Analysis of the Blunting Anti-Wrapping Strategy

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Interval methods for ODEs often face two obstacles in practical computations: the dependency problem and the wrapping effect. Taylor model methods, which have been developed by Berz and his group, have recently attracted attention. By combining interval arithmetic with symbolic calculations, these methods suffer far less from the dependency problem than traditional interval methods for ODEs. By allowing nonconvex enclosure sets for the flow of a given initial value problem, Taylor model methods have also a high potential for suppressing the wrapping effect.

Makino and Berz [1] advocate the so-called blunting method. In this paper, we analyze the blunting method (as an interval method) for a linear model ODE. We compare its convergence behavior with that of the well-known QR interval method.

1 Introduction

We consider a Taylor series method with constant stepsize and order on the test problem

$$y' = Ay, \quad y(0) = y_0 \in \boldsymbol{y}_0,$$

where $y \in \mathbb{R}^n$, $A \in \mathbb{R}^{n\times n}$, $n \geq 2$, and $\boldsymbol{y}_0$ is a given interval vector, accounting for uncertainty in initial conditions.

We represent the enclosure of

$$y(t_j; 0, \boldsymbol{y}_0) = \{y(t_j; 0, y_0) \mid y_0 \in \boldsymbol{y}_0\}$$

as

$$\big\{u_j + S_j\alpha + B_j r \mid \alpha \in \boldsymbol{\alpha},\, r \in \boldsymbol{r}_j\big\},$$

where $u_j, \alpha, r \in \mathbb{R}^n$, $\boldsymbol{r}_j \in \mathbb{IR}^n$; $S_j, B_j \in \mathbb{R}^{n\times n}$, B_j nonsingular, and $\boldsymbol{\alpha} = \boldsymbol{y}_0 - \mathrm{m}(\boldsymbol{y}_0)$. For an interval vector $\boldsymbol{z}$, $\mathrm{m}(\boldsymbol{z})$ denotes its midpoint.

Initially, when $j = 0$,

$$u_0 = \mathrm{m}(\boldsymbol{y}_0), \quad S_0 = I, \quad B_0 = I, \quad \text{and } \boldsymbol{r}_0 = 0.$$

In

$$y(t_j; 0, \boldsymbol{y}_0) \in \big\{u_j + S_j\alpha + B_j r \mid \alpha \in \boldsymbol{\alpha},\, r \in \boldsymbol{r}_j\big\},$$

- $\big\{u_j + S_j\alpha \mid \alpha \in \boldsymbol{\alpha}\big\}$ is an approximation to $y(t_j; 0, \boldsymbol{y}_0)$, and
- $\{B_j r \mid r \in \boldsymbol{r}_j\}$ is an approximation of the overestimation, or excess, accumulated in the integration process from 0 to t_j.

2 The blunting method

In

$$\boldsymbol{r}_j = (B_j^{-1}C_j)\boldsymbol{r}_{\boldsymbol{j-1}} + B_j^{-1}\boldsymbol{e}_j, \quad C_j = TB_{j-1}, \quad T = \sum_{i=0}^{k-1}\frac{(hA)^i}{i!},$$

we wish to select a nonsingular B_j such that

$$\{C_j r + e \mid r \in \boldsymbol{r}_{j-1},\, e \in \boldsymbol{e}_j\} \subseteq \{B_j r \mid r \in \boldsymbol{r}_j\},$$

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and $\{B_j r \mid r \in \boldsymbol{r}_j\}$ is a tight enclosure of

$$\{C_j r + e \mid r \in \boldsymbol{r}_{j-1},\, e \in \boldsymbol{e}_j\},$$

where $\boldsymbol{e}_j = \boldsymbol{z}_j - \mathrm{m}(\boldsymbol{z}_j)$, and $\boldsymbol{z}_j$ is the local error of the Taylor series method.

In the QR method, we perform a QR factorization $C_j = Q_j R_j$ and select $B_j = Q_j$. This choice leads to the simultaneous iteration $Q_j R_j = TQ_{j-1}$ [2]. In the blunting method, we select B_j from $C_j = TB_{j-1} = Q_j^* R_j^*$ (QR factorization of TB_{j-1}), $\widehat{B}_j = C_j D_j + Q_j G_j$, and $B_j = \widehat{B}_j F_j$. D_j is a diagonal matrix such that $C_j D_j$ is normalized (each column is of length 1 in $\|\cdot\|_2$). G_j is a diagonal matrix with blunting factors > 0 [1]. F_j is a diagonal matrix such that $B_j = \widehat{B}_j F_j$ is normalized. Letting $V_j = (R_j^* D_j + G_j) F_j$, we obtain the simultaneous iteration $Q_j^*(R_j^* V_{j-1}^{-1}) = TQ_{j-1}^*$. Choosing $Q_0 = Q_0^* = I$ (where I is the identity matrix), the relations between the respective matrices in the QR and in the blunting methods are $Q_j = Q_j^*$, $R_j = R_j^* V_{j-1}^{-1}$.

We are interested in the excess propagation in

$$(B_j^{-1} C_j)\boldsymbol{r}_{j-1} = (B_j^{-1} T B_{j-1})\boldsymbol{r}_{j-1}.$$

In the QR method, we have $B_j^{-1} T B_{j-1} = Q_j^T T Q_{j-1} = R_j$, whereas the blunting method reads

$$B_j^{-1} T B_{j-1} = V_j^{-1} Q_j^T Q_j R_j^* = V_j^{-1} R_j^* = V_j^{-1} R_j V_{j-1}.$$

Since the width of $\boldsymbol{r}_j$ is

$$\mathrm{w}(\boldsymbol{r}_j) = |B_j^{-1} T B_{j-1}| \mathrm{w}(\boldsymbol{r}_j) + |B_j^{-1}| \mathrm{w}(\boldsymbol{z}_j),$$

the excess propagation depends on the spectral radius of a certain matrix. In the QR method, this matrix is [2]

$$H_{j,i} = |Q_j^T T Q_{j-1}|\,|Q_{j-1}^T T Q_{j-2}| \cdots |Q_{i+1}^T T Q_i| = |R_j|\,|R_{j-1}| \cdots |R_{i+1}|,$$

whereas in the blunting method, it is

$$P_{j,i} = |B_j^{-1} T B_{j-1}|\,|B_{j-1}^{-1} T B_{j-2}| \cdots |B_{i+1}^{-1} T B_i| = |V_j^{-1} R_j V_{j-1}|\,|V_{j-1}^{-1} R_{j-1} V_{j-2}| \cdots |V_{i+1}^{-1} R_{i+1} V_i|.$$

Now we consider the case that T has eigenvalues λ_i of distinct magnitudes, i.e. $|\lambda_1| > |\lambda_2| > \cdots |\lambda_n| > 0$. In the QR method, the diagonal of $|R_j|$ converges to $(|\lambda_1|, |\lambda_2|, \cdots, |\lambda_n|)$, as $j \to \infty$. Thus, as j becomes sufficiently large, the diagonal of the upper triangular matrix $H_{j,i}$ behaves like

$$\big(\,|\lambda_1|^{j-i+1}, |\lambda_2|^{j-i+1}, \ldots, |\lambda_n|^{j-i+1}\,\big).$$

On the other hand, the diagonal of the upper triangular matrix $P_{j,i}$ behaves like

$$\big(\,|\lambda_1|^{j-i+1}, \alpha_{i,j}^{(2)} |\lambda_2|^{j-i+1}, \ldots, \alpha_{i,j}^{(n)} |\lambda_n|^{j-i+1}\,\big),$$

where

$$\alpha_{i,j}^{(k)} = \frac{(V_i)_{k,k}}{(V_j)_{k,k}}.$$

Since the $\alpha_{i,j}^{(k)}$ can be bounded above, the spectral radius of $P_{j,i}$ and the spectral radius of $H_{j,i}$ both tend to $|\lambda_1|^{j-i+1}$, as $j \to \infty$, so that the excess propagation in both methods should be similar, for j sufficiently large.

3 Remarks

- The blunting method and the QR method both work well for our simple test problem $y' = Ay$ (assuming T has eigenvalues of distinct magnitude).
- The suggested blunting factor 10^{-3} [1] may not always be a good choice. It seems reasonable for $y' = Ay$ to start with small blunting factors and increase them as j increases.
- At present, we do not know how to analyze the case that T has two or more eigenvalues of the same magnitude (this includes the important case the T has a pair of complex conjugate eigenvalues) or how to accommodate permutations in the QR and blunting methods. This question will be the subject of future research.

References

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