A Logic for Hypothetical Reasoning*

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Abstract

This paper shows that classical logic is inappropriate for hypothetical reasoning and develops an alternative logic for this purpose. The paper focuses on a form of hypothetical reasoning which appears computationally tractable. Specifically, Horn-clause logic is augmented with rules, called *embedded implications*, which can hypothetically add atomic formulas to a rulebase. By introducing the notion of *rulebase independence*, we show that these rules can express hypothetical queries which classical logic cannot; and by adopting methods from modal logic, we show these rules to be intuitionistic. In particular, they form a subset of intuitionistic logic having semantic properties similar to those of Horn-clause logic.

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1 Introduction

Several researchers in the logic-programming community have pointed out the utility of augmenting Prolog with the ability to hypothetically add facts to a rulebase. Miller, for instance, has shown how such rules can structure the runtime environment of a logic program [13], and Warren and Manchanda have proposed such logics for reasoning about database updates [17, 9]. The legal domain has inspired much work into this kind of hypothetical reasoning. Gabbay, for example, has reported a need to augment Prolog with hypothetical rules in order to encode the British Nationality Act. The act contains rules such as, "You are eligible for citizenship if your father would be eligible if he were still alive" [6]. In addition, McCarty has developed a wide class of hypothetical rules for the purpose of constructing computer-based legal consultation systems, especially systems for reasoning about contract law and corporate tax law [10, 12].

Hypothetical reasoning is complicated if hypothetical assertions can make the rulebase inconsistent [7]. The work described above, however, focuses on a special case in which inconsistencies do not occur. In particular, Horn-clause logic is augmented with rules called *hypothetical embedded-implications.* These are rules of the form $A \leftarrow (B \leftarrow C)$ which mean, "A is true if adding C to the rulebase would cause B to be true." Adding an atomic formula C to a Horn-clause rulebase does not cause inconsistencies.

The formal properties of hypothetical embedded-implications are still being explored. Gabbay has shown that they have an intuitionistic semantics [5], and Miller has developed fixpoint semantics for the predicate case [13]. McCarty has extended this work to a larger class of formulas and established interesting semantic results [10]. Bonner has shown that query processing in such systems is PSPACE-complete in the function-free predicate case (EXPTIME-complete when hypothetical deletions are allowed) [2].

This paper continues this line of theoretical research in two ways. First, it formulates a precise sense in which classical logic is incapable of expressing hypothetical queries and rules. Specifically, queries are required to be *rulebase independent.* That is, a user should not need a detailed knowledge of the rulebase in order to formulate a query, and a query should not need to be reformulated whenever the rulebase changes. An example is given of a simple, hypothetical query which cannot be expressed in classical logic in a rulebase-independent way.

Second, this paper presents a new development of the intuitionistic semantics of propositional embedded-implications. Instead of developing fixpoint semantics, the techniques of modal logic are used to construct a *canonical model.* The aim is to provide a new perspective, and hopefully gain new insight into the intuitionistic basis of hypothetical reasoning. We use modal techniques to show that hypothetical embedded-implications are intuitionistic, and that they form a subset of intuitionistic logic with semantic properties similar to those of Horn clauses. We also describe a close relationship which exists between the intuitionistic semantics of embedded implications and the classical semantics of Horn clauses.

2 Examples

This section gives several examples of hypothetical queries and rules. The examples are centered on the rules and regulations of a university, and on its graduation policy in particular.

In general, a rulebase contains two types of information: *facts* and *rules*. Facts are ground atomic formulas, and in our examples they include information about what courses a student has taken, what his major is, etc. For instance, the atomic formula $take(s, c)$ means that student s has taken course c, and $grad(s)$ means that s is elidgible for graduation. The rulebase contains facts such as $take(tony, his101)$ and $take(sue, eng345)$, which describe the status of individual students. The rulebase also contains rules such as

$$grad(s) \leftarrow take(s, his101), take(s, eng201)$$

which describe the graduation requirements of the university.

2.1 Hypothetical Queries

This section gives two examples of hypothetical queries. They are both described in three ways: (i) informally in English, (ii) formally at the meta-level, and (iii) formally at the object-level in our language of hypotheticals.[1] In these examples, the notation $R \vdash \psi$ means that query ψ is true when applied to rulebase R. When ψ is a hypothetical query, it has the form $B \leftarrow C$, meaning, "Could B be inferred if C were added to the rulebase R?"

- "Would Tony be eligible to graduate if he took cs452?"

  That is, if $take(tony, cs452)$ were added to the rulebase, could $grad(tony)$ be inferred? At the meta-level this can be expressed as follows:

  $$R \cup \{take(tony, cs452)\} \vdash grad(tony) \tag{1}$$

  In our language for hypotheticals, it is expressed at the object-level by the expression $\psi = grad(tony) \leftarrow take(tony, cs452)$. That is, ψ is an expression such that $R \vdash \psi$ iff condition (1) is satisfied.

- "Retrieve those students who could graduate if they took (at most) one more course."

  At the meta-level, this query can be expressed as follows:

  $$\exists c\ [R \cup \{take(s, c)\} \vdash grad(s)] \tag{2}$$

  In our language of hypotheticals, it is expressed at the object level by the expression $\psi(s) = \exists c\ [grad(s) \leftarrow take(s, c)]$. That is, $\psi(s)$ is an expression such that $R \vdash \psi(s)$ iff s satisfies condition (2).

[1] See [8] for a description of meta-level and object-level reasoning.

2.2 Hypothetical Rules

Having introduced hypothetical queries, we can use them in the premises of rules. Such rules have the form $A \leftarrow (B \leftarrow C)$, meaning, "$A$ is true if adding C to the rulebase would cause B to become true." These rules turn our query language into a logic for building rulebases. Three examples of hypothetical rules are given, including recursive rules.

- Suppose the university has rules for deciding eligibility for financial aid. In particular, the university has several types of aid, to which the following rule applies:

  > "If a student is not eligible for primary financial aid but is within one course of graduation, then he is eligible for secondary financial aid."

  This hypothetical rule can be represented by the following two rules:[2]

  $$\begin{array}{l} aid2(s) \leftarrow \sim aid1(s), grad1(s). \\ grad1(s) \leftarrow \exists c\ [grad(s) \leftarrow take(s,c)] \end{array}$$

  Here $aid1(s)$ and $aid2(s)$ mean that student s is eligible for primary and secondary aid, respectively. $grad1(s)$ means that s is within one course of graduation.

- Suppose the university wishes to enact the following rule:

  > "If a student can graduate by taking a foreign language course, then he can graduate if he is fluent in that language (i.e., he need not take the course)."

  This policy can be represented by adding the following two rules to the rulebase:

  $$\begin{array}{l} grad(s) \leftarrow \exists c \exists l\ [grad0(s,c),\ type(c,l),\ language(l),\ fluent(s,l)] \\ grad0(s,c) \leftarrow [grad(s) \leftarrow take(s,c)] \end{array}$$

  Here $grad0(s,c)$ means that student s can graduate by taking course c, and $language(l)$ means that variable l takes values from $\{french, german, russian, ...\}$.

- Suppose the rulebase contains facts of the form $prereq(c_1,c_2)$, meaning that course c_1 is a prerequisite for course c_2. We wish to define a predicate $gradseq(s,c)$ which is true iff student s can graduate by taking a sequence of courses leading up to course c. That is, $gradseq(s,c)$ is true iff there is a sequence of courses $c_1, c_2, ..., c_k, c$ such that

  - $prereq(c_1,c_2)$, $prereq(c_2,c_3)$, ... $prereq(c_k,c)$ are all true, and
  - if $take(s,c_1), take(s,c_2), ...take(s,c)$ were all true, then $grad(s)$ would be true.

---

[2] Note that the first rule uses *negation-by-failure* [8]. The use of negation-by-failure in hypothetical reasoning is addressed in [2] and [5].

$gradseq(s, c)$ can be defined by the following recursive, hypothetical rules:

$$gradseq(s, c_2) \leftarrow prereq(c_1, c_2),\ [gradseq(s, c_1) \leftarrow take(s, c_2)].$$
$$gradseq(s, c) \leftarrow [grad(s) \leftarrow take(s, c)].$$

3 Expressibility

This section defines a new notion of expressibility based on the idea that queries should be independent of the rulebase. Using this definition, a hypothetical query is constructed which cannot be expressed in classical logic.

3.1 Rulebase Independence

The term "rulebase query" is a generalization of "database query" and refers to a question that is posed to a system containing a large number of rules as well as facts. In general, there are two aspects to such a query: (i) a question that the user has in mind, and (ii) an expression which he constructs to represent it. "Query formulation" is the task of constructing this expression. This section argues that query formulation should be independent of the rulebase. In particular,

- A user should be able to construct an expression to represent a query without having a detailed knowledge of the rulebase.
- If the rulebase is updated, the expression representing a query should not have to change.

Similar constraints exist in database systems; e.g., a user should be able to formulate a database query without knowing the contents of the database.

Rulebase independence is important for a variety of reasons. First, it permits casual use of the rulebase: One does not have to be an expert in the contents of the rulebase in order to formulate queries. Second, it increases reliability: Without rulebase independence, if a user's knowledge of the rulebase were inaccurate, then his queries could be formulated incorrectly. Third, it makes changes less expensive: If queries are not rulebase independent, then changes in the rulebase can propagate throughout the user community. For instance, utility programs which query the rulebase, and libraries of commonly used queries would have to be updated whenever the rulebase changes. Finally, the notion of rulebase independence is important not only to the formulation of queries but also to the construction of rulebases themselves. The premise of a rule is nothing more than a rulebase query. Thus, keeping rule premises independent of the rest of the rulebase has all of the advantages just listed for queries: it makes rules easier to formulate and more reliable, and it prevents small changes from propagating throughout the rulebase.

Rulebase independence is nicely captured in the following definition:

A rulebase query is expressible in a language iff it is possible to construct a single expression ψ which returns the correct answer for all rulebases R.

We illustrate the idea of rulebase independence with a negative example, that is, by an expression which is rulebase *dependent.* Consider the rulebase R_1 consisting of the following, single rule:

$$grad(s) \leftarrow take(s, eng101), take(s, cs250).$$

Suppose we wish to ask the query, "Is student s within one course of graduation?" Knowing the contents of R_1, we could use $\psi = take(s, eng101) \vee take(s, cs250)$ to represent the query. This expression is *database*-independent, because it assumes nothing about the contents of the database of students and courses. When the database is updated (which may happen frequently), ψ will continue to give correct answers.

Nevertheless, ψ does not represent the query in a *rulebase-independent* way, for when the rulebase changes, ψ may return incorrect answers. For example, ψ will *not* return those students who are within one course of graduation when given the rulebase R_2 consisting of the following, single rule:

$$grad(s) \leftarrow take(s, eng101), take(s, cs250), take(s, math140).$$

3.2 Classical Logic

Because of the properties of material implication, classical logic cannot represent some hypothetical queries in a rulebase-independent way. For example, consider the query: "If one of B_1 or B_2 were added to the rulebase, would C be true?" To represent this query, we need an expression ψ such that for all rulebases R,

$$R \vdash \psi \quad \textit{iff} \quad R \cup \{B_1\} \vdash C \quad \textit{or} \quad R \cup \{B_2\} \vdash C \tag{3}$$

The obvious candidate for ψ is the expression $(C \leftarrow B_1) \vee (C \leftarrow B_2)$. Indeed, in intuitionistic logic, this expression does represent the query in a rulebase-independent way. In classical logic, however, material implication leads to the following equivalences:

$$\begin{aligned}
\psi &\equiv (C \leftarrow B_1) \vee (C \leftarrow B_2) \\
&\equiv (C \vee \sim B_1) \vee (C \vee \sim B_2) \\
&\equiv C \vee \sim B_1 \vee \sim B_2 \\
&\equiv C \leftarrow B_1,\ B_2
\end{aligned}$$

Hence,

$$R \vdash \psi \quad \textit{iff} \quad R \cup \{B_1, B_2\} \vdash C$$

by the deduction theorem. Classically, then, ψ does not represent query (3). Is there is some other expression which does? The following theorem assures us that there is not.

Theorem 1 *Classically, there is no expression ψ such that for any set R of propositional Horn clauses,*[3] $R \models \psi$ *iff* $R \cup \{B_1\} \models C$ *or* $R \cup \{B_2\} \models C$.

Proof: (by contradiction) Suppose such a ψ exists. We show that it cannot give correct answers for all three of the following rulebases:

$$R_1 = \{B_1 \to C\}, \quad R_2 = \{B_2 \to C\}, \quad R_3 = \{B_1, B_2 \to C\}.$$

In particular, we show that if ψ returns correct answers for R_1 and R_2, then it must return incorrect answers for R_3. The proof exploits the above equivalences, i.e., $R_1 \vee R_2 \equiv R_3$.

$$\begin{array}{lllll}
\text{Clearly,} & R_1 \cup \{B_1\} \models C & \text{and} & R_2 \cup \{B_2\} \models C & \\
\text{thus} & R_1 \models \psi & \text{and} & R_2 \models \psi & \text{by definition of } \psi, \\
\text{thus} & (R_1 \vee R_2) \models \psi & & & \\
\text{thus} & R_3 \models \psi & \text{since } R_1 \vee R_2 \equiv R_3, & & \\
\text{thus} & R_3 \cup \{B_1\} \models C & \text{or} & R_3 \cup \{B_2\} \models C & \text{by definition of } \psi, \\
\text{i.e.,} & \{B_1, B_2 \to C\} \cup \{B_1\} \models C & \text{or} & \{B_1, B_2 \to C\} \cup \{B_2\} \models C &
\end{array}$$

which is false. *QED*

These ideas extend to the construction of rulebases. In particular, suppose R is a rulebase not containing the atom A, and suppose we wish to add new rules R_o to R so that for all R,

$$R \cup R_o \vdash A \quad \textit{iff} \quad R \cup \{B_1\} \vdash C \quad \textit{or} \quad R \cup \{B_2\} \vdash C$$

Classically, there is no set of formulas R_o (Horn or otherwise) which satisfies this condition. For if R_o existed, then we could contradict theorem 1 using $\psi = A \leftarrow R_o$. As the next section shows, however, R_o can be constructed using hypothetical embedded-implications. In particular, $R_o = \{A \leftarrow (C \leftarrow B_1), \;\; A \leftarrow (C \leftarrow B_2)\}$.

4 Hypothetical Inference

Because classical logic is inadequate for hypothetical inference, new inference mechanisms are needed, i.e., proof procedures for embedded implications. Such procedures have been developed by several researchers [6, 13, 11], and this section defines a simplified version of them. This propositional version retains the essential properties of the more elaborate systems while admitting a clean theoretical analysis.

Definition 1 *A Horn rule is an expression of the form* $B \leftarrow B_1, B_2, ..., B_k$ *where* $k \geq 0$ *and* B *and each* B_i *are atomic.*

[3] This theorem remains true if R is restricted to Horn clauses built from a given set $\mathcal{L}$ of atoms, so long as $\{B_1, B_2, C\} \subseteq \mathcal{L}$.

Definition 2 *An embedded implication is an expression of the form* $B \leftarrow \phi_1, \phi_2, ..., \phi_k$ *where* $k \geq 0$, *B is atomic, and each* ϕ_i *is a Horn rule.*

Note that embedded implications include Horn rules as a special case, and Horn rules include atomic formulas as a special case.

Definition 3 *Suppose R is a set of embedded implications. If B and* B_i *are atomic, then*

1. $R \vdash B$ *if* $B \in R$
2. $R \vdash B$ *if there is a rule* $B \leftarrow \phi_1, ..., \phi_k$ *in R, and* $R \vdash \phi_i$ *for each i.*
3. $R \vdash B \leftarrow B_1, ..., B_k$ *if* $R \cup \{B_1, ..., B_k\} \vdash B$

If R is a set of Horn rules, then this inference system is equivalent to classical Horn-clause logic. In general, however, these inference rules do not have a classical semantics: Although they are sound classically, they are not complete. To see this, consider the rulebase $R = \{A \leftarrow (B \leftarrow C), \quad D \leftarrow A, \quad D \leftarrow C\}$. Classically, D can be inferred from R[4], but using the above inference system, it cannot.[5]

5 Intuitionistic Logic

The rules of hypothetical inference defined above are non classical. Indeed, they were introduced precisely to overcome the shortcomings of classical logic described in section 3.2. The next section shows that they have an intuitionistic semantics. This section provides a brief development of propositional intuitionistic logic adapted from [4] and [10].

Definition 4 *Suppose* $\mathcal{L}$ *is a finite or countably infinite set of propositional atoms. A substate is a subset of* $\mathcal{L}$, *and an intuitionistic structure is a set of substates. Furthermore, if* s_1 *and* s_2 *are substates, then* $s_1 \leq s_2$ *iff* $s_1 \subseteq s_2$.

Definition 5 *(Satisfaction) Suppose* ψ *is a formula, M is an intuitionistic structure, and s is a substate of M. Then* $s, M \models \psi$ *is read, "M satisfies* ψ *at s," and is defined recursively as follows:*

- *If A is atomic, then* $s, M \models A$ *iff* $A \in s$
- $s, M \models (\psi_1 \wedge \psi_2)$ *iff* $s, M \models \psi_1$ *and* $s, M \models \psi_2$
- $s, M \models (\psi_1 \vee \psi_2)$ *iff* $s, M \models \psi_1$ *or* $s, M \models \psi_2$

[4]Classically, the two Horn rules in R are equivalent to $D \leftarrow (A \vee C)$, and the embedded implication is equivalent to $(A \vee C) \wedge (A \vee \sim B)$. Thus R entails D classically.

[5]because there are no rules in R for inferring B or C.

- $s, M \models \sim \psi$ *iff* $r, M \not\models \psi$ *for all* $r \geq s$ *in* M

- $s, M \models \psi_2 \leftarrow \psi_1$ *iff* $r, M \models \psi_1 \Rightarrow r, M \models \psi_2$ *for all* $r \geq s$ *in* M*.*

Note that unlike classical logic, intuitionistic implication is not defined in terms of disjunction and negation. Instead, it has an independent semantic definition. This is why intuitionistic logic does not give rise to the problems described in section 3.2.

Definition 6 *(Models)* $M \models \psi$ *iff* $s, M \models \psi$ *for all substate* s *of* M*.*

If $M \models \psi$ then we say that M *satisfies* ψ or that M *is a model of* ψ.

Definition 7 *(Entailment) Suppose* ψ_1 *and* ψ_2 *are formulas. Then* $\psi_1 \models \psi_2$ *iff every model of* ψ_1 *is also a model of* ψ_2*.*

The following lemma gives basic properties of intuitionistic logic which are easily derived from the above definitions. These properties are used extensively in the rest of the paper.

Lemma 1 *Suppose* M *is an intuitionistic structure. Then*

- $s, M \models \psi$ *iff* $r, M \models \psi$ *for all* $r \geq s$ *in* M

- $M \models \psi_2 \leftarrow \psi_1$ *iff* $\forall_{s \in M}\ s, M \models \psi_1 \Rightarrow s, M \models \psi_2$

- $\psi_3 \models \psi_2 \leftarrow \psi_1$ *iff* $\psi_3 \wedge \psi_1 \models \psi_2$ *(Deduction Theorem)*

- *If* $\psi_3 \models \psi_2 \leftarrow \psi_1$ *and* $\psi_3 \models \psi_1$ *then* $\psi_3 \models \psi_2$ *(Modus Ponens)*

6 Semantics

Several researchers have developed fixpoint semantics for inference rules like those of section 4. Miller, for instance, has developed fixpoints semantics for such rules based on intuitionistic and minimal logic [13]. McCarty has considered a larger class of formulas involving negation and embedded universal quantifiers[6], developing an intuitionistic fixpoint semantics and establishing interesting semantic results [10]. Recently, Manchanda has considered hypothetical deletions as well as additions, developing a fixpoint semantics based on dynamic logic [9]. The use of fixpoint semantics to establish completeness results has been common in logic programming since the seminal work of [1] and [16].

This section presents an alternative development for the propositional case. It is based on completeness techniques used in modal logic. The aim is to add a new perspective, and hopefully gain new insight into the intuitionistic basis of hypothetical reasoning. To this end,

[6]A quantifier is *embedded* if it appears in the premise of a rule, as in $A(x) \leftarrow \forall y\ [B(x,y) \leftarrow C(x,y)]$

we introduce an intuitionistic structure called the *canonical model*. This structure, defined proof-theoretically, provides the necessary link between inference and semantics. It also plays a central semantic role, comparable to that of the unique minimal model in Horn-clause logic.

Finally, we compare the intuitionistic semantics of embedded implications to the classical semantics of Horn clauses, outlining the close relationship that exists between them. In particular, some well-known properties of Horn clauses are shown to be a special case of properties of the canonical model.

6.1 Soundness and Completeness

To show that the hypothetical inference rules of section 4 are intuitionistic, we must prove that they are sound and complete with respect to intuitionistic semantics. In particular, we must prove the following two theorems:

Theorem 2 *(Soundness) If R is a set of embedded implications and ϕ is a Horn rule, then if $R \vdash \phi$ then $R \models \phi$.*

Theorem 3 *(Completeness) If R is a set of embedded implications and ϕ is a Horn rule, then if $R \models \phi$ then $R \vdash \phi$.*

Soundness follows immediately because the rules of hypothetical inference given in definition 3 are intuitionistically valid. In particular, they correspond to modus ponens and the deduction theorem.

Proving completeness is more complex. The approach taken here is an adaptation of techniques used in modal logic and centers on the notion of a canonical model [3]. Our proof strategy is as follows: Given a set of embedded implications R,

1. Construct an intuitionistic structure M_R, called the *canonical model* of R.

2. Show that $M_R \models R$.

3. Show that if $M_R \models A$ then $R \vdash A$, for every atom A.

Thus, if $R \models A$, then A is true in all models of R. In particular, it is true in M_R, and so by point 3, $R \vdash A$. This establishes the completeness of *atomic* inference; i.e., $R \models A \;\Rightarrow\; R \vdash A$ for any atom A. By the deduction theorem, however, it follows that $R \models \phi \;\Rightarrow\; R \vdash \phi$ for any Horn-rule ϕ. This would prove theorem 3.

The central question, however, is how to construct the canonical model M_R. We define it to be the range of a proof-theoretic operator cl_R. This operator provides the essential link between hypothetical reasoning and intuitionistic semantics. In the following, $\mathcal{L}$ is a (possibly infinite) set of propositional atoms from which rulebases and queries are constructed.

Definition 8 *If R is a set of embedded implications constructed from the atoms in $\mathcal{L}$, and s is a (possibly infinite) subset of $\mathcal{L}$, then $cl_R(s)$ is the atomic closure of R and s and is defined as follows:*

$$cl_R(s) = \{A \in \mathcal{L} \mid R \cup s \vdash A\}$$

Lemma 2 *cl_R has the following properties:*

- *it is inflationary: $s \leq cl_R(s)$.*
- *it is monotonic: if $s_1 \leq s_2$ then $cl_R(s_1) \leq cl_R(s_2)$.*
- *it is idempotent: $cl_R(s) = cl_R(cl_R(s))$*

These three properties are enjoyed by every (monotonic) inference system [14]. The third property, in particular, expresses the fact that cl_R is a logical closure operator; that is, $cl_R(s)$ contains all the atoms derivable from R and s, so further applications of the inference rules will not derive any more.

Definition 9 *If R is a set of embedded implications constructed from the atoms in $\mathcal{L}$, then M_R is the canonical model of R and is defined as follows:*

$$M_R = \{cl_R(s) \mid s \subseteq \mathcal{L}\}$$

Lemma 3 *$s \in M_R$ iff $s = cl_R(s)$.*

Proof: The *if* direction is trivial. In the other direction, suppose that $s \in M_R$. Then there is a subset r of $\mathcal{L}$ such that $s = cl_R(r)$. But $cl_R(r) = cl_R(cl_R(r))$ because cl_R is idempotent. Hence $s = cl_R(s)$. *QED*

Definition 9 establishes point 1 above. Point 3 is established by the following simple lemma.

Lemma 4 *For any atom A, if $M_R \models A$ then $R \vdash A$.*

Proof: Suppose $M_R \models A$. Let $s_o = cl_R(\{\})$. Then

$s_o \in M_R$ by definition of M_R.
$s_o, M_R \models A$ since $M_R \models A$.
$A \in s_o$
$A \in cl_R(\{\})$
$R \cup \{\} \vdash A$ by definition of cl_R.
$R \vdash A$

QED

Point 2 is established by the following two lemmas.

Lemma 5 *For any Horn-rule ϕ, if $s, M_R \models \phi$ then $R \cup s \vdash \phi$.*

Proof: Let $\phi = B \leftarrow B_1, ..., B_k$ and let $r_o = cl_R(s \cup \{B_1, ..., B_k\})$. Note that $r_o \in M_R$.

$$
\begin{array}{ll}
\textit{if} & s, M_R \models B \leftarrow B_1, ..., B_k \\
\textit{then} & r, M_R \models B_1, ..., B_k \ \Rightarrow \ r, M_R \models B \quad \text{for all } r \geq s \text{ in } M_R, \quad \text{by definition 5.} \\
& r_o, M_R \models B_1, ..., B_k \ \Rightarrow \ r_o, M_R \models B \quad \text{since } r_o \geq s \text{ and } r_o \in M_R. \\
& \{B_1, ..., B_k\} \subseteq r_o \ \Rightarrow \ B \in r_o \quad \text{since the } B\text{'s are atomic.} \\
& B \in r_o \quad \text{since } \{B_1, ..., B_k\} \subseteq r_o \\
& B \in cl_R(s \cup \{B_1, ..., B_k\}) \\
& R \cup s \cup \{B_1, ..., B_k\} \vdash B \\
& R \cup s \vdash B \leftarrow B_1, ..., B_k \quad \text{by definition 3.}
\end{array}
$$

QED

Lemma 6 *Suppose R contains the rule $A \leftarrow \phi_1, ..., \phi_k$ where each ϕ_i is a Horn rule; then $M_R \models A \leftarrow \phi_1, ..., \phi_k$.*

Proof: We show that for each substate s in M_R, if $s, M_R \models \phi_1, ..., \phi_k$ then $s, M_R \models A$. The result then follows by the second item in lemma 1.

$$
\begin{array}{ll}
\textit{if} & s, M_R \models \phi_1, ..., \phi_k \\
\textit{then} & R \cup s \vdash \phi_1, ..., \phi_k \quad \text{by lemma 5} \\
& R \cup s \vdash A \quad \text{since } A \leftarrow \phi_1, ..., \phi_k \in R. \\
& A \in cl_R(s) \\
& A \in s \quad \text{since } s = cl_R(s), \text{ by lemma 3.} \\
& s, M_R \models A \quad \text{by definition.}
\end{array}
$$

QED

M_R is thus a model of every rule in R.

Corollary 1 *If R is a set of embedded implications, then $M_R \models R$.*

The canonical model M_R thus satisfies points 1-3 above, thereby proving Theorem 3. The rules of hypothetical inference defined in section 4 are therefore sound and complete with respect to intuitionistic semantics. In other words, this form of hypothetical reasoning is intuitionistic reasoning.

6.2 Fundamental Semantic Properties

Although our system for hypothetical inference is intuitionistic, it is not equivalent to the full intuitionistic logic. Disjunctions, for instance, cannot be expressed. In the rest of the paper, we establish some properties of this subset of intuitionistic logic. The present section establishes two results which are the basis for the rest. In particular, it is shown (i) that M_R has a unique minimal substate, and (ii) that M_R is the unique maximal model of the rulebase R.

The first result is a simple consequence of definition 9 and the monotonicity of cl_R.

Theorem 4 *M_R has a unique minimal substate $s_o = cl_R(\{\})$. That is, $s_o \in M_R$ and $s_o \leq s$ for all $s \in M_R$.*

Corollary 2 *If ψ is an arbitrary formula, then $M_R \models \psi$ iff $s_o, M_R \models \psi$.*

Proof:

$$
\begin{array}{ll}
 & M_R \models \psi \\
\textit{iff} & s, M_R \models \psi \text{ for all } s \text{ in } M_R, \quad \text{by definition 6.} \\
\textit{iff} & s, M_R \models \psi \text{ for all } s \geq s_o \text{ in } M_R. \\
\textit{iff} & s_o, M_R \models \psi \quad \text{by lemma 1.}
\end{array}
$$

QED

Actually, theorem 4 can be generalized to the *substate intersection property*. That is, the intersection of any collection of substates of M_R is also a substate of M_R. In particular, the intersection of all substates of M_R is the unique minimal substate. Since cl_R is monotonic, this property follows from the proof of the Tarski fixpoint theorem [15]; however, because cl_R is also inflationary, a short version of this proof exists.

Theorem 5 *If $M \subseteq M_R$ then $\cap M \in M_R$.*

Proof: If $s \in M$ then $s \in M_R$ and so $s = cl_R(s)$ by lemma 3. Furthermore $\cap M \subseteq s$, so $cl_R(\cap M) \subseteq cl_R(s) = s$. Thus, intersecting over all s in M, we get $cl_R(\cap M) \subseteq \cap M$. But $\cap M \subseteq cl_R(\cap M)$, since cl_R is inflationary. Thus $cl_R(\cap M) = \cap M$. Hence $\cap M \in M_R$ by definition. *QED*

Showing that M_R is the largest intuitionistic model of R is less simple but still straightforward. The proof requires the following lemma, which is a generalization of soundness.[7]

Lemma 7 *Suppose that $s, M \models R$. Then for any Horn rule ϕ, if $R \cup s \vdash \phi$ then $s, M \models \phi$*

Proof: We define $R \overset{n}{\vdash} \phi$ to mean that $R \vdash \phi$ can be derived in n applications (or less) of the inference rules of definition 3. Clearly, $R \vdash \phi$ iff $R \overset{n}{\vdash} \phi$ for some finite n. Assuming that $s, M \models R$, we show by induction on n that $R \cup s \overset{n}{\vdash} \phi \Rightarrow s, M \models \phi$.

[7]This lemma would be equivalent to soundness if s were restricted to be finite.

Basis: If $R \cup s \overset{1}{\vdash} \phi$, then ϕ is atomic and $\phi \in R \cup s$, by inference rule 1 of definition 3. There are two cases: (i) if $\phi \in R$, then $s, M \models \phi$ since $s, M \models R$; else (ii) if $\phi \in s$, then $s, M \models \phi$ by definition.

Induction: Suppose the lemma is true for $\overset{n}{\vdash}$. We show that it must also be true for $\overset{n+1}{\vdash}$. There are two cases, depending on whether rule 2 or 3 is the last inference rule used in the derivation of $R \cup s \vdash \phi$.

Case (i): If $R \cup s \overset{n+1}{\vdash} \phi$ and rule 2 is the last inference rule, then ϕ is atomic; furthermore, there is a rule $\phi \leftarrow \phi_1, ..., \phi_k$ in R such that $R \cup s \overset{n}{\vdash} \phi_i$ for each i. Thus, by induction hypothesis $s, M \models \phi_i$ for each i. But $s, M \models \phi \leftarrow \phi_1, ..., \phi_k$ and so $s, M \models \phi$.

Case (ii): If $R \cup s \overset{n+1}{\vdash} \phi$ and rule 3 is the last inference rule, then ϕ is a Horn rule $B \leftarrow B_1, ..., B_k$; furthermore, $R \cup s \cup \{B_1, ..., B_k\} \overset{n}{\vdash} B$. Thus,

$R \cup r \overset{n}{\vdash} B$ for all $r \geq s \cup \{B_1, ..., B_k\}$, by monotonicity.
$r, M \models B$ for all $r \geq s \cup \{B_1, ..., B_k\}$ in M, by induction hypothesis.
$\{B_1, ..., B_k\} \subseteq r \Rightarrow r, M \models B$ for all $r \geq s$ in M.
$r, M \models B_1, ..., B_k \Rightarrow r, M \models B$ for all $r \geq s$ in M.
$s, M \models B \leftarrow B_1, ..., B_k$ by definition 5.
$s, M \models \phi$.

QED

Theorem 6 *M_R is the unique maximal model of M_R. That is, if $M \models R$ then $M \subseteq M_R$*

Proof: Suppose that $M \models R$. We show that if $s \in M$ then $s \in M_R$.

if $s \in M$
then $R \cup s \vdash A \Rightarrow s, M \models A$ for every atom A, by lemma 7.
$A \in cl_R(s) \Rightarrow A \in s$ for every atom A.
$cl_R(s) \subseteq s$
$cl_R(s) = s$ since cl_R is inflationary.
$s \in M_R$ by definition 9.

QED

Theorems 6 and 4 show that M_R is the unique maximal model of R and that M_R has a unique minimal substate. The existence of such a model was originally established by McCarty in [10]. Indeed, McCarty defines an intuitionistic structure called K^* which he shows is the unique maximal model of a set of embedded implications. Theorem 6 assures us that McCarty's K^* is identical to our M_R, at least in the propositional case. Conceptually, however, the main difference between them is that K^* is defined semantically whereas M_R is defined proof-theoretically.

6.3 Inferring Horn Rules

The derivation of Horn rules is central to the system of hypothetical inference defined in section 4. This section establishes results on such derivations. The first result shows that the canonical model contains all the information necessary to perform hypothetical inference.

Corollary 3 *If ϕ is a Horn rule, then* $R \models \phi$ *iff* $M_R \models \phi$.

Proof: The *only if* direction follows trivially since M_R is a model of R. In the other direction, we show that $s, M_R \models \phi \;\Rightarrow\; s, M \models \phi$ for all models M of R. Central to the argument is the maximality of M_R, i.e., that $M \subseteq M_R$. Letting $\phi = B \leftarrow B_1, ..., B_k$, we get

$$
\begin{array}{ll}
\textit{if} & M_R \models B \leftarrow B_1, ..., B_k \\
\textit{then} & \forall_{s \in M_R}\ s, M_R \models B_1, ..., B_k \;\Rightarrow\; s, M_R \models B \quad \text{by lemma 1} \\
 & \forall_{s \in M_R}\ \{B_1, ..., B_k\} \subseteq s \;\Rightarrow\; B \in s \\
 & \forall_{s \in M}\ \{B_1, ..., B_k\} \subseteq s \;\Rightarrow\; B \in s \quad \text{since } M \subseteq M_R. \\
 & \forall_{s \in M}\ s, M \models B_1, ..., B_k \;\Rightarrow\; s, M \models B \\
 & M \models B \leftarrow B_1, ..., B_k \quad \text{by lemma 1}
\end{array}
$$

QED

This theorem provides a semantic interpretation of negation-as-failure for hypothetical inference; i.e., ϕ cannot be inferred from a rulebase R iff ϕ is not true in the maximal model of R. In other words, invoking the closed-world assumption on R is equivalent to reasoning in the maximal model.

Note the similarity of this to the semantics of failure in the classical logic of Horn clauses. Indeed, there is a theorem of Horn logic comparable to corollary 3: an atom can be inferred from a set of Horn clauses iff it is true in the unique minimal model. Furthermore, invoking the closed-world assumption is equivalent to reasoning in the minimal model. In this sense, the canonical model is an intuitionistic analogue of the unique minimal model of Horn logic. The analogy can be taken one step further. Because of the unique minimal model, Horn clauses have the attractive property that they entail a disjunction of atoms iff they entail one of the atoms individually. Similarly, as the next corollary shows, a set of embedded implications entails a disjunction of Horn rules iff it entails one of the Horn rules individually. This follows directly from the unique maximality and unique minimal substate properties of the canonical model.

Corollary 4 *If $\phi_1, ..., \phi_k$ are Horn rules, then* $R \models \phi_1 \vee \cdots \vee \phi_k$ *iff* $R \models \phi_i$ *for some i.*

Proof: The if direction is trivial. In the other direction, let s_o be the unique minimal substate of M_R.

$$\begin{array}{ll} \textit{if} & R \models \phi_1 \vee \cdots \vee \phi_k \\ \textit{then} & M_R \models \phi_1 \vee \cdots \vee \phi_k \quad \text{since } M_R \models R. \\ & s_o, M_R \models \phi_1 \vee \cdots \vee \phi_k \quad \text{by corollary 2.} \\ & s_o, M_R \models \phi_i \ \text{ for some } i, \quad \text{by definition 5.} \\ & M_R \models \phi_i \ \text{ for some } i, \quad \text{by corollary 2.} \\ & R \models \phi_i \ \text{ for some } i, \quad \text{by corollary 3.} \end{array}$$

QED

This corollary means that a theorem prover for intuitionistic embedded-implications does not need extensive modification to deal with disjunctive goals. Indeed, the disjuncts are non-interacting and a theorem prover can work on each one separately. This suggests adding the following rule of hypothetical inference to those of section 4:

- *For any collection of Horn rules $\phi_1, ..., \phi_k$,*

  $R \vdash \phi_1 \vee \cdots \vee \phi_k$ *if* $R \vdash \phi_i$ *for some i.*

6.4 Relationship to Horn-Clause Logic

The intuitionistic semantics of embedded implications seems to have little relation to the classical semantics of Horn clauses. This is somewhat surprising since Horn rules are a special case of embedded implications both syntactically and proof-theoretically. This section resolves the apparent incompatibility, showing that when R is a set of Horn rules, its classical and intuitionistic semantics are closely related. Indeed, many of the well-known properties of classical Horn clauses, such as the existence of a unique minimal model, are special cases of the properties of canonical models.

First, we note that it does not matter whether Horn rules are treated classically or intuitionistically. In both cases, the inference rules of section 4 form a sound and complete inference system. Thus,

Lemma 8 *If R is a set of Horn rules, and ϕ is a Horn rule, then $R \models_i \phi$ iff $R \models_c \phi$.*[8]

Second, we note that a classical model can be interpreted as an intuitionistic substate, and that a collection of classical models can be interpreted as an intuitionistic model. This forms the basis of the following lemma.

Lemma 9 *If ϕ is a Horn rule, then M is an intuitionistic model of ϕ iff every substate of M is a classical model of ϕ.*

[8] In this corollary, and in the rest of this section, $\models_i$ stands for intuitionistic entailment and $\models_c$ stands for classical entailment.

Proof: Let $\phi = B \leftarrow B_1, ..., B_k$. Then

$$
\begin{array}{ll}
 & M \models_i B \leftarrow B_1, ..., B_k \\
\textit{iff} & \forall_{s \in M}\ s, M \models_i B_1, ..., B_k \;\Rightarrow\; s, M \models_i B \quad \text{by lemma 1} \\
\textit{iff} & \forall_{s \in M}\ \{B_1, ..., B_k\} \subseteq s \;\Rightarrow\; B \in s \\
\textit{iff} & \forall_{s \in M}\ s \models_c B \leftarrow B_1, ..., B_k
\end{array}
$$

QED

Corollary 5 *Suppose R is a set of Horn rules. Then M is an intuitionistic model of R iff M is a collection of classical models of R. In particular, the canonical model M_R is the collection of all classical models of R.*

Many semantic properties of classical Horn clauses can now be seen to be special cases of the properties of canonical models. For instance, the model intersection property of Horn clauses is a special case of the substate intersection property of canonical models; and the unique minimal model property is a special case of the unique minimal substate property. In Horn-clause logic, the unique minimal model plays a central semantic role: an atom is entailed by a set of Horn clauses iff it is in the unique minimal model. This property too is a special case of a more general property of canonical models:

Corollary 6 *If R is a set of embedded implications, and A is an atom, then $R \vdash A$ iff A is in s_o, the unique minimal substate of M_R.*

Proof: $R \vdash A$ *iff* $A \in cl_R(\{\})$ *iff* $A \in s_o$. *QED*

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