model, can capture credit-state migration, is the affine Markov model proposed by Hurd and Kuznetsov [16] and used by them in pricing CDOs in [15].

Some exact and approximate methods for loss-distribution evaluation have been studied in [22]. There are also Fourier transform related methods. In particular, the saddle point approximation method[2] was applied by Antonov, Mechkov, and Misirpashaev [3] and Yang, Hurd and Zhang [32], for pricing CDOs. Here we apply the exponential-approximation method, as described in the previous section, to synthetic CDO pricing.

3.2 The pricing equation and the Gaussian copula model

To illustrate our method, we use a simple one-factor Gaussian copula model. Examples of other models, to which the EAP method is applicable, are given in the Conclusion, Section 4. It is assumed that the interest rates are deterministic and the recovery rate corresponding to

[2]In principle, this method is applicable to large pools, as the pool size plays the role of the large parameter (albeit not in the classical multiplicative manner) in the inverse Fourier integrand's exponent to which the saddle point method is applied.

each underlying name is constant. Let $0 < t_1 < t_2 < \cdots < t_n = T$ be the set of premium dates, with T denoting the maturity of the CDO, and $d_1, d_2, \ldots, d_n$ be the set of corresponding discount factors. Suppose there are K names in the pool with recovery-adjusted notional values $LGD_1, LGD_2, \ldots, LGD_K$ in properly chosen units. Let $\mathscr{L}_i^P$ be the pool's cumulative losses up to time t_i and ℓ be the attachment point of a specified tranche of thickness S. An attachment point of a tranche is a threshold that determines whether some losses of the pool shall be absorbed by this tranche, *i.e.,* if the realized losses of the pool are less than ℓ, then the tranche will not suffer any loss, otherwise it will absorb an amount up to S. Accordingly, the detachment point of the tranche is $u = S + \ell$. Thus the loss absorbed by the specified tranche up to time t_i is $L_i = \min\left(S, (\mathscr{L}_i^P - \ell)^+\right)$. If we further assume the fair spread s for the tranche is a constant, then it can be calculated from the equation (see e.g., [8], [20])

$$s = \frac{\mathbb{E}\left[\sum_{i=1}^n (L_i - L_{i-1}) d_i\right]}{\mathbb{E}\left[\sum_{i=1}^n (S - L_i)(t_i - t_{i-1}) d_i\right]} = \frac{\sum_{i=1}^n \mathbb{E}\left[(L_i - L_{i-1}) d_i\right]}{\sum_{i=1}^n \mathbb{E}\left[(S - L_i)(t_i - t_{i-1}) d_i\right]}, \tag{10}$$

with $t_0 = 0$ and $\mathbb{E}\left[L_0\right] = 0$.

Assuming the discount factors d_i are independent of the $\mathscr{L}_j^P$, it follows from (10) that the problem of computing the fair spread s reduces to evaluating the expected cumulative losses $\mathbb{E}\left[L_i\right]$, $i = 1, 2, \ldots, n$. In order to compute this expectation, we have to specify the default processes for each name and the correlation structure of the default events. One-factor models were first introduced by Vasicek [31] to estimate the loan loss distribution and then generalized by Li [26], Gordy and Jones [12], Hull and White [14], Iscoe, Kreinin and Rosen [21], Laurent and Gregory [25], and Schönbucher [30], to name a few.

Let τ_k be the default time of name k. Assume the risk-neutral default probabilities $\pi_k(t) = \mathbb{P}\left(\tau_k < t\right)$, $k = 1, 2, \ldots, K$, that describe the default-time distributions of all K names are available, where τ_k and t take discrete values from $\{t_1, t_2, \ldots, t_n\}$. The dependence structure of the default times are determined in terms of their creditworthiness indices Y_k, which are defined by

$$Y_k = \beta_k X + \sigma_k \varepsilon_k, \qquad k = 1, 2, \ldots, K, \tag{11}$$

where X is the systematic risk factor, ε_k are idiosyncratic factors that are mutually independent and are also independent of X; β_k and σ_k are constants satisfying the relation $\beta_k^2 + \sigma_k^2 = 1$.

These risk-neutral default probabilities and the creditworthiness indices are related by the threshold model

$$\pi_k(t) = \mathbb{P}(\tau_k < t) = \mathbb{P}(Y_k < H_k(t)), \tag{12}$$

where $H_k(t)$ is the default barrier of the k-th name at time t.

Thus the correlation structure of default events is captured by the systematic risk factor X. Conditional on a given value x of X, all default events are independent. If we further assume, as we do, that X and ε_k follow the standard normal distribution, then Y_k is a standard normal random variable and from (12) we have $H_k(t) = \Phi^{-1}(\pi_k(t))$. Furthermore, the correlation between two different indices Y_i and Y_j is $\beta_i\beta_j$.

The conditional, risk-neutral default-time distribution is defined by

$$\pi_k(t;x) = \mathbb{P}(Y_k < H_k(t) | X = x). \tag{13}$$

Thus from (11) and (13) we have

$$\pi_k(t;x) = \Phi\left(\frac{H_k(t) - \beta_k x}{\sigma_k}\right). \tag{14}$$

The conditional and unconditional risk-neutral default-time probabilities at the premium date t_i are denoted by $\pi_k(i;x)$ and $\pi_k(i)$, respectively.

In this conditional independence framework, the expected cumulative tranche losses $\mathbb{E}[L_i]$ can be computed as

$$\mathbb{E}[L_i] = \int_{-\infty}^{\infty} \mathbb{E}_x[L_i] \, \mathrm{d}\Phi(x), \tag{15}$$

where $\mathbb{E}_x[L_i] = \mathbb{E}_x\left[\min\left(S, (\mathscr{L}_i^P - \ell)^+\right)\right]$ is the expectation of L_i conditional on the value of X being x, where $\mathscr{L}_i^P = \sum_{k=1}^K LGD_k \mathbf{1}_{\{Y_k < H_k(t_i)\}}$, and the indicators $\mathbf{1}_{\{Y_k < H_k(t_i)\}}$ are mutually independent conditional on X. Generally, the integration in (15) needs to be evaluated numerically using an appropriate quadrature rule:

$$\mathbb{E}[L_i] \approx \sum_{m=1}^{M} \mathrm{w}_m \mathbb{E}_{x_m}\left[\min\left(S, (\mathscr{L}_i^P - \ell)^+\right)\right]. \tag{16}$$

In the notation of Section 2, $Z = \mathscr{L}_i^P$, $Z_k = LGD_k \mathbf{1}_{\{Y_k < H_k(t_i)\}}$, and $\mathscr{M} = X$.

3.3 CDO pricing based on the exponential approximation

Notice from formula (16) that the fundamental problem in CDO pricing is how to evaluate the conditional expected loss $\mathbb{E}_{x_m}\left[\min\left(S, (\mathscr{L}_i^P - \ell)^+\right)\right]$ with a given value x_m of X. Since

$$\min\left(S, (\mathscr{L}_i^P - \ell)^+\right) = f(\mathscr{L}_i^P; \ell, \ell + S), \tag{17}$$

from (7) we see that

$$\begin{aligned}
\min\left(S, (\mathscr{L}_i^P - \ell)^+\right) \approx &(\ell + S)\left[1 - \sum_{n=1}^{N} \omega_n \exp\left(\gamma_n \frac{\mathscr{L}_i^P}{\ell + S}\right)\right] - \ell\left[1 - \sum_{n=1}^{N} \omega_n \exp\left(\gamma_n \frac{\mathscr{L}_i^P}{\ell}\right)\right] \\
=&S - (\ell + S)\sum_{n=1}^{N} \omega_n \exp\left(\frac{\gamma_n}{\ell + S}\mathscr{L}_i^P\right) + \ell \sum_{n=1}^{N} \omega_n \exp\left(\frac{\gamma_n}{\ell}\mathscr{L}_i^P\right).
\end{aligned}$$

As a special case of (8) we have

$$\begin{aligned}
\mathbb{E}_{x_m}\left[\min\left(S, (\mathscr{L}_i^P - \ell)^+\right)\right] \approx &S - (\ell + S)\sum_{n=1}^{N} \omega_n \mathbb{E}_{x_m}\left[\exp\left(\frac{\gamma_n}{\ell + S}\sum_{k=1}^{K} LGD_k \mathbf{1}_{\{Y_k < H_k(t_i)\}}\right)\right] \\
&+ \ell \sum_{n=1}^{N} \omega_n \mathbb{E}_{x_m}\left[\exp\left(\frac{\gamma_n}{\ell}\sum_{k=1}^{K} LGD_k \mathbf{1}_{\{Y_k < H_k(t_i)\}}\right)\right] \\
=&S - (\ell + S)\sum_{n=1}^{N} \omega_n \prod_{k=1}^{K} \mathbb{E}_{x_m}\left[\exp\left(\frac{\gamma_n}{\ell + S} LGD_k \mathbf{1}_{\{Y_k < H_k(t_i)\}}\right)\right] \\
&+ \ell \sum_{n=1}^{N} \omega_n \prod_{k=1}^{K} \mathbb{E}_{x_m}\left[\exp\left(\frac{\gamma_n}{\ell} LGD_k \mathbf{1}_{\{Y_k < H_k(t_i)\}}\right)\right],
\end{aligned} \tag{18}$$

where

$$\mathbb{E}_{x_m}\left[\exp\left(\frac{\gamma_n}{\ell + S} LGD_k \mathbf{1}_{\{Y_k < H_k(t_i)\}}\right)\right] = \pi_k(i; x_m)\exp\left(\frac{\gamma_n}{\ell + S} LGD_k\right) + (1 - \pi_k(i; x_m)),$$

$$\mathbb{E}_{x_m}\left[\exp\left(\frac{\gamma_n}{\ell} LGD_k \mathbf{1}_{\{Y_k < H_k(t_i)\}}\right)\right] = \pi_k(i; x_m)\exp\left(\frac{\gamma_n}{\ell} LGD_k\right) + (1 - \pi_k(i; x_m)),$$

since $\mathbf{1}_{\{Y_k < H_k(t_i)\}} = 1$ with probability $\pi_k(i; x_m)$ and 0 with probability $1 - \pi_k(i; x_m)$, and LGD_k is a constant.

3.4 Numerical results

In this section we present numerical results comparing the accuracy and the computational time for EAP and the exact method, which we denote by JKM, proposed in [22]. The results presented below are based on a sample of $15 = 3 \times 5$ pools: three pool sizes (100, 200, 400) and, for each size, five different types in terms of the notional values of the underlying reference entities. The possible notional values for each pool are summarized in Table 3. The number of reference entities is the same in each of the homogeneous subgroups in a given pool. For example, for a 200-name pool of type 3, there are four homogeneous subgroups each with 50 reference entities of notional values 50, 100, 150, and 200, respectively. Thus, the higher the pool type, the more heterogeneous is the pool, with pool type 1 indicating a completely homogeneous pool in terms of notional values. In the figures and tables that follow, the label K-m denotes a pool of size K and pool type m.

Table 3: Selection of notional values of K-name pools

Pool Type	1	2	3	4	5
Notional values	100	50, 100	50, 100, 150, 200	20, 50, 100, 150, 200	$10, 20, \ldots, K$

For each name, the common risk-neutral cumulative default probabilities are given in Table 4.

Table 4: Risk-neutral cumulative default probabilities

Time	1 yr.	2 yrs.	3 yrs.	4 yrs.	5 yrs.
Prob.	0.0072	0.0185	0.0328	0.0495	0.0680

The recovery rate is assumed to be 40% for all names. Thus the LGD of name k is $0.6N_k$. The maturity of a CDO deal is five years (*i.e.*, $T = 5$) and the premium dates are $t_i = i, i = 1, \ldots, 5$ years from today ($t_0 = 0$). The continuously compounded interest rates are $r_1 = 4.6\%, r_2 = 5\%, r_3 = 5.6\%, r_4 = 5.8\%$ and $r_5 = 6\%$. Thus the corresponding discount factors, defined by $d_i = \exp(-t_i r_i)$, are $0.9550, 0.9048, 0.8454, 0.7929$ and 0.7408, respectively. All CDO pools have five tranches that are determined by the attachment points (ℓ's) of the tranches. For this experiment, the five attachment points are: 0, 3%, 7%, 10% and 15%, with the last detachment point being 30%. The constants β_k are all 0.5. (In practice, the β_k's are known as sensitivities and are calibrated to market data. Here they are simply taken as input to the model, for the purpose of numerical experimentation.)

All methods for this experiment were coded in Matlab and the programs were run on a Pentium D 925 PC. The results are presented in Tables 5, 6 and 7.

The accuracy comparison results are presented in Table 5. The four numbers in each pair of brackets in the main part of the table are the spreads in basis points for the first four tranches of the corresponding pool. For example, $(2248.16, 635.22, 268.22, 118.34)$ are the spreads evaluated by the JKM method for the first four tranches of the 200-name homogeneous pool with type 1 reference entity composition. The entries under "25-term", "100-term", and "400-term" are the spreads evaluated using the exponential-approximation method with 25, 100 and 400 terms, respectively. From the table we can see that, as the number of terms increases, the accuracy of the spreads improves.

As a side-experiment, a comparison was made between the spreads for the third tranche, $[7\%, 10\%]$, and its thickening, $[7\%, 10.1\%]$, to check for monotonicity. In all cases, the spreads decreased (as they should) and by the amounts shown in Table 6. For the EAP method, 25 terms were used, as that represents the least accurate of the approximations used in the experiments and therefore provides the most stringent test.

The ratio of CPU times used by the JKM method and the exponential-approximation method using different numbers of terms, for the 15 test pools are presented in Table 7. The numbers in the table represent the ratio, EAP/JKM, of average timings over 50 runs, for the calcuation of the spread for the fourth tranche, $[10\%, 15\%]$. Note that, for the exponential-approximation method, its CPU time is independent of the pool structure: its computational cost depends only on the number of names and the number of terms in the exponential approximation. This feature is not shared by most of the other commonly used methods for pricing CDOs. From the table we see that, as the pool becomes more heterogeneous, the ratio of (EAP time)/(JKM time) decreases; as the pool becomes larger, the ratio of (EAP time)/(JKM time) decreases. It is interesting to note from (7) that, for a given pool, to evaluate any single tranche using the exponential-approximation method takes about as much

Table 5: Accuracy comparison between the exact JKM method and the EAP method using 25, 100 and 400 terms

Pool ID	Exact	25-term	100-term	400-term
100-1	(2167.69, 642.44, 276.38, 123.50)	(2165.42, 643.11, 276.90, 123.25)	(2167.77, 642.49, 276.30, 123.47)	(2167.73, 642.44, 276.38, 123.50)
100-2	(2142.13, 647.07, 278.40, 124.34)	(2141.24, 647.51, 278.12, 124.6)	(2142.41, 647.03, 278.29, 124.45)	(2142.20, 647.06, 278.38, 124.35)
100-3	(2128.39, 648.42, 279.39, 125.38)	(2128.97, 648.18, 279.51, 125.56)	(2128.52, 648.44, 279.44, 125.38)	(2128.44, 648.42, 279.39, 125.39)
100-4	(2097.58, 651.38, 282.49, 127.35)	(2097.14, 651.60, 282.59, 127.57)	(2097.61, 651.45, 282.49, 127.33)	(2097.6, 651.39, 282.50, 127.35)
100-5	(3069.39, 688.44, 167.46, 35.80)	(3069.46, 688.63, 167.44, 35.88)	(3069.51, 688.44, 167.47, 35.80)	(3069.43, 688.44, 167.46, 35.81)
200-1	(2248.16, 635.22, 268.22, 118.34)	(2246.75, 635.79, 268.36, 118.55)	(2248.17, 635.29, 268.22, 118.28)	(2248.18, 635.22, 268.22, 118.34)
200-2	(2237.60, 636.69, 269.06, 118.85)	(2236.93, 636.98, 269.28, 119.09)	(2237.69, 636.69, 268.99, 118.91)	(2237.62, 636.69, 269.06, 118.85)
200-3	(2229.45, 637.58, 269.84, 119.32)	(2229.11, 637.87, 270.00, 119.53)	(2229.45, 637.63, 269.85, 119.32)	(2229.46, 637.58, 269.84, 119.32)
200-4	(2212.52, 639.43, 271.42, 120.30)	(2212.44, 639.65, 271.62, 120.48)	(2212.59, 639.48, 271.39, 120.30)	(2212.53, 639.44, 271.42, 120.3)
200-5	(3350.42, 662.54, 136.15, 24.47)	(3350.43, 662.82, 136.14, 24.51)	(3350.50, 662.54, 136.14, 24.47)	(3350.43, 662.54, 136.15, 24.47)
400-1	(2291.12, 30.91, 264.05, 115.78)	(2290.68, 631.3, 264.34, 116.07)	(2291.17, 630.99, 263.98, 115.77)	(2291.12, 630.92, 264.05, 115.78)
400-1	(2285.92, 31.56, 264.50, 116.05)	(2285.76, 631.84, 264.79, 116.35)	(2285.95, 631.57, 264.51, 116.04)	(2285.93, 631.56, 264.50, 116.06)
400-3	(2281.84, 632.00, 264.88, 116.29)	(2281.82, 632.26, 265.15, 116.57)	(2281.82, 632.05, 264.88, 116.26)	(2281.85, 632.01, 264.89, 116.29)
400-4	(2273.15, 632.96, 265.69, 116.78)	(2273.19, 633.20, 265.95, 117.05)	(2273.22, 632.96, 265.69, 116.76)	(2273.16, 632.96, 265.69, 116.78)
400-5	(3427.70, 649.51, 130.86, 23.59)	(3427.86, 649.78, 130.87, 23.65)	(3427.79, 649.50, 130.85, 23.58)	(3427.71, 649.51, 130.86, 23.59)

Table 6: Tranche spread-differences between tranches $[7, 10]$ and $[7, 10.1]$, to check monotonicity of spreads; 25 terms used in EAP method

Pool ID	JKM	EAP-25
100-1	2.46	2.69
100-2	2.88	2.66
100-3	2.62	2.67
100-4	2.69	2.69
100-5	2.73	2.73
200-1	2.62	2.62
200-2	2.69	2.62
200-3	2.62	2.63
200-4	2.63	2.64
200-5	2.42	2.42
400-1	2.60	2.60
400-2	2.60	2.61
400-3	2.60	2.61
400-4	2.61	2.61
400-5	2.33	2.33

time as to evaluate any other tranche, except for the the lowest tranche. On the other hand, for the exact method, calculating the spread for the j-th tranche takes as much time as calculating the spreads for the first j tranches, provided that expected losses are calculated for cumulative tranches, 0% to the various detachment points, and then differenced to obtain results for individual tranches. This is the most efficient way to handle multiple tranches, using the JKM method.

For a real deal with LGDs plotted in Figure 1, the exponential approximation method runs faster than the recursive method JKM. Calculated with the JKM method, the spreads for the first four tranches $[0, 3]$, $[3, 7]$, $[7, 10]$ and $[10, 15]$, are $(2176.5, 643.3, 274.8, 122.4)$, with

Table 7: Ratio of CPU times used by the EAP method over those used by the JKM method, to evaluate the fourth tranche, [10%, 15%], on the test pools

	# terms in EAP				
Pool ID	25	50	100	200	400
100-1	1.65	2.75	5.69	10.53	21.12
100-2	1.43	2.57	5.32	9.85	19.77
100-3	1.11	2.01	4.15	7.68	15.42
100-4	0.44	0.79	1.64	3.05	6.11
100-5	0.40	0.72	1.49	2.76	5.53
200-1	0.83	1.25	2.08	3.66	7.01
200-2	0.77	1.18	1.91	3.43	6.52
200-3	0.66	1.03	1.68	2.91	5.64
200-4	0.43	0.66	1.07	1.90	3.65
200-5	0.27	0.41	0.67	1.20	2.29
400-1	0.44	0.72	1.14	2.08	4.14
400-2	0.42	0.62	1.07	1.96	3.87
400-3	0.35	0.56	0.87	1.59	3.18
400-4	0.17	0.27	0.42	0.77	1.54
400-5	0.06	0.08	0.14	0.26	0.52

the calculation taking 7 seconds. Using the 100-term approximation method, we obtained the same spreads, but used only 1.18 seconds; using the 50-term approximation we obtained the spreads $(2176.5, 643.2, 274.9, 122.5)$ in 0.65 second.

Comparisons of the accuracy of other methods, when applied to the main examples of this section, are provided in [18]; see [22] for a description of the methods and other similar examples.

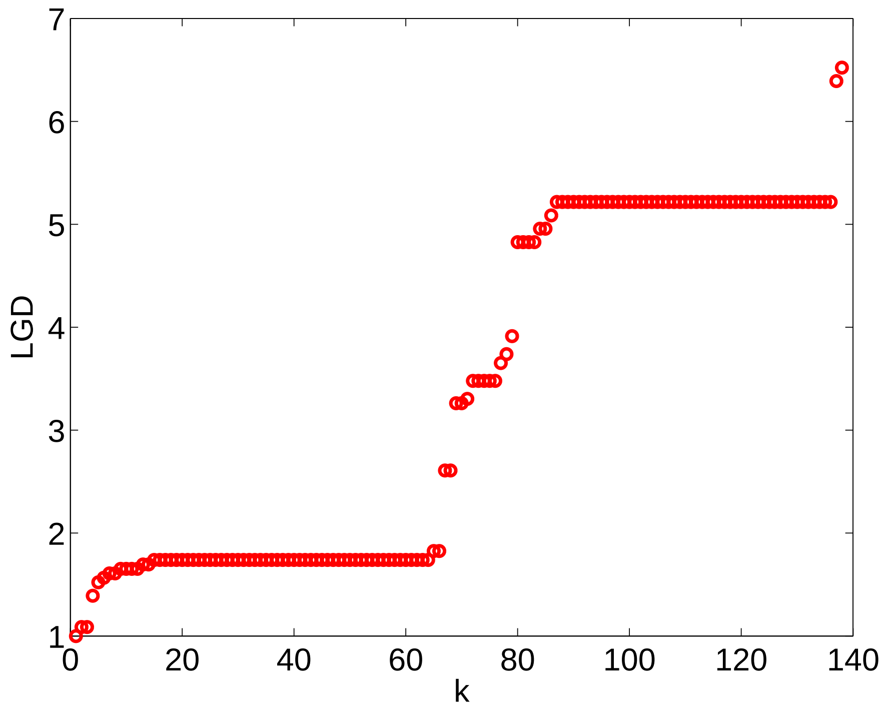

Figure 1: LGDs of the reference entities in a real portfolio, the values are relative to the smallest LGD

3.5 Sensitivities

It is of practical interest to see how sensitive a tranche's value or spread (when the latter is not an input) is, to small changes in various parameters of the adopted model. Generally, there are two approaches to sensitivity calculations: infinitesimal and macroscopic. In the infinitesimal approach, one calculates the derivative of the value or spread with respect to the parameter in closed form and then uses the derivative in a linear approximation. In the macroscopic approach, the value or spread is recalculated at a slightly perturbed level of the chosen parameter, and then the sensitivity is taken as the difference between the new and original values or spreads. In principle, the infinitesimal approach can be carried out for the EAP method, based on the formulae (15)-(18), just using the usual rules of calculus, especially the chain rule. However, the resulting expression may be very complicated and it is quite possible that the large number of arithmetic operations involved, will be slower than a simple revaluation, especially if the invariant parts of the expression (18) are cached and available during the revaluation. Of course, the identity of the parameter being tweaked, has a strong influence on the result of the comparison of the two approaches. For example, the formula resulting from tweaking a LGD is much less complicated than one resulting from tweaking a PD. For the sake of generality, we will follow the macroscopic approach and restrict attention to the accuracy of spread sensitivities calculated with the EAP method, for the sake of concreteness, comparing the results to the benchmark JKM method.

Of all the model parameters that can be perturbed or *tweaked*, the probabilities of default are the most interesting ones, especially as they affect all tranches.[3] Specifically, we will continue with the 15-pool examples of the previous section and study numerically the result of subjecting the probablity of default (PD) curve to a positive parallel shift of 20 bp. The results for the alternative choices of 10 bp and 5 bp, for the parallel shift, are similar and described in detail in [19]. In Figures 2–5, the pool size 100 (respectively, 200, 400) is indicated by a solid (respectively, dashed, dot-dashed) line while the pool type 1 (respectively, 2, 3, 4, 5) is indicated by the marker ■ (respectively, ⧫, ▲, ▼, ►).

We start with a 50-term EAP. Reported errors are relative and expressed as percentages, in absolute value. The term *base* refers to prior to tweaking the PD curve while the term *tweaked* refers to after tweaking the PD curve. The errors in the base values of the spreads of the five tranches, for each of the 15 pools, are given in Figure 2. Next, the errors in the tweaked values of the spreads of the five tranches, for each of the 15 pools, are given in Figure 3, along with the errors in the sensitivities. The most striking feature is the reduction of the error in the sensitivities, relative to the size of the errors in the base and tweaked values.

Since the largest errors are for the highest tranche, it is of interest to see how it behaves as the number of terms in the EAP is increased. In Figures 4–5, the signed relative errors are plotted for the base and tweaked spreads and the spread sensitivities.

[3]In practice, the default probabilities are themselves calculated quantities, being functions of hazard rates which in turn are bootstrapped from market-quoted CDS spreads. For simplicity we will tweak the default probabilities directly.

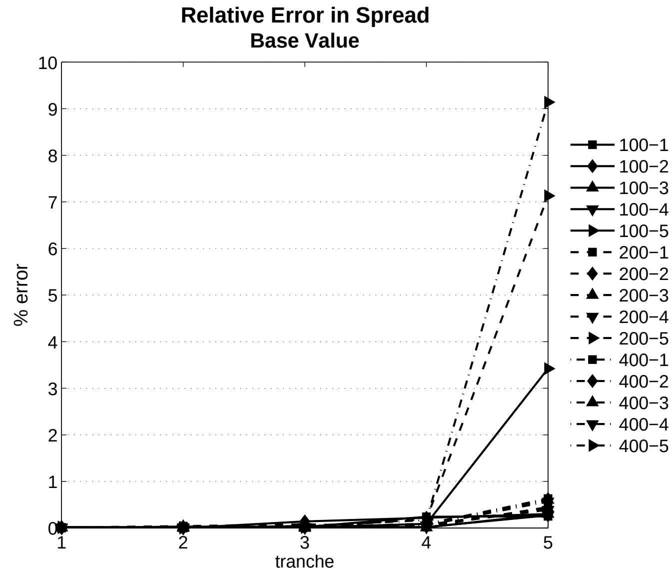

Figure 2: Comparative results in percentage error for the 50-term EAP method for all five tranches and 15 pools, K-m, where $K = 100, 200, 400$ and $m = 1, 2, 3, 4, 5$: base values.

4 Conclusion

A new method for pricing correlation-dependent derivatives has been proposed. The method is based on an exponential approximation to the payoff function. With this approximation, the evaluation of the conditional expectation of the stop-loss function of the credit portfolio can be computed by calculating a series of conditional expectations for individual obligors. In Section 3, we applied this method to synthetic CDO pricing where the correlation structure of the underlying obligors is specified through a simple one-factor Gaussian copula model. This method could be applied to other models belonging to the conditional independence framework. From formula (8), we see that there are no restrictions on the distribution of each Z_k, other than the ease of calculation of its conditional Laplace transform.

The EAP method applies to the basic affine jump diffusion intensity-based models introduced by Duffie and Gârleanu [10] and extended succesively by Mortensen [27] and then Eckner [11]. These models belong to the conditional independence framework with the auxiliary factor $\mathcal{M}$ (see Section 2 herein) being the integrated common component of the default

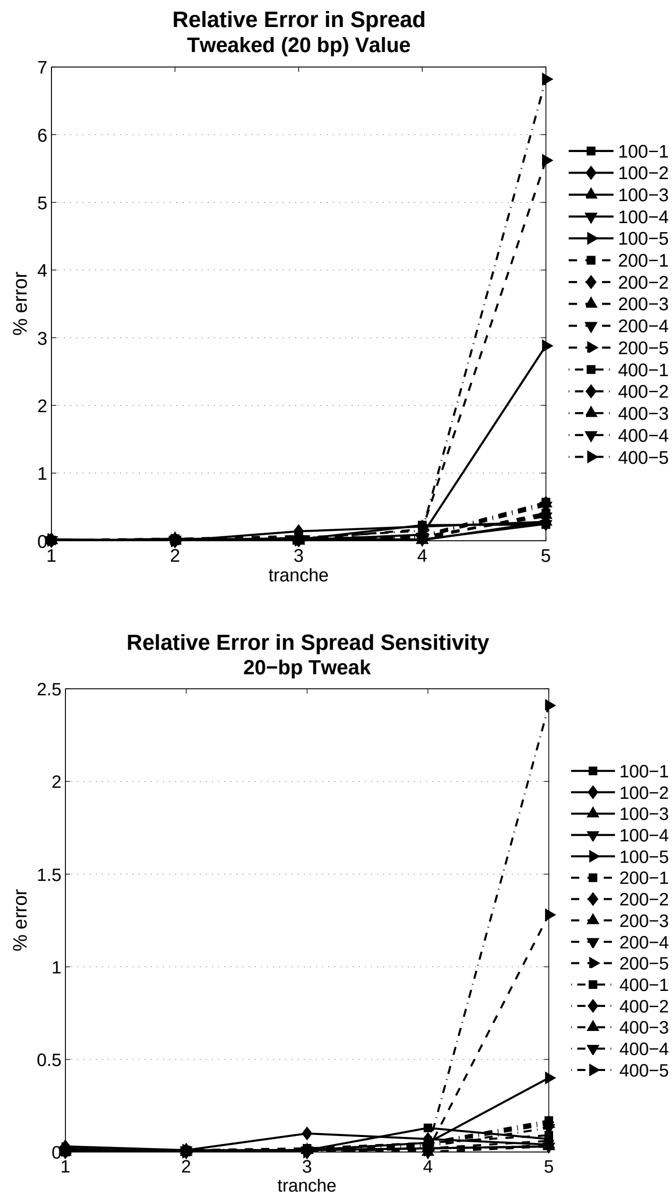

Figure 3: Comparative results in percentage error for the 50-term EAP method for all five tranches and 15 pools, K-m, where $K = 100, 200, 400$ and $m = 1, 2, 3, 4, 5$; tweak size: 20 bp.

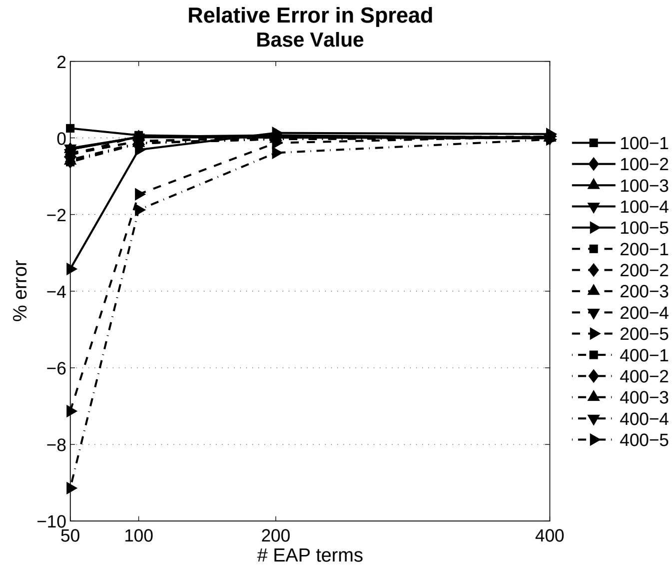

Figure 4: Comparative results in percentage signed error for the 50-, 100-, 200-, and 400-term EAP method for the Senior tranche and 15 pools, K-m, where $K = 100, 200, 400$ and $m = 1, 2, 3, 4, 5$: base values.

intensities. There are explicit formulas available for the conditional default probabilities and the probability density function of $\mathcal{M}$ can be obtained by inverting its Fourier transform, which is known in a semi-analytic form – see the cited references for details. (In [10], there is also a so-called sectoral extension of the model. In that case, $\mathcal{M}$ would be a random vector, consisting of the independent, integrated sectoral default intensities as well as the single global one. As such, a multidimensional numerical integration would be required in the last unconditioning step of either approach.) For the calibration of these models, it is efficient to use a low-accuracy EAP (e.g., a 25-term one), in the first stage of the calibration. Once the solution is close to the optimal one, one can use a high-accuracy EAP to speed up the convergence. Note that the EAP approach avoids the necessity of rounding losses, which is a property of the algorithm of [2] that is used in [27] and [11] to calculate the pool loss distribution.

The EAP approach also applies to more general models that can, e.g., incorporate stochastic recovery rates as in [1] or [24]. In [1], the recovery rate is constant, conditional on the auxilliary factor (latent credit driver), X. In [24], although LGD and $\mathbf{1}_{\{\tau \leq t\}}$ are correlated, for

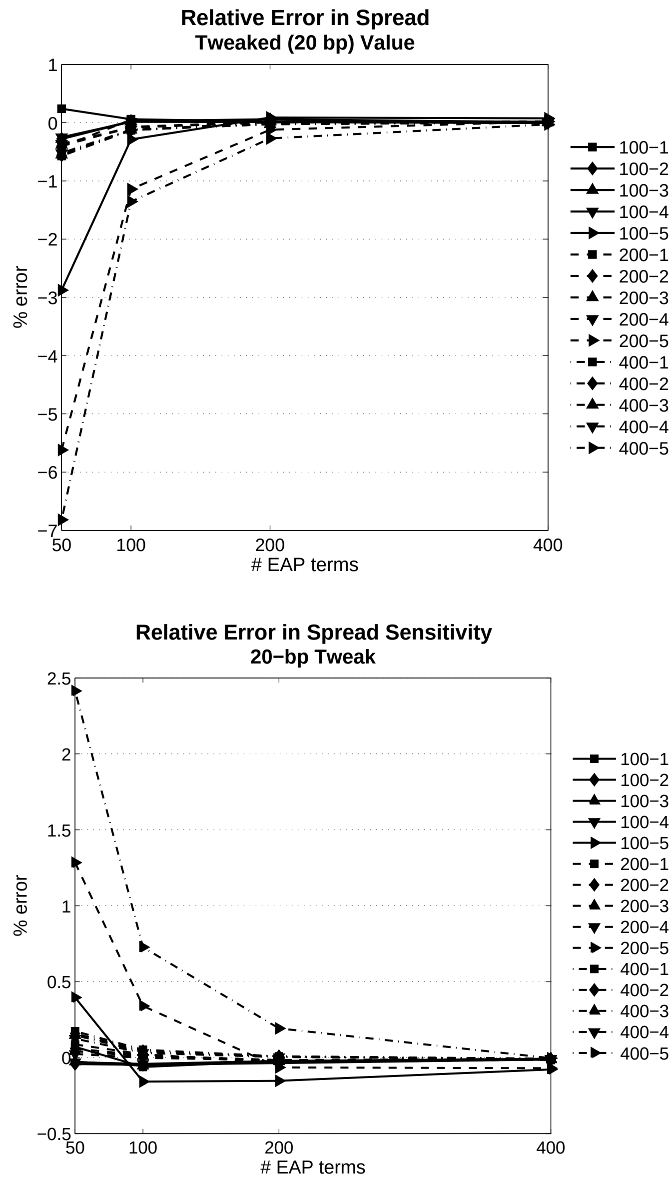

Figure 5: Comparative results in percentage signed error for the 50-, 100-, 200-, and 400-term EAP method for Senior tranche and 15 pools, K-m, where $K = 100, 200, 400$ and $m = 1, 2, 3, 4, 5$; tweak size: 20 bp.

each obligor, the LGD and the indicator function depend on the same Y, the creditworthiness of the obligor; and the definition of LGD is explicit. Thus we can compute the conditional expectation, $\mathbb{E}[\text{LGD}\, \mathbf{1}_{\{\tau \leq t\}} \,|\, X]$, explicitly.

Here is a summary of the advantages and disadvantages of the EAP method, as applied to synthetic CDOs.

Advantages of EAP

- It is faster than the traditional (loss distribution) approach in any (hence, especially simultaneously all) of the following situations: single tranches; very heterogeneous pools; large pools. The CPU time for the EAP method is independent of the pool structure: its computational cost depends only on the number of names in the pool and the number of terms in the exponential approximation.

- It is quite accurate; e.g., with 50 exponential terms, spreads are observed to be correct to within 1 bp, representing a 0.5% error for all but the highest tranches where the errors are of the same order of magnitude as the spreads.

- It does not round losses, as in many versions of the traditional approach which round losses to values on a regular grid (e.g., for use with FFT).

- The weights and exponents can be calculated once, stored, then used for many pools and tranches. Compared to the saddlepoint approximation method, this is the main advantage of EAP, as the saddlepoint method must compute some parameters dynamically.

- In the EAP method, the number of terms used can be tailored to each tranche with a higher number of terms used for higher tranches, to improve accuracy.

- It easily handles numerical sensitivities (e.g., to PDs) and is quite accurate, even for the highest tranches.

Disadvantages (Scope) of EAP

- It is slower than the traditional approach in any of the following situations: evaluation of multiple (> 3) tranches on a single pool; the highest tranches (requiring around 200 terms for comparable accuracy, ceteris paribus [such as no rounding of losses]); very homogeneous pools.

References

[1] Salah Amraoui, Laurent Cousot, Sébastien Hitier, and Jean-Paul Laurent. Pricing CDOs with state dependent stochastic recovery rates. Available from http://www.defaultrisk.com/pp_cdo_85.htm, September 2009.

[2] Leif Andersen, Jakob Sidenius, and Susanta Basu. All your hedges in one basket. *Risk*, pages 61–72, November 2003.

[3] Alexandre Antonov, Serguei Mechkov, and Timur Misirpashaev. Analytical techniques for synthetic CDOs and credit default risk measures. Available from http://www.defaultrisk.com/pp_crdrv_77.htm, May 2005.

[4] Robert Eric Beard, Teino Pentikäinen, and Erkki Pesonen. *Risk Theory: The Stochastic Basis of Insurance*. Monographs on Statistics and Applied Probability. Chapman and Hall, 3rd edition, 1984.

[5] Richard Bruyere and Christophe Jaeck. *Collateralized Debt Obligations*, volume 1 of *Encyclopedia of Quantitative Finance*, pages 278–284. John Wiley & Sons, 2010.

[6] Xavier Burtschell, Jon Gregory, and Jean-Paul Laurent. A comparative analysis of CDO pricing models. *The Journal of Derivatives*, 16(4):9–37, 2009.

[7] Nelson De Pril. On the exact computation of the aggregate claims distribution in the individual life model. *ASTIN Bulletin*, 16(2):109–112, 1986.

[8] Ben De Prisco, Ian Iscoe, and Alex Kreinin. Loss in translation. *Risk*, pages 77–82, June 2005.

[9] Alain Debuysscher and Marco Szegö. The Fourier transform method – technical document. Working report, Moody's Investors Service, January 2003.

[10] Darrell Duffie and Nicholae Gârleanu. Risk and valuation of collateralized debt obligations. *Financial Analysts Journal*, 57:41–59, January/February 2001.

[11] Andreas Eckner. Computational techniques for basic affine models of portfolio credit risk. *Journal of Computational Finance*, 13(1):1–35, Summer 2009.

[12] Michael Gordy and David Jones. Random tranches. *Risk*, 16(3):78–83, March 2003.

[13] Christian Hipp. Improved approximations for the aggregate claims distribution in the individual model. *ASTIN Bulletin*, 16(2):89–100, 1986.

[14] John Hull and Alan White. Valuation of a CDO and an n^{th} to default CDS without Monte Carlo simulation. *Journal of Derivatives*, 12(2):8–23, 2004.

[15] Tom Hurd and Alexey Kuznetsov. Fast CDO computations in the affine Markov chain model. Available from http://www.math.mcmaster.ca/tom/NewAMCCDO.pdf, October 2006.

[16] Tom Hurd and Alexey Kuznetsov. Affine Markov chain models of multifirm credit migration. *Journal of Credit Risk*, 3(1):3–29, Spring 2007.

[17] Ian Iscoe, Ken Jackson, Alex Kreinin, and Xiaofang Ma. An exponential approximation to the hockey-stick function. Submitted to *Journal of Applied Numerical Mathematics.* Available from http://www.cs.toronto.edu/NA/reports.html#IJKM.paper2, 2010.

[18] Ian Iscoe, Ken Jackson, Alex Kreinin, and Xiaofang Ma. Supplemental numerical examples for CDO pricing methodology. Available from http://www.cs.toronto.edu/pub/reports/na/suppl_ex.pdf, 2011.

[19] Ian Iscoe, Ken Jackson, Alex Kreinin, and Xiaofang Ma. Supplemental numerical examples for CDO pricing methodology: Sensitivities. Available from http://www.cs.toronto.edu/pub/reports/na/suppl_sens.pdf, 2011.

[20] Ian Iscoe and Alex Kreinin. Valuation of synthetic CDOs. *Journal of Banking & Finance*, 31(11):3357–3376, November 2007.

[21] Ian Iscoe, Alex Kreinin, and Dan Rosen. An integrated market and credit risk portfolio model. *Algo Research Quarterly*, 2(3):21–38, September 1999.

[22] Ken Jackson, Alex Kreinin, and Xiaofang Ma. Loss distribution evaluation for synthetic CDOs. Available from http://www.cs.toronto.edu/pub/reports/na/JKM.paper1.pdf, February 2007.

[23] Stuart A. Klugman, Harry H. Panjer, and Gordon E. Willmot. *Loss Models from Data to Decisions*. John Wiley & Sons, Inc., 1998.

[24] Martin Krekel. Pricing distressed CDOs with base correlation and stochastic recovery. Available from http://www.defaultrisk.com/pp_cdo_60.htm, May 2008.

[25] Jean-Paul Laurent and Jon Gregory. Basket default swaps, CDOs and factor copulas. *Journal of Risk*, 17:103–122, 2005.

[26] David X Li. On default correlation: A copula function approach. *Journal of Fixed Income*, 9(43-54), 2000.

[27] Allan Mortensen. Semi-analytical valuation of basket credit derivatives in intensity-based models. *The Journal of Derivatives*, 13:8–26, Summer 2006.

[28] Harry H Panjer. Recursive evaluation of a family of compound distributions. *ASTIN Bulletin*, 12(1):22–26, 1981.

[29] Susan M Pitts. A functional approach to approximations for the individual risk model. *ASTIN Bulletin*, 34(2):379–397, 2004.

[30] Philipp J Schönbucher. *Credit Derivatives Pricing Models*. Wiley Finance Series. John Wiley & Sons Canada Ltd., 2003.

[31] Oldrich Alfons Vasicek. The distribution of loan portfolio value. *Risk*, 15(12):160–162, December 2002.

[32] Jingping Yang, Tom Hurd, and Xuping Zhang. Saddlepoint approximation method for pricing CDOs. *Journal of Computational Finance*, 10:1–20, 2006.