The Herbrand Semantics of Transaction Logic

A *state* is a class of equivalent first-order classical formulas. A state will be denoted by a single formula in the class. A *path* is any finite sequence of states. A *split* of a path $\pi = \langle s_1, ..., s_n \rangle$ is any pair of subpaths, $\pi_1 = \langle s_1, ..., s_i \rangle$ and $\pi_2 = \langle s_i, ..., s_n \rangle$. In this case, we write $\pi = \pi_1 \circ \pi_2$. Finally, we define $\top$ to be a special classical Herbrand structure that satisfies every first-order formula.

Definition 1 **[Structures]** *A* Herbrand path structure *is a mapping,* $\mathbf{M}$*, from paths to classical Herbrand structures. For every state, s, the structure* $\mathbf{M}(\langle s \rangle)$ *must satisfy all the formulas in s.*

A *variable assignment* is a mapping that takes a variable as input, and returns a Hebrand term as output. We extend the mapping from variables to function terms in the usual way. As usual in logic, we omit the variable assignment for *sentences*, *i.e.*, for formulas with no free variables.

Definition 2 **[Satisfaction]** *Let* $\mathbf{M}$ *be a Herbrand path structure, π be a a path, and ν be a a variable assignment. Then,*

- $\mathbf{M}, \pi \models_\nu p$ *iff* $\mathbf{M}(\pi) \models^c_\nu p$, *for any atomic formula p, where "$\models^c_\nu$" denotes classical satisfaction.*
- $\mathbf{M}, \pi \models_\nu \neg\phi$ *iff it is not the case that* $\mathbf{M}, \pi \models_\nu \phi$.
- $\mathbf{M}, \pi \models_\nu \phi \wedge \psi$ *iff* $\mathbf{M}, \pi \models_\nu \phi$ *and* $\mathbf{M}, \pi \models_\nu \psi$. $\vee$ *is the dual of* $\wedge$.
- $\mathbf{M}, \pi \models_\nu \phi \otimes \psi$ *iff* $\mathbf{M}, \pi_1 \models_\nu \phi$ *and* $\mathbf{M}, \pi_2 \models_\nu \psi$ *for* some *split* $\pi_1 \circ \pi_2$ *of path* π.
- $\mathbf{M}, \pi \models_\nu \phi \oplus \psi$ *iff* $\mathbf{M}, \pi_1 \models_\nu \phi$ *or* $\mathbf{M}, \pi_2 \models_\nu \psi$ *for* every *split* $\pi_1 \circ \pi_2$ *of path* π.
- $\mathbf{M}, \pi \models_\nu (\forall X)\phi$ *iff* $\mathbf{M}, \pi \models_\mu \phi$ *for* every *variable assignment μ that agrees with ν everywhere except on X. $\exists X$ is the dual of $\forall X$.*

For transaction formulas, satisfaction in path structures is defined as follows:

Definition 3 **[Models – Transaction Formulas]** *A path structure* $\mathbf{M}$ satisfies *a transaction formula* ϕ *iff* $\mathbf{M}, \pi \models \phi$ *for every path π.* $\mathbf{M}$ *is a* model *of a set of formulas iff it satisfies every formula in the set.*

For elementary transitions (*i.e.*, formulas of the form $\langle \mathbf{D}_1, \mathbf{D}_2 \rangle u$, where u is a ground atomic formula and $\mathbf{D}_1$ and $\mathbf{D}_2$ are first-order formulas) satisfaction is defined as follows.

Definition 4 **[Models – Elementary Transitions]** *A path structure* $\mathbf{M}$ satisfies *an elementary transition* $\langle \mathbf{D}_1, \mathbf{D}_2 \rangle u$ *if and only if* $\mathbf{M}, \langle \mathbf{D}_1, \mathbf{D}_2 \rangle \models u$. $\mathbf{M}$ *is a* model *of a set of transitions iff it satisfies every transition in the set.*

Definition 5 **[Executional Entailment]** *Let $\mathcal{B}$ be a set of elementary transitions,* $\mathbf{P}$ *be a set of transaction formulas, ϕ be a transaction formula, and* $\mathbf{D}_0, \mathbf{D}_1, \ldots, \mathbf{D}_n$ *be a sequence of first-order formulas. Then,*

$$\mathcal{B}, \mathbf{P}, \mathbf{D}_0, \mathbf{D}_1, \ldots, \mathbf{D}_n \models \phi$$

if and only if $\mathbf{M}, \langle \mathbf{D}_0, \mathbf{D}_1, \ldots, \mathbf{D}_n \rangle \models \phi$ *for every model,* $\mathbf{M}$*, of* $\mathcal{B}$ *and* $\mathbf{P}$.