The strong chromatic index of graphs

by

Mohammad Mahdian

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Graduate Department of Computer Science
University of Toronto

Abstract

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A strong edge-colouring of a graph G is an assignment of colours to the edges of G such that every colour class is an induced matching. The minimum number of colours in such a colouring is called the strong chromatic index of G. In 1985, Erdős and Nešetřil conjectured that the strong chromatic index of every graph of maximum degree Δ is at most $\frac{5}{4}\Delta^2$. In this thesis, we present a survey of known results related to strong edge-colourings, and an introduction to the probabilistic method. We will use the probabilistic method to prove that the strong chromatic index of a C_4-free graph (i.e., a graph which does not contain a 4-cycle as a subgraph) of maximum degree Δ is at most $(2+o(1))\frac{\Delta^2}{\ln \Delta}$. This implies that the conjecture of Erdős and Nešetřil is true for C_4-free graphs with large maximum degree. We will show that our bound is asymptotically the best possible, up to a constant multiple. Also, we will investigate the algorithmic aspects of the strong edge-colouring problem, and will prove that it is **NP**-complete even in a very restricted setting. Finally, we present a list of open problems and conjectures related to strong edge-colourings.

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Contents

Chapter 1

Introduction

The probabilistic method is a powerful method in combinatorics. It has found many applications in various areas of combinatorics, especially in graph colourings. The main subject of this thesis is to apply this method to a special variety of graph colouring, namely strong edge-colourings.

A *strong edge-colouring* of a graph G is an assignment of colours to the edges of G such that every two edges *of distance at most two* receive different colours. Two edges are of distance at most two iff either they share an endpoint (Figure 1.1(a)), or one of their endpoints are adjacent (Figure 1.1(b)).

An *induced matching* (also called a *strong matching*) in a graph G is a set A of edges such that no two edges in A are of distance at most two. In other words, an induced matching of G is the edge-set of an induced subgraph of G that is also a matching. Clearly, a strong edge-colouring can also be defined as a partition of edges into induced matchings.

It is also possible to define a strong edge-colouring as a vertex colouring of a special type of graph. For every graph G, the *line graph* of G, which is denoted by $L(G)$, is a graph

Figure 1.1: Two edges of distance at most two

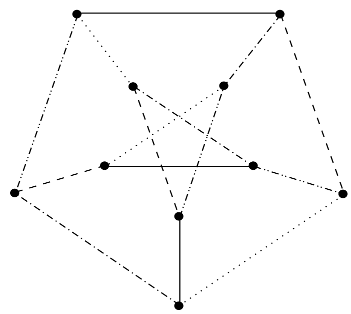

Figure 1.2: A strong edge-colouring of the Petersen graph with five colours

which has a vertex corresponding to each edge of G, and two vertices are adjacent if the corresponding edges in G are adjacent (i.e., share an endpoint). The *square* of a graph H, which is denoted by H^2, is a graph which has one vertex corresponding to each vertex of H, and two vertices are adjacent if their corresponding vertices in H are either adjacent or have a common neighbour (i.e., if they are of distance at most two in H). By the above definitions, it is clear that a strong edge-colouring of G can be viewed as a proper vertex colouring of the square of the line graph of G (i.e., $L(G)^2$).

The *strong chromatic index* of G, usually denoted by $s\chi'(G)$, is the minimum number of colours in a strong edge-colouring of G. For example, it is easy to see that the strong chromatic index of the Petersen graph is five (see Figure 1.2).

We denote the maximum degree of a graph G by $\Delta(G)$, or by Δ when there is no ambiguity. For most types of graph colouring, the first question that is usually asked is to find an upper bound for the minimum number of colours necessary to colour the graph in terms of Δ. For strong edge-colourings, this problem was first considered by Erdős and Nešetřil. They conjectured that the strong chromatic index of every graph of maximum degree Δ is at most $\frac{5}{4}\Delta^2$. This conjecture is still widely open.

The motivation of this research is to try to apply a probabilistic graph colouring procedure, that was previously used to solve other graph colouring problems, to the strong edge-colouring problem. It turns out that we need an extra assumption about the structure of the graph in order to be able to use the probabilistic procedure. As a result, we found an upper bound for the strong chromatic index of graphs which do not contain a cycle of length four as a subgraph. The upper bound is asymptotically less than $\frac{5}{4}\Delta^2$; therefore, our result implies the conjecture of Erdős and Nešetřil for graphs without a 4-cycle.

We will use another probabilistic argument to show that our upper bound is asymptotically best possible, up to a constant multiple. Also, we will consider several algorithmic questions related to strong edge-colourings, and prove some **NP**-completeness results.

The rest of this thesis is organized as follows: In Chapter 2 we present a survey of known results related to strong edge-colourings. Chapter 3 is an introduction to the probabilistic method. Most of the important tools of the probabilistic method, such as the first moment method, the linearity of expectation, Markov's inequality, the alteration method, the Lovasz Local Lemma, the Chernoff bound, Talagrand's inequality, and the semirandom method are illustrated by several examples from graph theory and combinatorics. The main results of the thesis are presented in Chapter 4. In this chapter, an upper bound and a lower bound for the strong chromatic index of C_4-free graphs are proved. Most of the results of this chapter have appeared in [32]. Chapter 5 is devoted to algorithmic questions regarding strong edge-colourings. Finally, in Chapter 6 we have some concluding remarks and a discussion about other open problems and some ideas for future works.

Throughout the thesis, we assume that the reader is familiar with the very basic concepts and notations of graph theory and elementary probability theory, such as graph and subgraph, path, cycle, event, and random variable. To see the definitions of the basic concepts of graph theory, refer to any textbook on graph theory (such as [7, 52]). For an introduction to the probability theory concepts that are used in this thesis refer to [37].

Chapter 2

The strong edge-colouring conjecture

2.1 Introduction and history

At a conference in Prague in 1985, P. Erdős and J. Nešetřil presented the following problem (See [12] or [13]).

Problem 2.1 *Find the least $t(k)$ so that if $\Delta(G) = k$, then $s\chi'(G) \leq t(k)$.*

It is trivial that for any graph G of maximum degree Δ, $s\chi'(G) \leq 2\Delta^2 - 2\Delta + 1$. This implies that $t(k) \leq 2k^2 - 2k + 1$. On the other hand, Erdős and Nešetřil constructed a graph G_k of maximum degree k such that

$$s\chi'(G_k) = \begin{cases} 5k^2/4 & k \text{ is even} \\ 5k^2/4 - k/2 + 1/4 & k \text{ is odd} \end{cases}$$

This example shows that $t(k) \geq s\chi'(G_k)$. For k even, the example can be obtained by replacing each vertex of a 5-cycle by an independent set of size $k/2$, and connecting two vertices if the corresponding vertices in the 5-cycle are adjacent (The operation of replacing each vertex in a graph with an independent set is called *vertex multiplication* and the resulting graph is denoted by $C_5^{k/2}$). For k odd, the example can be obtained by replacing two adjacent vertices of the 5-cycle by independent sets of size $(k+1)/2$, and others by independent sets of size $(k-1)/2$ (See Figure 2.1). In each case, it can be seen

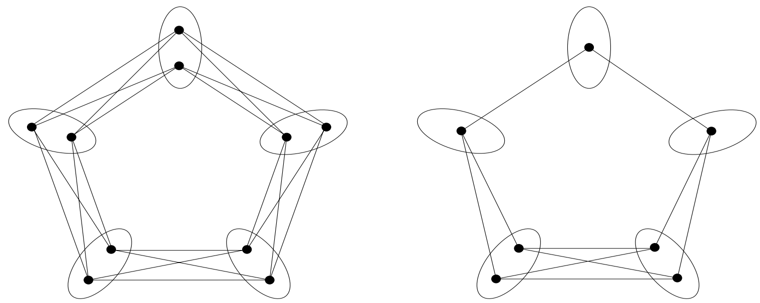

Figure 2.1: Graphs with (a) $\Delta = 4, s\chi' = 20$ (b) $\Delta = 3, s\chi' = 10$

easily that there is not any induced matching of size two in the graph, and therefore the colour of the edges must be distinct.

Erdős and Nešetřil conjectured that in fact $t(k) = s\chi'(G_k)$.

Conjecture 2.1 (Strong edge-colouring conjecture) *For any graph G of maximum degree Δ, $s\chi'(G) \leq f(\Delta)$, where $f(\Delta)$ is defined as follows.*

$$f(\Delta) = \begin{cases} 5\Delta^2/4 & \text{if } \Delta \text{ is even} \\ 5\Delta^2/4 - \Delta/2 + 1/4 & \text{if } \Delta \text{ is odd} \end{cases} \tag{2.1}$$

In 1989, R. J. Faudree et al. [14] posed the following conjecture for the strong chromatic index of bipartite graphs. The complete bipartite graph $K_{\Delta,\Delta}$ shows that the following conjecture, if true, is the best possible.

Conjecture 2.2 (Bipartite strong edge-colouring conjecture) *For any bipartite graph G of maximum degree Δ, $s\chi'(G) \leq \Delta^2$.*

R. Brualdi and J. Quinn [8] generalized the above conjecture as follows.

Conjecture 2.3 (Generalized bipartite strong edge-colouring conjecture) *For any bipartite graph G with bipartition X, Y such that the maximum degree of a vertex in X is α and of Y is β, $s\chi'(G) \leq \alpha\beta$.*

Notice that the concept of strong edge-colouring, had been defined by J. L. Fouquet and J. L. Jolivent [16, 17] before the conjecture of Erdős and Nešetřil. They had studied the strong chromatic index of cubic planar graphs.

In this chapter we give a survey of the known results about the above conjectures and related problems. In Section 2.2 a general bound on the strong chromatic index due to M. Molloy and B. Reed is presented. Section 2.3 contains a summary of results about some conjectures that are implied by the strong edge-colouring conjectures. These results provide evidence for the correctness of the strong edge-colouring conjecture. In Section 2.4, you will see a survey of results about the strong chromatic index of graphs with small maximum degree. Section 2.5 contains some results about the strong chromatic index of special classes of graphs. Some algorithmic questions and results related to the strong edge-colouring are mentioned in Section 2.6. Finally, in Section 2.7, we list several conjectures related to the strong edge-colouring.

2.2 A general bound

Horák, Qing, and Trotter (See [22, 23]) remarked that it appears very hard even to prove $s\chi'(G) \leq (2-\epsilon)\Delta^2$ for some $\epsilon > 0$. In [35], M. Molloy and B. Reed settled this problem by the following theorem. This is the only known general bound for the strong chromatic index.

Theorem 2.1 *If G has maximum degree Δ sufficiently large, then $s\chi'(G) \leq 1.998\Delta^2$.*

The proof consists of two parts. In the first part, it is shown that for any graph G, the square of the line graph of G is $\frac{1}{36}$-sparse, i.e. there are at most $(1-\frac{1}{36})\binom{\Delta(L(G)^2)}{2}$ edges in the neighbourhood of any vertex of $L(G)^2$. In the second part, it is proved that for any $\delta > 0$, there is a $\gamma > 0$ such that if a graph H is δ-sparse, its chromatic number is at most $(1-\gamma)\Delta(H)$. The proof of this part is probabilistic and is based on a single iteration of a random colouring procedure similar to the random procedure which we will use in Chapter 4 (The idea of the proof of this part is mentioned in Section 3.4.2). In fact, the result in Chapter 4 is motivated by the proof of the above theorem.

2.3 Some consequences of the conjectures

The strong edge-colouring conjecture implies the following two conjectures. Here $sm(G)$ denotes the size of the largest induced matching of G, and $am(G)$ denotes the size of the largest *antimatching* of G. An antimatching is a set A of edges such that any two elements of A are of distance at most two. It is clear that $sm(G)$ and $am(G)$ correspond to the size of the maximum independent set and the maximum clique of the square of the line graph of G.

Conjecture 2.4 *For any graph G, $|E(G)| \leq f(\Delta)sm(G)$, where f is the function defined by Equation 2.1.*

Conjecture 2.5 *For any graph G, $am(G) \leq f(\Delta)$, where f is the function defined by Equation 2.1.*

A special case of Conjecture 2.4 can be formulated in terms of $2K_2$-free graphs. A connected graph is called $2K_2$-free if it does not contain an *induced* subgraph isomorphic to $2K_2$. The class of $2K_2$-free graphs has many interesting properties (See, for example, [51] or [11]) and arises in many applications in graph theory, such as the perfect graph conjecture and the strong chromatic index conjecture.

When $sm(G) = 1$, Conjecture 2.4 can be formulated as follows.

Conjecture 2.6 (Bermond's conjecture) *Every $2K_2$-free graph with maximum degree Δ has at most $f(\Delta)$ edges, where f is the function defined by Equation 2.1.*

The above conjecture is posed by Erdős and Nešetřil in [13]. Also, it had been previously conjectured by Bermond et al. [5] in 1983. Their conjecture was motivated by a problem in the design of bus interconnection networks (See [5, 4, 6]). The problem is about the maximum number of processors in a multiprocessor system. The processors communicate through the buses. Each processor is on two buses, and at most d processes can share a bus. The objective is to find the maximum number of processors for which we can design such a bus interconnection network, so that for every two processors P_1 and P_2, either P_1 and P_2 can communicate directly through a bus, or there is another processor P such that P_1 and P, and also P_2 and P, can communicate directly. It is not hard to see that the maximum number of processors in such a system is equal to the maximum number of edges in a $2K_2$-free graph of maximum degree d.

In [5], it is remarked that D. J. Kleitman has proved Bermond's conjecture for Δ even, but apparently, his proof was never published. The first published proof of Bermond's conjecture is due to Chung et al. [10] who also proved that the only extremal $2K_2$-free graphs can be obtained from a 5-cycle by vertex multiplication (See Figure 2.1). Another proof of this fact in the case of regular graphs, and some results about related problems can be found in [38]. Also, some interesting Hall-type necessary and sufficient conditions for the existence of strong matchings are proved in [31].

The following is a corollary of Bermond's theorem.

Theorem 2.2 *For every graph G of maximum degree Δ, $s\chi'(G) \leq 2\Delta(\Delta-1)$.*

Proof: By Brook's theorem, for every graph G,

$$s\chi'(G) = \chi(L(G)^2) \leq \Delta(L(G)^2) = 2\Delta(\Delta-1),$$

unless $L(G)^2$ is a complete graph or an odd cycle. It is easy to verify that the square of a graph cannot be an odd cycle of length more than three. If $L(G)^2$ is a complete graph, it means that G is a $2K_2$-free graph. But in this case, by Bermond's theorem, G has at most $f(\Delta)$ edges, and therefore $s\chi'(G) \leq |E(G)| \leq f(\Delta) \leq 2\Delta(\Delta-1)$. ■

Conjecture 2.5 is still open. The following weaker form of this conjecture is proved by Faudree et al. [15].

Theorem 2.3 *There is a constant $\epsilon > 0$ such that $am(G) \leq (2-\epsilon)\Delta^2$.*

Note that the above theorem can also be obtained from the result of Molloy and Reed (Theorem 2.1).

For the bipartite case, the corresponding statements are proved. Faudree et al. [15] proved the following theorem

Theorem 2.4 *For any bipartite graph G, $am(G) \leq \Delta^2$.*

The proof is very short and simple. Also, they proved in [14] that every bipartite $(k+1)K_2$-free graph with maximum degree Δ has at most $k\Delta^2$ edges. This implies the following theorem.

Theorem 2.5 *For every bipartite graph G, $|E(G)| \leq \Delta^2 sm(G)$.*

They also proved that the only extremal graphs are of the form $mC_8^{\Delta/2} \cup nK_{\Delta,\Delta}$, where $2m+n=k$.

2.4 Graphs with small maximum degree

The inequality $s\chi'(G) \leq 2\Delta^2 - 2\Delta + 1$ implies that the strong edge-colouring conjecture is true for $\Delta \leq 2$. Therefore, the first nontrivial case is $\Delta = 3$. L. Andersen [3], and independently, P. Horák et al. [22] settled this problem by proving the following theorem.

Theorem 2.6 *If G is a graph with maximum degree $\Delta \leq 3$, then $s\chi'(G) \leq 10$.*

In [3], it is also proved that for any graph G with maximum degree $\Delta \leq 3$, there is a linear time algorithm for finding a strong edge-colouring using 10 colours.

For $\Delta = 4$, by the strong edge-colouring conjecture every graph must have a strong edge-colouring using 20 colours. By Theorem 2.2, we know that every graph has a strong edge-colouring using 24 colours. This bound has been improved by Horák [21] as follows.

Theorem 2.7 *If G is a graph with maximum degree $\Delta \leq 4$, then $s\chi'(G) \leq 23$.*

This is the only known result for $\Delta = 4$. For $\Delta \geq 5$, the problem is unsolved.

For the bipartite case, the problem is solved for $\Delta = 3$ by A. Steger and M. Yu [49].

Theorem 2.8 *If G is a bipartite graph with maximum degree $\Delta \leq 3$, then $s\chi'(G) \leq 9$.*

Conjecture 2.2 is open for $\Delta \geq 4$. For Conjecture 2.3, the only known result other than the above theorem (which solves this conjecture in the case $\alpha = \beta = 3$) is the following. This result is proved in [8].

Theorem 2.9 *Let G be a bipartite graph with bipartition X, Y, and without any cycle of length four. If the maximum degree of a vertex of X is two and the maximum degree of a vertex of Y is Δ, then $s\chi'(G) \leq 2\Delta$.*

In fact, in [8], it is conjectured that 2Δ in the above theorem can be replaced with $\Delta + 2$. This conjecture is still open and it seems to be very hard.

2.5 Special classes of graphs

The strong edge-colouring conjecture is proved for many classes of graphs. In this section, we list several such classes of graphs.

2.5.1 Trees

A trivial lower bound for the strong chromatic index of a graph G is the following.

$$s\chi'(G) \geq am(G) \geq \max_{uv \in E(G)} \{\deg(u) + \deg(v) - 1\}$$

We define $\sigma(G) := \max_{uv \in E(G)} \{\deg(u) + \deg(v) - 1\}$. Therefore for any graph $s\chi'(G) \geq \sigma(G)$. The following theorem, which is proved in [15], proves that the equality holds for trees.

Theorem 2.10 *If G is a tree then $s\chi'(G) = \sigma(G)$.*

This clearly implies that Conjecture 2.2 and also Conjecture 2.3 hold for trees.

2.5.2 d-dimensional cubes

The d-dimensional cube Q_d is a graph whose vertex set is the set of all binary sequences of length d, and two vertices are adjacent iff the corresponding sequences differ in exactly one position. It is easy to see that Q_d is a bipartite graph. The following theorem, which is proved in [15], implies that Conjecture 2.2 is true for d-dimensional cubes.

Theorem 2.11 *For the d-dimensional cube Q_d,*

$$s\chi'(Q_d) = am(Q_d) = 2d \qquad \text{if} \quad d \geq 2$$

2.5.3 Graphs with all cycle lengths divisible by four

The important property of a graph G all of whose cycle lengths are divisible by four is that none of the cycles of G has a chord. The following theorem, which is proved in [8], states that Conjecture 2.3 is true for such graphs. Note that every such graph is bipartite.

Theorem 2.12 *Let G be a bipartite graph with bipartition X, Y. Let the maximum degree of a vertex in X be α and that in Y be β. Assume that all cycle lengths are divisible by four. Then $s\chi'(G) \leq \alpha\beta$.*

Previously, it had been proved in [15] that Conjecture 2.2 holds for such graphs. It is also conjectured that in fact, the strong chromatic index of any such graph is bounded by a linear function of Δ.

2.5.4 Planar graphs

The following theorem is proved in [15]. The proof makes use of the four colour theorem.

Theorem 2.13 *If G is a planar graph, then $s\chi'(G) \leq 4\Delta + 4$. Moreover for every $\Delta \geq 2$, there exists a planar graph G with maximum degree Δ and $s\chi'(G) = 4\Delta - 4$.*

There are some open problems and conjectures about strong edge-colouring of planar graphs, which will be mentioned in Section 2.7.

2.5.5 Chordal graphs

A graph G is called *chordal* if for every cycle C of length at least four in G, there is an edge other than the edges of C connecting two vertices of C (such an edge is called a *chord* of the cycle). The following two theorems are proved in [9].

Theorem 2.14 *If G is chordal, then $L(G)^2$ is chordal.*

Theorem 2.15 *In a chordal graph, every maximal antimatching is a maximal clique-neighbourhood, where a clique-neighbourhood is the set K of edges of a clique together with some edges each of which is incident to a member of K.*

Theorem 2.14 implies that for every chordal graph G, $L(G)^2$ is perfect and therefore its chromatic number, i.e., the strong chromatic index of G, is equal to the size of its maximum clique, which equals to the size of the maximum antimatching of G. By Theorem 2.15, the size of an antimatching in a chordal graph G is at most

$$\max_k \{ \binom{k}{2} + k(\Delta - k + 1) \} = \frac{\Delta(\Delta + 1)}{2},$$

and therefore chordal graphs satisfy the strong edge-colouring conjecture.

2.5.6 Kneser graphs

The Kneser graph KN_n^m is defined for $m \geq 2n$ as follows. The vertices are n-elements subsets of a fixed m-element set, and two vertices are adjacent if and only if the corresponding sets are disjoint. Faudree et al. [15] have presented two short and elegant proofs for the following theorem.

Theorem 2.16 *The strong chromatic index of the Kneser graph KN_n^m is equal to $\binom{m}{2n}$ for every $m \geq 2n$.*

This together with the combinatorial inequality $\binom{m}{2n} \leq \binom{m-n}{n}^2$ imply that the strong edge-colouring conjecture holds for Kneser graphs.

2.5.7 Subset graphs and revolving door graphs

The *revolving door* graph RD^d is a bipartite graph with bipartition X, Y defined as follows: X is the set of $(d-1)$-element subsets and Y is the set of d-element subsets of a $(2d-1)$-element set. A vertex x in X is adjacent to a vertex y in Y if the set corresponding to x is a subset of the set corresponding to y. The strong chromatic index of RD^d is computed in [15].

In [39], the *subset graph* $S_m(k,l)$ which is a generalization of the revolving door graph is defined and its chromatic index is computed. The subset graph $S_m(k,l)$ is a bipartite graph with one part consisting of k-element subsets of an m-element ground set and the other consisting of l-element subsets of the same ground set. Two vertices are adjacent if one of their corresponding sets is a subset of the other. According to this definition, the revolving door graph RD^d is $S_{2d-1}(d-1,d)$.

The following theorem is proved in [39]. The proof is a simple generalization of the proof of the theorem about the strong chromatic index of RD^d in [15]. Also, a special case of the following theorem (when $l = 1$) is proved in [40].

Theorem 2.17 *The strong chromatic index of the graph $S_m(k,l)$ $(m \geq l \geq k)$ is equal to*

$$s\chi'(S_m(k,l)) = \binom{m}{l-k} \leq \binom{m-k}{l-k}\binom{l}{k},$$

which satisfies Conjecture 2.3.

It is noted in [39] that if we define the *disjoint subset* graph $D_m(k,l)$ as a bipartite graph with one part consisting of k-element subsets and the other part consisting of l-element subsets of an m-element set, and two vertices adjacent if and only if the corresponding subsets are disjoint, then $D_m(k,l)$ is isomorphic to $S_m(k, m-l)$ and therefore its strong chromatic index is equal to $\binom{m}{l+k}$, which satisfies the strong edge-colouring conjecture. Note that $D_m(k,l)$ can be viewed as a bipartite generalization of the Kneser graph.

The definition of $S_m(k,l)$ can be generalized as follows: The graph $S_m(k,l,\lambda)$ is a bipartite graph with the vertex set consisting of k-element and l-element subsets of an m-element set, and two vertices adjacent if and only if the corresponding subsets intersect in exactly λ elements. The following theorem is proved in [39].

Theorem 2.18 *The strong chromatic index of the graph $S_m(k,l,\lambda)$ satisfies*

$$s\chi'(S_m(k,l,\lambda)) \leq \binom{m}{k+l-2\lambda} \min\left\{\binom{m-k-l+2\lambda}{\lambda}, \binom{k+l-2\lambda}{k-\lambda}\right\}$$

and Conjecture 2.3 is true for $S_m(k,l,\lambda)$.

The problem of finding the exact value of $s\chi'(S_m(k,l,\lambda))$ is still open.

2.6 Algorithmic questions

Determining the computational complexity of finding the strong chromatic index is considered for the first time in the present thesis, and it is proved that this problem is **NP**-complete even in a very restricted setting (See Section 5.1). However, there have been some results about some special cases of this problem, and also about two closely related problems, namely the induced matching problem and the distance-2 colouring problem. In this section, we present a survey of these results.

2.6.1 Polynomially solvable special cases

In [9], the problem of finding the strong chromatic index is studied for the class of chordal graphs (See Section 2.5.5), and it is proved that both problems can be solved in polynomial time, using the well-known algorithms for finding the maximum clique in chordal graphs.

Also, for all of the classes of graphs mentioned in Section 2.5 for which the strong chromatic index is determined (i.e., trees, d-dimensional cubes, Kneser graphs, and subset graphs $S_m(k,l)$), it is easy to turn the proof into an algorithm for finding the strong chromatic index, as well as the optimal strong edge-colouring.

2.6.2 The induced matching problem

One problem that is closely related to the strong edge-colouring problem is the problem of finding the largest induced matching in the given graph. Because of the similarities between this problem and the maximum matching problem, one may think that this problem is polynomially solvable. But in [9], K. Cameron proved that this problem is **NP**-complete. Her proof is based on a reduction from the independent set problem.

She also considered this problem for the class of chordal graphs, and proved that it is polynomially solvable for chordal graphs. The algorithm can also be parallelized.

2.6.3 Distance-2 colouring

Another related problem is the problem of colouring the square of the given graph. The problem of determining the strong chromatic index is clearly a special case of this problem. The problem of colouring the square of the given graph is called *distance-2 colouring*. More generally, a *distance-k colouring* of a graph G is a colouring of the vertices of G in such a way that any two vertices with distance at most k receive different colours. The distance-2 colouring problem has many practical applications. It has originally arisen in approximating the Hessian matrices of certain nonlinear functions using a minimum number of gradient evaluations (See [26, 33]). It also has interesting applications in frequency assignment (broadcast scheduling) in packet radio networks (See [30, 42, 44]).

In [33], a reduction from the satisfiability problem is used to prove that the distance-k colouring problem is **NP**-complete for any fixed k. Also, in [41], it is proved that the distance-2 colouring problem is **NP**-complete even for planar graphs of bounded degree. But for graphs which are simultaneously treewidth bounded and degree bounded, a polynomial time algorithm is presented in [30].

Also, there has been some research for finding good approximation algorithms for the distance-2 colouring problem. In [33], it is proved that the simple greedy algorithm provides a performance guarantee of $O(\sqrt{n})$ for the distance-2 colouring problem for any graph with n vertices. Another algorithm with performance guarantee of $O(\theta)$ is provided in [42], where θ is the thickness of the graph. Note that these approximate algorithms provide some (not-so-good) approximation algorithms for the strong edge-colouring problem.

For the distance-2 colouring of some special classes of graphs, better approximation algorithms have been found; for example, a 3-approximation algorithm for (r, s)-civilized graphs [30], some constant performance guarantee approximation algorithms for intersection graphs of circles and unit disk graphs [45], a 9-approximation algorithm for planar graphs [42], and a 3-approximation algorithm for planar graphs of bounded degree [30].

2.7 Related conjectures

There are many open problems and conjectures regarding strong edge-colourings and induced matchings. In this section, we list some conjectures.

Faudree et al. [14] proved that maximum number of edges of a $(k+1)K_2$-free graph is $k\Delta^2$. They also proved that if we impose the additional restriction that the graph is connected, the maximum number of edges is at most $k\Delta^2 - \Delta$, and conjectured the following.

Conjecture 2.7 *There is a constant c such that for k and Δ sufficiently large, every connected $(k+1)K_2$-free graph with maximum degree Δ has at most $k\Delta^2 - ck\Delta$ edges.*

We mentioned the following conjecture in Section 2.4. It is an interesting and probably hard conjecture, which is motivated by the concept of *incidence colouring* defined in [8].

Conjecture 2.8 *Every bipartite graph with bipartition X, Y such that the degree of each vertex in X is at most two and the degree of each vertex in Y is at most Δ has strong chromatic index at most $\Delta + 2$.*

The following conjecture of Faudree et al. [15] was mentioned in Section 2.5.3.

Conjecture 2.9 *The strong chromatic index of every graph with all cycle lengths divisible by four, is bounded by a linear function of Δ.*

The following conjecture is mentioned in [15, 23]. It is also remarked that the complement of C_6 shows that this conjecture, if true, is the best possible.

Conjecture 2.10 *If G is cubic and planar, then $s\chi'(G) \leq 9$.*

The following two conjectures are also mentioned in [15, 23].

Conjecture 2.11 *If G is a bipartite cubic C_4-free graph, then $s\chi'(G) \leq 7$.*

Conjecture 2.12 *If G is a bipartite cubic graph and its girth is sufficiently large, then $s\chi'(G) \leq 5$.*

Chapter 3

Introduction to the probabilistic method

The probabilistic method is one of the most powerful and modern tools in combinatorics. The basic idea of this method is simple and elegant: *In order to prove the existence of an object with a specific property, prove that a randomly chosen object satisfies that property with positive probability.*

The first use of the probabilistic method is in the proof of a theorem by Szele in 1943 which will be mentioned in Section 3.2. However, Erdős is usually credited as the inventor of the probabilistic method, because he was the first one who recognized the power of this method and applied it to a variety of problems. The first problem that was solved by Erdős using this method was a problem about Ramsey numbers which is mentioned in Section 3.1.

One of the most common uses of the probabilistic method is to provide counterexamples to combinatorial conjectures. In addition, today, the probabilistic method has found numerous applications in different areas such as combinatorics and graph theory, combinatorial geometry, computational complexity, and combinatorial number theory. In the next chapter, we will use this method to find lower and upper bounds on the strong chromatic index of C_4-free graphs.

In this chapter, we will present an introduction to the probabilistic method and the tools that we will use in the next chapter. In Section 3.1, the basic probabilistic method is illustrated by some examples. In Section 3.2 the use of expectation in probabilistic arguments (which is also known as the first moment method) is studied, and some basic tools such as linearity of expectation and Markov's inequality are presented. Section 3.3

is about the Lovasz Local Lemma, one of the most important tools in the probabilistic method. In Section 3.4, we will introduce some other important tools in the probabilistic method, which are concentration inequalities. We present two concentration inequalities, namely the Chernoff bound and Talagrand's inequality, in this section. Finally, in Section 3.5 we will briefly introduce the idea of the semirandom method, which is the basic idea of the proof of the main theorem of the next chapter.

3.1 The basic method

In this section, we present two simple and classic examples of the probabilistic method. In these examples, we prove the existence of an object (in the first example, a graph, and in the second example, a 2-colouring) with a desired property P, by proving that a random object satisfies P with positive probability. In these examples the random object is constructed in the most obvious way (in the first example, by putting an edge between each pair of vertices independently and with probability $\frac{1}{2}$, and in the second example, by colouring each vertex with one of two colours, each with probability $\frac{1}{2}$). In later sections, we will see more complex examples in which we have to consider more sophisticated random procedures for constructing the random object.

An important fact about the probabilities that is used in many probabilistic arguments (including the examples in this section) is the following principle. The proof is simple and can be found in any probability theory book.

Theorem 3.1 (Subadditivity of probabilities) *Let $E_1, E_2, \ldots, E_k$ be arbitrary events. Then,*

$$\mathbf{Pr}(E_1 \vee E_2 \vee \cdots \vee E_k) \leq \mathbf{Pr}(E_1) + \mathbf{Pr}(E_2) + \cdots + \mathbf{Pr}(E_k).$$

3.1.1 Lower bound for Ramsey numbers

This example is the subject of a paper by Erdős in 1947, in which Erdős used the probabilistic method for the first time. Let the Ramsey number $R(k, l)$ be the least integer n such that every graph with at least n vertices either contains a clique of size k, or contains an independent set of size l. Szekeres proved that the Ramsey number $R(k, l)$ satisfies the following inequality (see [20] for the proof and history of this theorem).

$$R(k, l) \leq \binom{k + l - 2}{k - 1}$$

For proving a lower bound on $R(k,l)$, one must find a graph without a clique of size k or an independent set of size l. This problem has proved to be a very difficult task. In 1947, Erdős proved the following theorem by a probabilistic argument. No constructive proof for this theorem has been found.

Theorem 3.2 *For every $k \geq 3$, $R(k,k) \geq 2^{k/2}$.*

Proof: Let $N \leq 2^{k/2}$. We show that there is a graph of size N without any clique of size k or an independent set of size k. Consider a random graph $G_{N,\frac{1}{2}}$, which has N vertices and there is an edge between two vertices with probability $\frac{1}{2}$. For every subset S of vertices of size k, let E_S be the event that S is a clique in $G_{N,\frac{1}{2}}$. It is clear that the probability of E_S is $2^{-\binom{k}{2}}$. Therefore, by subadditivity of probabilities, the probability that $G_{N,\frac{1}{2}}$ contains a clique of size k is at most

$$\sum_S \mathbf{Pr}(E_S) = \binom{N}{k} 2^{-\binom{k}{2}} < \frac{N^k}{k!} 2^{-\binom{k}{2}} \leq \frac{2^{k^2/2}}{k!} 2^{-\binom{k}{2}} = \frac{2^{k/2}}{k!} < \frac{1}{2},$$

since for every $k \geq 3$, $\frac{2^{k/2}}{k!} < \frac{1}{2}$. Similarly, the probability that $G_{N,\frac{1}{2}}$ has an independent set of size k is less than $\frac{1}{2}$. Therefore, the probability that $G_{N,\frac{1}{2}}$ has an independent set or a clique of size k is less than $\frac{1}{2} + \frac{1}{2} = 1$. Thus, the probability that $G_{N,\frac{1}{2}}$ has neither an independent set nor a clique of size k is positive. This means that there is a graph on N vertices without a clique or an independent set of size k. ■

In fact, the above argument with a more careful analysis yields

$$R(k,k) > \frac{1}{e\sqrt{2}}(1+o(1))k2^{k/2}.$$

Notice that the above argument is a nonconstructive existence argument and does not provide a general method for constructing a graph with $2^{k/2}$ vertices without any independent set or clique of size k. Finding a constructive proof for Theorem 3.2 is still an open question.[1] In general, probabilistic arguments are essentially existence arguments, although some methods for turning probabilistic arguments into constructive proofs have been developed, which can be used in many cases.

We will use more powerful tools (alteration method and the Local Lemma) to derive better lower bounds on $R(k,k)$ in Sections 3.2.2 and 3.3.1.

[1] Erdős has offered \$100 for a solution to this problem.

3.1.2 2-Colouring hypergraphs

A *hypergraph* H is a pair $H = (V, E)$, where V is a set whose elements are called the vertices of H, and E is a collection of subsets of V, called edges. A hypergraph is called *n-uniform*, if every edge of the hypergraph is a subset of size n of the vertices. A hypergraph whose vertices can be coloured using two colours, such that no edge is monochromatic, is called a *2-colourable hypergraph.* In 1963, Erdős raised the question of finding the minimum number of edges in an n-uniform hypergraph which is *not* 2-colourable. He proved the following theorem.

Theorem 3.3 *Every n-uniform hypergraph with less than 2^{n-1} edges is 2-colourable.*

Proof: Let $H = (V, E)$ be an n-uniform hypergraph with fewer than 2^{n-1} edges. Consider a random 2-colouring of V. For every fixed edge $e \in E$, the probability that e is monochromatic is $2^{-(n-1)}$. Therefore, by subadditivity of probabilities, the probability that at least one of the edges of H is monochromatic is at most

$$|E| \times 2^{-(n-1)} < 2^{n-1} \times 2^{-(n-1)} = 1.$$

Thus, the probability that no edge of H is monochromatic is more than zero. So, there is a 2-colouring of H without any monochromatic edge. ∎

In 1964, Erdős used another simple probabilistic argument to show that there are n-uniform hypergraphs with $(1 + o(1))\frac{e \ln 2}{4} n^2 2^n$ edges which are not 2-colourable, and so Theorem 3.3 is nearly best possible. Refer to [2] or [48] to see the proof.

3.1.3 Probabilistic method or counting argument?

The proof of Theorem 3.2 in the original paper of Erdős is a bit different from the proof that we presented in Section 3.1.1. The proof that is mentioned in the paper is the following counting argument.

Another proof of Theorem 3.2: Let $N \leq 2^{k/2}$. There are $2^{N(N-1)/2}$ different labeled graphs on N vertices. The number of graphs on N vertices which have a clique of size k is at most

$$\binom{N}{k} 2^{\binom{N}{2} - \binom{k}{2}} < \frac{N^k}{k!} \times \frac{2^{N(N-1)/2}}{2^{k(k-1)/2}} \leq \frac{2^{k/2}}{k!} \times 2^{N(N-1)/2} < \frac{1}{2} \times 2^{N(N-1)/2}.$$

Similarly, the number of graphs with an independent set of size k is less than $\frac{1}{2} \times 2^{N(N-1)/2}$. Therefore, the number of graphs which have either an independent set or a clique of size

k is less than $2^{N(N-1)/2}$, and hence, there must be a graph on N vertices without any clique or independent set of size k. ∎

Similarly, it is possible to prove Theorem 3.3 by a counting argument. At first glance, it seems that all probabilistic arguments can be expressed as *more natural* counting arguments. So, why do we bother to use probabilities?

The reason is that the situation is not always so simple. In many problems, we need to consider more complicated probability spaces. For example for finding a lower bound for non-diagonal Ramsey numbers (i.e., $R(k,l)$, where $k \neq l$), it is usually necessary to take a random graph in which there is an edge between each pair of vertices with a fixed probability other than $\frac{1}{2}$. Furthermore, if we use the language of probabilities, we can use many probability tools such as the Lovasz Local Lemma, linearity of expectation, correlation inequalities, the Chernoff bound, and Chebyschev's, Azuma's, and Talagrand's inequalities. We will see several examples in the next sections in which it is almost impossible to express the proof by counting arguments. Erdős used both counting and probability languages in his results, but today the probabilistic language is usually used.

3.2 Using expectation

The expected value (or expectation) of a random variable X is defined as

$$\mathbf{E}(X) = \sum_{a \in S} a \times \mathbf{Pr}(X = a),$$

where S is the set of all possible values of X. The expected value is also called the first moment. The following theorem is the most important property of expectation. The proof can be found in any probability theory textbook.

Theorem 3.4 (Linearity of expectation) *Let $X_1, X_2, \ldots, X_n$ be random variables and $X = X_1 + X_2 + \cdots + X_n$. Then*

$$\mathbf{E}(X) = \mathbf{E}(X_1) + \mathbf{E}(X_2) + \cdots + \mathbf{E}(X_n).$$

The power and importance of the above theorem is due to the fact that it does not need the random variables $X_1, X_2, \ldots, X_n$ to be independent. This usually makes $\mathbf{E}(X)$ easy to compute, and hence it is very convenient for probabilistic arguments. If you want to appreciate the power of the above theorem, try to solve the following example *without* using the linearity of expectation.

Example 3.1 Let X be the number of fixed points of a random permutation π of $\{1, 2, \ldots, n\}$. For computing $\mathbf{E}(X)$, we define the indicator random variable X_i ($1 \leq i \leq n$) as follows:

$$X_i = \begin{cases} 1 & \text{if } \pi(i) = i \\ 0 & \text{otherwise} \end{cases}$$

It is clear that $X = X_1 + X_2 + \cdots + X_n$. Furthermore, we have:

$$\mathbf{E}(X_i) = 1 \times \mathbf{Pr}(\pi(i) = i) + 0 \times \mathbf{Pr}(\pi(i) \neq i) = \frac{1}{n}$$

Therefore, $\mathbf{E}(X) = \mathbf{E}(X_1) + \mathbf{E}(X_2) + \cdots + \mathbf{E}(X_n) = 1$.

The method that is used in the above example is the general method that is usually used to compute the expected value of a random variable. The method is based on defining indicator random variables and expressing the random variable X as a summation of indicator random variables. The expected value of an indicator random variable is equal to the probability that it is equal to one. Therefore, by the linearity of expectation, it is sufficient to compute these probabilities.

The following trivial principle is the basis of many probabilistic arguments.

The First Moment Principle: *Let X be a random variable. If $\mathbf{E}(X) \leq t$, then $\mathbf{Pr}(X \leq t) > 0$, and if $\mathbf{E}(X) \geq t$, then $\mathbf{Pr}(X \geq t) > 0$.*

Notice that we can express basic applications of the probabilistic method (such as the two examples that were mentioned in Sections 3.1.1 and 3.1.2), using the above principle. For example the proof of Theorem 3.3 can be stated as follows.

Proof of Theorem 3.3: In a random 2-colouring of a hypergraph H, the probability that a fixed edge e is monochromatic is $2^{-(n-1)}$. Let X be the number of monochromatic edges in a random 2-colouring. By linearity of expectation,

$$\mathbf{E}(X) = |E| \times 2^{-(n-1)} < 2^{n-1} \times 2^{-(n-1)} = 1.$$

Thus, by the first moment principle, $\mathbf{Pr}(X < 1) > 0$. But since X is an integer valued variable, $\mathbf{Pr}(X < 1) = \mathbf{Pr}(X = 0)$. So, the probability that a random 2-colouring of H does not have any monochromatic edge is positive. ■

We present two other applications of the first moment principle in Sections 3.2.1 and 3.2.2. The example in Section 3.2.2 also illustrates another technique, known as the *alteration method* or *deletion method* which is usually used in more complex probabilistic arguments. In Section 3.2.3 we state Markov's inequality which is in fact a generalization of the first moment principle. An application of this inequality, together with the alteration method, is mentioned in Section 3.2.4. Several other examples of using expectation (together with some other tools) are mentioned in Section 3.4.

3.2.1 Number of Hamiltonian paths in a tournament

In this section we present a theorem of Szele (1943) which is sometimes considered to be the first application of the probabilistic method. The theorem proves the existence of a tournament with a large number of Hamiltonian paths using the first moment principle. A *tournament* is a complete directed graph, i.e., a directed graph which has exactly one edge between each pair of vertices. One can think of the vertices of the tournament as players, and an edge from the vertex u to the vertex v as an indication that player u has beaten player v (There is exactly one match between each pair of players). A Hamiltonian path is a path which traverses each vertex exactly once.

Theorem 3.5 *For every n, there is a tournament with n players and at least $\frac{n!}{2^{n-1}}$ Hamiltonian paths.*

Proof: We use the method that is used in Example 3.1 to find the expected number of Hamiltonian paths in a random tournament. For every permutation σ of the vertices, let X_σ be the indicator random variable that is one if σ is a Hamiltonian path, and zero otherwise. It is clear that

$$\mathbf{E}(X_\sigma) = \mathbf{Pr}(\sigma \text{ is a Hamiltonian path}) = \frac{1}{2^{n-1}}.$$

Therefore, by the linearity of expectation, the expected value of the number of Hamiltonian paths in a random tournament is equal to the summation of $\mathbf{E}(X_\sigma)$ for every σ, which is equal to $\frac{n!}{2^{n-1}}$. The first moment principle completes the proof. ■

3.2.2 A better lower bound for the Ramsey number

In some problems, it may be impossible to prove that the random object satisfies the desired property with positive probability, but sometimes, we can prove that a random object *mostly* satisfies the desired property, with positive probability. In these cases, one approach is to take such a random object, and change it (for example, delete some of the vertices), in such a way that the resulting object satisfies the desired property. Here we apply this idea to the problem of finding a lower bound for Ramsey numbers, which is mentioned in Section 3.1.1.

Theorem 3.6 *For every k, n,*

$$R(k,k) > n - \binom{n}{k} 2^{1-\binom{k}{2}}$$

Proof: Consider a random graph on n vertices. For every set S of k vertices, the probability that S is either a clique or an independent set is $2^{1-\binom{k}{2}}$. Therefore, (by an argument similar to the argument of Example 3.1) the expected number of sets of size k, which are either a clique or an independent set in the random graph is equal to $\binom{n}{k}2^{1-\binom{k}{2}}$. So, by the first moment principle, there is a graph on n vertices with at most $\binom{n}{k}2^{1-\binom{k}{2}}$ cliques and independent sets. Now, we delete one vertex from each of the cliques or independent sets of this graph. The resulting graph has $n - \binom{n}{k}2^{1-\binom{k}{2}}$ vertices and does not have any clique or independent set of size k. ∎

Using some calculus, it is not difficult to choose a value for n which maximizes $n - \binom{n}{k}2^{1-\binom{k}{2}}$. This yields the following corollary.

Corollary 3.1 $R(k,k) > \frac{1}{e}(1+o(1))k2^{k/2}$.

Recall that the careful analysis of the arguments of Theorem 3.2 yields the lower bound $\frac{1}{e\sqrt{2}}(1+o(1))k2^{k/2}$, and therefore the above corollary improves the lower bound by a factor of $\sqrt{2}$.

3.2.3 Markov's Inequality

The first moment principle states that the probability that a random variable is more than its expected value is less than one. The following inequality, known as *Markov's inequality* is a generalization of this fact.

Theorem 3.7 (Markov's Inequality) *For every positive random variable X and every positive number t,*

$$\mathbf{Pr}(X \geq t) \leq \frac{\mathbf{E}(X)}{t}.$$

Proof: By the definition of $\mathbf{E}(X)$, we have:

$$\begin{aligned}
\mathbf{E}(X) &= \sum_{a \in S} a \times \mathbf{Pr}(X = a) \\
&\geq \sum_{a \geq t} a \times \mathbf{Pr}(X = a) \\
&\geq \sum_{a \geq t} t \times \mathbf{Pr}(X = a) \\
&= t \times \mathbf{Pr}(X \geq t)
\end{aligned}$$

Therefore, $\mathbf{Pr}(X \geq t) \leq \frac{\mathbf{E}(X)}{t}$. ∎

The advantage of the Markov inequality is that it is very simple, and it only requires X to be a *positive* random variable. In the next section, we will see one of the applications of this inequality.

3.2.4 Graphs with high chromatic number and high girth

A fundamental question in the theory of graph colouring is the relationship between the subgraphs of a graph and its chromatic number. For example, can we prove that every triangle-free graph has a chromatic number less than some constant?

B. Descartes was the first one to construct triangle-free graphs with arbitrarily high chromatic numbers (See [7]). But this example contains many short cycles. This raised another question: Is there any graph without any short cycle, and with arbitrarily high chromatic number? In 1959, Erdős used the probabilistic method to prove the existence of graphs with high girth and high chromatic number. The girth of a graph is defined as the length of its shortest cycle. Erdős proved the following theorem. Both Markov's inequality and the alteration technique are used in the proof.

Theorem 3.8 *For every g, k, there is a graph of girth more than g and chromatic number more than k.*

Proof: Consider the random graph $G_{n,p}$ on n vertices which has an edge between each pair of vertices with probability p (n and p are two parameters that will be fixed later). We prove that with high probability, $G_{n,p}$ has high chromatic number. The idea is that if a graph can be coloured with k colours, then at least one of the colour classes is an independent set of size at least $\frac{n}{k}$. Therefore, in order to show that $G_{n,p}$ has chromatic number more than k (with high probability), it suffices to prove that it does not contain an independent set of size $\frac{n}{k}$ (with high probability). For some reason (which will become clear later), we prove the stronger statement that the probability that $G_{n,p}$ contains an independent set of size $\frac{n}{2k}$ is small. By subadditivity of probabilities, this probability is at most

$$\begin{aligned}\mathbf{Pr}(G_{n,p} \text{ has an independent set of size } \frac{n}{2k}) &\leq \binom{n}{n/2k}(1-p)^{\binom{n/2k}{2}} \\ &< 2^n \times e^{-p\frac{n^2}{8k^2}}\end{aligned}$$

Therefore if we take $p = n^{\epsilon-1}$, for some positive ϵ, we have $-p\frac{n^2}{8k^2} = -\frac{n^{1+\epsilon}}{8k^2}$, and therefore, $2^n \times e^{-p\frac{n^2}{8k^2}} = \exp((\ln 2)n - (\frac{1}{8k^2})n^{1+\epsilon})$ tends to zero as n tends to infinity. Therefore, for n sufficiently large, the above probability is less than $\frac{1}{2}$.

Now, we use the linearity of expectation to compute the expected number of cycles of length less than or equal to g in $G_{n,p}$. We denote this number by X. It is not difficult to see that this expected value is equal to

$$\mathbf{E}(X) = \sum_{i=3}^{g} \frac{n!}{2i(n-i)!}p^i < \sum_{i=3}^{g}(np)^i < g(np)^g = gn^{\epsilon g}.$$

Therefore if we choose ϵ so that $\epsilon g < 1$, we have $\mathbf{E}(X) = o(n)$. Thus, for sufficiently large n, the expected number of cycles of length at most g is less than $\frac{n}{4}$, and hence, by Markov's inequality,

$$\mathbf{Pr}(X \geq \frac{n}{2}) < \frac{1}{2}.$$

Therefore, if we choose n sufficiently large, and $p = n^{\epsilon-1}$, where $0 < \epsilon < \frac{1}{g}$, the probability that $G_{n,p}$ has an independent set of size $\frac{n}{2k}$, or the number of cycles of length at most g in it is at least $\frac{n}{2}$, is less than $\frac{1}{2} + \frac{1}{2} = 1$. Thus, there is a graph G without any independent set of size $\frac{n}{2k}$ and with less than $\frac{n}{2}$ cycles of length at most g. Now we delete one vertex from each cycle of length at most g in G, and call the resulting graph G'. It is clear that G' has at least $\frac{n}{2}$ vertices and does not contain an independent set of size $\frac{n}{2k}$. Therefore, its chromatic number is more than $\frac{n}{2} \Big/ \frac{n}{2k} = k$. ■

3.3 The Lovasz Local Lemma

In most of the examples of the probabilistic method, we have a set of "bad" events $P_1, P_2, \ldots, P_n$, each occurring with probability at most p, and we want to prove that with positive probability, none of these bad events occur. In previous sections, the method that we used in such situations was based on the fact that the probability that at least one of P_i's occur, is at most np, and therefore if we prove $np < 1$, it implies that the probability that none of P_i's occur is positive.

In some situations, this is not a good method. For example, consider the case where the events are independent. In this case, no matter how large p is (as long as it is less than one), the probability that none of P_i's occur is $(1-p)^n$ and therefore is positive, even if np is much greater than one. In general, if each event is independent of most of the other events, the above method does not provide a good estimate on the probability that at least one of P_i's occur. In such situations, the Lovasz Local Lemma comes into play. The Lovasz Local Lemma provides a sufficient condition for the probability that none of the P_i's hold to be positive. The main advantage of the condition that the Lovasz Local Lemma provides, over the condition $np < 1$ which we used in previous sections, is that the condition of the Local Lemma is in terms of the degree of dependency between the events, not in terms of n.

Another important point is that in most of the examples in previous sections, we actually proved that the probability that the random object satisfies the desired property is not only positive, but also high (usually tending to one). In other words, in most examples, we proved that almost every random object satisfies the desired property. But for example in the situation that the bad events are independent, the probability that no bad event occurs is $(1-p)^n$, which is a positive, but small number (it tends to zero as n tends to infinity). The situation is the same for the Lovasz Local Lemma: The Local Lemma proves that the probability that no bad event occurs is positive, while this probability is small. This justifies the power of the Local Lemma. The Local Lemma can be used in some situations when no other tool can be used.

The Local Lemma has various forms. Here, we present the symmetric form. To see the general form and the asymmetric form of the Local Lemma, refer to [36] or [2]. In some rare applications, we have to use the more general forms of the Local Lemma, rather than the symmetric form, but usually the symmetric form does the job. (Refer to [36] to see some examples which need other forms of the Local Lemma.) To see the proof of the Local Lemma, refer to [2].

Before, stating the Local Lemma, we need some definitions about the independence of events. Two events A and B are called *independent* if $\mathbf{Pr}(A|B) = \mathbf{Pr}(A)$, or equivalently if $\mathbf{Pr}(A \wedge B) = \mathbf{Pr}(A)\mathbf{Pr}(B)$. An event A is called *mutually independent* of a set $\{A_1, A_2, \ldots, A_n\}$ of events if $\mathbf{Pr}(A|A_{i_1} \wedge A_{i_2} \wedge \cdots \wedge A_{i_k}) = \mathbf{Pr}(A)$ for all distinct indices $i_1, i_2, \ldots, i_k$. Intuitively, this means that A is mutually independent of a set S

of events if *any* information about the occurrence of the events in S does not give *any* information about the probability of the occurrence of the event A. Notice that if A and A_1 are independent and also A and A_2 are independent, it cannot be deduced that A is mutually independent of $\{A_1, A_2\}$. To see this, consider the experiment of flipping three coins independently, and let the event A_{ij} $(1 \leq i, j \leq 3)$ be the event that the outcome of the ith and jth coin flips are the same. It is easy to see that every two of the events A_{12}, A_{23}, A_{31} are independent, but none of them is mutually independent of the set of the other two events.

Theorem 3.9 (Lovasz Local Lemma) *Let $E_1, E_2, \ldots, E_n$ be a set of events, such that each event is mutually independent of the set of all but at most d other events, and each event occurs with probability at most p. If $ep(d+1) < 1$ (e is the base of natural logarithm), then with positive probability, none of E_i's occur.*

If we replace the condition $ep(d+1) < 1$ by $pd < \frac{1}{4}$, we get a simpler (and of course weaker) version of the Local Lemma, which is still applicable in most of the situations.

In this section, we present three examples of using the Local Lemma in probabilistic arguments. The first two examples, mentioned in Sections 3.3.1 and 3.3.3, are about the two problems that we discussed before, using the first moment method, namely providing a lower bound for the Ramsey number $R(k,k)$ and 2-colouring hypergraphs. The third example (Section 3.3.4) is a nice theorem of B. Reed which will be used in the next chapter. Also, in Section 3.3.2 we will present a simple theorem that is very useful in proving the mutual independence of events.

3.3.1 Ramsey numbers revisited

We can use the Local Lemma to prove the following theorem of [20] which is an improvement over Theorem 3.2.

Theorem 3.10 *If $e\left(\binom{k}{2}\binom{n}{k-2}+1\right)2^{1-\binom{k}{2}} < 1$, then $R(k,k) > n$.*

Proof: In the proof of Theorem 3.2, for each subset S of size k of the vertices of the random graph $G_{n,\frac{1}{2}}$, we have the bad event that S is either an independent set or a clique. Let's call this event E_S. We claim that every event E_S is mutually independent of the set of events $\{E_{S'} : |S' \cap S| \leq 1\}$. This claim is intuitively clear, but we mention the proof here because the rigorous proof is not so trivial.

We must prove $\mathbf{Pr}(E_S \mid E_{S_1} \wedge E_{S_2} \wedge \cdots \wedge E_{S_r}) = \mathbf{Pr}(E_S)$ for every $S_1, S_2, \ldots, S_r$ such that $|S_i \cap S| \leq 1$ for all i. In order to construct the random graph $G_{n,\frac{1}{2}}$, we perform a random trial for each pair of vertices whose outcome is either to put an edge between the pair of vertices, or not to put an edge between them. Let T be the set of trials corresponding to the pairs of the vertices of S, and T' be the set of all other trials. It is clear that every outcome of the trials of T', completely determines whether the events $E_{S_1}, E_{S_2}, \ldots, E_{S_r}$ occur or not. Therefore, if m denotes the number of outcomes of the trials of T' which make *all* of the events $E_{S_1}, E_{S_2}, \ldots, E_{S_r}$ occur, the number of outcomes of all of the trials which make these events occur is $m2^{\binom{k}{2}}$, and the number of outcomes of all of the trials which make these events, and also the event E_S occur is $2m$ (because there are exactly two possible outcomes for the trials in T which make E_S occur). Therefore, $\mathbf{Pr}(E_S \mid E_{S_1} \wedge E_{S_2} \wedge \cdots \wedge E_{S_r})$ is equal to $2m \big/ m2^{\binom{k}{2}} = 2^{1-\binom{k}{2}} = \mathbf{Pr}(E_S)$. This completes the proof of our claim.

Therefore, every event is mutually independent of all but at most $\binom{k}{2}\binom{n}{k-2}$ other events. So, the Local Lemma with $p = 2^{1-\binom{k}{2}}$ and $d = \binom{k}{2}\binom{n}{k-2}$ completes the proof. ■

We can derive the following corollary from the above theorem by some computation.

Corollary 3.2 $R(k,k) > \frac{\sqrt{2}}{e}(1+o(1))k2^{k/2}$.

Note that this is a factor $\sqrt{2}$ improvement over Theorem 3.6 and a factor 2 improvement over Theorem 3.2. The reason that the improvement is not substantial in this example is that there are a lot of dependencies between the bad events.

3.3.2 The mutual independence principle

In most applications of the Lovasz Local Lemma, we prove the mutual independence of events by a straightforward argument similar to the argument that is used in the proof of Theorem 3.10. In order to avoid repeating this simple argument, we use the following principle that is stated in [37]. The proof is essentially the same as the argument in the proof of Theorem 3.10.

Theorem 3.11 (The Mutual Independence Principle) *Suppose that $\mathcal{X} = X_1, X_2, \ldots, X_m$ is a sequence of independent random experiments. Suppose further that $A_1, \ldots, A_n$ is a set of events, where each A_i is determined by a subset $F_i \subseteq \mathcal{X}$ of random experiments. If $F_i \cap (F_{i_1} \cup \cdots \cup F_{i_k}) = \emptyset$, then A_i is mutually independent of $\{A_{i_1}, \ldots, A_{i_k}\}$.*

Intuitively, the above theorem states that if an event A is determined by a set of random experiments which is disjoint from the set of random experiments that determine a set of events S, then A is mutually independent of S.

3.3.3 2-colouring hypergraphs revisited

In Section 3.1.2 we proved that every n-uniform hypergraph with less than 2^{n-1} edges can be 2-coloured. But what if the hypergraph has more edges, but the edges are mostly disjoint? In this case, we can use the Local Lemma to prove the following theorem.

Theorem 3.12 *Every n-uniform hypergraph in which each edge intersects at most 2^{n-3} other edges is 2-colourable.*

Proof: Consider a random colouring of the vertices of the hypergraph H, and let the event E_e be the event that the edge e is monochromatic. Clearly, $\mathbf{Pr}(E_e) = 2^{-(n-1)}$. By the mutual independence principle, every event E_e is mutually independent of the set of events $\{E_{e'} \mid e \text{ and } e' \text{ are disjoint}\}$. Therefore, using the Local Lemma with $p = 2^{-(n-1)}$ and $d = 2^{n-3}$ completes the proof. ∎

3.3.4 A theorem on list colouring

In this section, we will prove a sufficient condition for the existence of list colouring. It is clear that if there is a list of $\Delta + 1$ colours on each vertex, there is a proper list colouring (The list colouring can be found by a simple greedy algorithm). This is an obvious sufficient condition, and it is easy to see that actually, it is the best possible condition, if we do not impose any restriction on the lists (other than their size). But if we have the extra restriction that the lists have limited intersections, the following theorem can be proved. This theorem first appeared in [43].

Theorem 3.13 *If the list of each vertex has at least ℓ elements, and for every vertex v and every colour c in the list of v, c does not appear on the lists of more than $\frac{\ell}{2e}$ neighbours of v, then there is a proper list colouring.*

Proof: Without loss of generality, we may assume that the list of each vertex contains exactly ℓ colours. Consider a random colouring, i.e., a colouring in which each vertex receives a random colour from its list, with equal probability for each colour. We will prove that with positive probability, this colouring is a proper colouring. For every edge $e = uv$ and every colour c in the intersection of the lists of u and v, let $A_{e,c}$ be the event that both u and v are coloured with c. It is clear that if none of the events $A_{e,c}$ occur, the colouring is a proper colouring. It is clear that the probability of $A_{e,c}$ is exactly $\frac{1}{\ell^2}$. Also, by the mutual independence principle, for every such e, c, the event $A_{e,c}$ is independent of the set of events $\{A_{e',c'} \mid e' \text{ is not incident with } u \text{ or } v\}$. If e' is incident with u, c' must be on the list of u and also on the list of the other endpoint of e'. But by our assumption every colour in the list of u appears on the list of at most $\frac{\ell}{2e}$ neighbours of u. Therefore, there are at most $\frac{\ell^2}{2e}$ such pairs e', c'. Similarly, there are at most $\frac{\ell^2}{2e}$ pairs e', c' for which e' is incident with v. Therefore, the event $A_{e,c}$ is mutually independent of all other events but a set of at most $\frac{\ell^2}{2e} + \frac{\ell^2}{2e} - 1 = \frac{\ell^2}{e} - 1$ events (The -1 term is because we have counted $A_{e,c}$ twice). So, we can use the Local Lemma with $p = \frac{1}{\ell^2}$ and $d = \frac{\ell^2}{e} - 1$. ■

Notice that for proving Theorem 3.13, we must be careful in the choice of the bad events. For example, if instead of the event $A_{e,c}$ that we used in the above proof, we consider the bad event A_e which means that both endpoints of e get the same colour, we will not be able to use the assumption of the theorem to bound the dependencies between the bad events.

3.4 Concentration inequalities

Assume that we have a random variable X with expected value $\mathbf{E}(X) = a$. Can we deduce that the value of X is probably near a? Of course, this is not possible in general. For example, if the random variable X takes the values 100 and -100 each with probability $\frac{1}{2}$, the expected value of X would be zero, but X is never near its expected value. However, in many practical situations, it is possible to prove that the random variable is near its expected value with high probability. In such cases, we say that the random variable is *concentrated.*

Concentration inequalities (also called large deviation inequalities) are theorems which enable us to prove that a random variable is concentrated, under specific conditions. There are many concentration inequalities, such as the Chernoff bound, Chebyschev's inequality, Azuma's inequality, and Talagrand's inequality. Also, notice that Markov's inequality is actually a concentration inequality. Refer to [34] to see more inequalities.

In this section, we state two concentration inequalities: the Chernoff bound and Talagrand's inequality. The Chernoff bound proves the concentration of a special random variable, namely the random variable with binomial distribution. This random variable is very important since it appears in many problems. Talagrand's inequality applies to a broader range of random variables. We will use this inequality in the next chapter.

3.4.1 The Chernoff bound

Let $X_1, X_2, \ldots, X_n$ be n *independent* random variables which are each equal to one with probability p, and zero with probability $1-p$. The random variable $X = X_1 + X_2 + \cdots + X_n$ is called the binomial random variable (or the random variable with binomial distribution), and is denoted by $\mathrm{BIN}(n,p)$. It is clear that the expected value of $\mathrm{BIN}(n,p)$ is np. The Chernoff bound proves that this random variable is highly concentrated around its expected value.

Theorem 3.14 (The Chernoff bound) *For any $0 \le t \le np$,*

$$\mathbf{Pr}(|\mathrm{BIN}(n,p) - np| > t) < 2e^{-t^2/3np}.$$

We do not mention the proof here, but the idea of the proof is to use Markov's inequality on the random variable $e^{\lambda X}$, and then optimize λ to get the best result. Refer to [2] for a complete proof. Notice that the Chernoff bound has many different forms, but the above form is perhaps the most simple and still useful form. Here we present two applications of the Chernoff bound. To see other examples refer to [37] or [2].

Example 1: Discrepancy of hypergraphs

Consider a 2-colouring of an n-uniform hypergraph H. The discrepancy of an edge $e \in E(H)$ in this colouring is the absolute value of the difference between the number of vertices in e which are coloured with colour 1, and the number of vertices in e which are coloured with colour 2. The discrepancy of the colouring is the maximum of the discrepancies of the edges. In the following theorem we prove that every n-uniform hypergraph with n edges has a colouring with a relatively small discrepancy.

Theorem 3.15 *Every n-uniform hypergraph with n edges has a 2-colouring with discrepancy at most $\sqrt{8n\ln n}$.*

Proof: Consider a random 2-colouring of H. We prove that with positive probability, the discrepancy of the colouring is no more than $\sqrt{8n\ln n}$.

Consider any edge e of the hypergraph, and let X be the number of vertices in e which are coloured with the colour 1 in the random colouring. It is clear that $X = \mathrm{BIN}(n, \frac{1}{2})$. Therefore, the Chernoff bound with $t = \sqrt{2n \ln n}$ implies that

$$\mathbf{Pr}(|X - \frac{n}{2}| > t) < 2e^{-\frac{t^2}{3n/2}} = 2e^{-\frac{4}{3}\ln n} = 2n^{-\frac{4}{3}} < \frac{1}{n}.$$

Notice that the discrepancy of e is more than $\sqrt{8n \ln n}$ if and only if $|X - \frac{n}{2}| > \sqrt{2n \ln n}$. Therefore, the probability that the discrepancy of e is more than $\sqrt{8n \ln n}$ is less than $\frac{1}{n}$. So, by subadditivity of probabilities, the probability that at least one of the edges of H has discrepancy more than $\sqrt{8n \ln n}$ is less than $n \times \frac{1}{n} = 1$. Thus, with positive probability, the discrepancy of the random colouring is at most $\sqrt{8n \ln n}$. ■

Example 2: Unrankable Tournaments

Let T be a tournament with vertex set $\{1, 2, \ldots, n\}$, and σ be a permutation of $\{1, 2, \ldots, n\}$ (σ can be thought of as a ranking of the players). A pair $\{i, j\}$ is called an upset if i beats j in T, but $\sigma(j) > \sigma(i)$ (i.e. the ranking is not consistent with the result of their match), and is called a nonupset otherwise. The fit $f(T, \sigma)$ is the number of nonupsets minus the number of upsets. We usually want to find a ranking which maximizes the fit. The following theorem, proved by J. Moon and Erdős in 1965, says that there are tournaments for which no good ranking can be found.

Theorem 3.16 *There is a tournament T so that for all rankings σ,*

$$f(T, \sigma) \leq 2n^{3/2}\sqrt{\ln n}.$$

Proof: Consider a fixed permutation σ and a random tournament T, and let X_σ be the number of nonupsets. It is clear that $f(T, \sigma) = X_\sigma - (\binom{n}{2} - X_\sigma) = 2(X_\sigma - \frac{1}{2}\binom{n}{2})$. On the other hand, it is easy to see that X_σ is actually $\mathrm{BIN}(\binom{n}{2}, \frac{1}{2})$. Therefore, by the Chernoff bound with $t = n^{3/2}\sqrt{\ln n}$, we have

$$\mathbf{Pr}(|X_\sigma - \frac{1}{2}\binom{n}{2}| > t) < 2e^{-t^2/\frac{3}{2}\binom{n}{2}} = 2e^{-\frac{4n^3 \ln n}{3n(n-1)}} < 2e^{-\frac{4}{3}n \ln n} < n^{-n} < \frac{1}{n!}.$$

Therefore, $\mathbf{Pr}(f(T, \sigma) > 2t) = \mathbf{Pr}(X_\sigma - \frac{1}{2}\binom{n}{2} > t) < \frac{1}{n!}$. Since there are $n!$ permutations, by subadditivity of probabilities, the probability that there is a permutation σ for which $f(T, \sigma) > 2n^{3/2}\sqrt{\ln n}$ is at most

$$\sum_{\sigma} \mathbf{Pr}(f(T, \sigma) > 2t) < n! \times \frac{1}{n!} = 1.$$

Therefore, with positive probability, the random tournament does not have any ranking σ for which $f(T, \sigma) > 2n^{3/2}\sqrt{\ln n}$. In other words, there is a tournament T such that for all σ, $f(T, \sigma) \le 2n^{3/2}\sqrt{\ln n}$. ∎

J. Spencer in his doctoral dissertation (see [46]) showed that for every tournament T, there is a ranking σ with $f(T, \sigma) = O(n^{3/2})$. Later, he also showed that it is possible to prove Theorem 3.16 with $2n^{3/2}\sqrt{\ln n}$ replaced by a constant multiple of $n^{3/2}$ (see [47]).

3.4.2 Talagrand's inequality

Talagrand's inequality is a powerful concentration inequality that applies to a broad range of random variables. The original theorem of Talagrand, which is stated below, does not look like a concentration inequality at the first glance, and is hard to get used to, but there are some simple and still powerful corollaries, which are more convenient to use. However, it is important to notice that none of these inequalities has the full power of Talagrand's original inequality. In this section, we present Talagrand's inequality and a corollary which will be used in the next chapter.

Before we state Talagrand's inequality, we need some definitions. Let $X_1, X_2, \ldots, X_m$ be m independent random variables (the random trials). For every subset A of the possible outcomes and any real number ℓ, we define A_ℓ to be the set of all possible outcomes which are within distance ℓ of A in the following sense: $(x_1, x_2, \ldots, x_m) \in A_\ell$ if and only if for *any* real $a_1, a_2, \ldots, a_m$, there exists at least one element $(y_1, y_2, \ldots, y_m)$ of A such that

$$\sum_{x_i \neq y_i} a_i < \ell \left(\sum_{i=1}^{m} a_i^2 \right)^{1/2}.$$

Theorem 3.17 (Talagrand's Inequality) *For every set A and every real number ℓ,*

$$\mathbf{Pr}(A)(1 - \mathbf{Pr}(A_\ell)) \le e^{-\ell^2/4}.$$

See [50] or [34] for a proof. The following theorem is a corollary of Talagrand's inequality that will be used in the next chapter.

Theorem 3.18 *Let $R_1, R_2, \ldots, R_n$ be n binary random variables defined as a function of m independent random trials $X_1, X_2, \ldots, X_m$, and let $R = R_1 + R_2 + \cdots + R_n$. Let β, D be positive numbers. Suppose that for each outcome $x = (x_1, x_2, \ldots, x_m)$ of trials, we can find subsets $W_1, W_2, \ldots, W_n$ of the set of trials (trials in W_i are called witnesses of R_i), such that*

(a) *For every i, if $R_i(x) = 1$, the outcome of the trials in W_i certify that $R_i(x) = 1$; i.e., for every other outcome $y = (y_1, y_2, \ldots, y_m)$ such that $y_j = x_j$ for every $X_j \in W_i$, we have $R_i(y) = 1$.*

(b) *Each trial is a witness of at most β random variables (i.e., for every trial X_j, $|\{i \mid X_j \in W_i\}| \leq \beta$).*

(c) $\sum_{i=1}^{n} |W_i| \leq D$,

Then for all t,

$$\mathbf{Pr}(|R - \mathbf{E}(R)| > t) \leq 4e^{-\frac{t^2}{70\beta D}}.$$

Proof: First, we prove the following claim:

Claim 3.1 *For every real μ, τ,*

$$\mathbf{Pr}(R \leq \mu - \tau) \times \mathbf{Pr}(R \geq \mu) \leq e^{-\frac{\tau^2}{4\beta D}}.$$

Proof: Let $A = \{y : R(y) \leq \mu - \tau)$ and $B = \{x : R(x) \geq \mu\}$. For every $x \in B$, we show that $x \notin A_\ell$, where $\ell = \frac{\tau}{\sqrt{\beta D}}$. Assume, to the contrary, that there is some x such that $x \in B$ and $x \in A_\ell$. By the definition of A_ℓ, for any real $a_1, a_2, \ldots, a_m$, there exists at least one element $y = (y_1, y_2, \ldots, y_m)$ of A such that $\sum_{x_j \neq y_j} a_j < \ell \left(\sum_{j=1}^{m} a_j^2\right)^{1/2}$. Consider the witnesses corresponding to the outcome x, and for every $j = 1, \ldots, m$, let a_j be $|\{i \mid X_j \in W_i\}|$. Since $x \in A_\ell$, there is $y \in A$ such that

$$\sum_{x_j \neq y_j} a_j < \ell \left(\sum_{j=1}^{m} a_j^2\right)^{1/2}.$$

Because of the conditions of the concentration inequality, for every j, $a_j \leq \beta$ and $\sum_{j=1}^{m} a_j = \sum_{i=1}^{n} |W_i| \leq D$. Thus,

$$\sum_{x_j \neq y_j} a_j < \ell \left(\beta \sum_{j=1}^{m} a_j\right)^{1/2} \leq \ell\sqrt{\beta D} = \tau.$$

Notice that (by the definition of a_j) there are at most $\sum_{x_j \neq y_j} a_j$ values of i for which the condition "$x_j = y_j$ for every $X_j \in W_i$" is violated. Therefore, there are at most $\sum_{x_j \neq y_j} a_j$ values of i such that $R_i(x) = 1$, but $R_i(y) = 0$. Thus,

$$R(y) \geq R(x) - \sum_{x_j \neq y_j} a_j > R(x) - \tau \geq \mu - \tau,$$

contradicting with $y \in A$. Therefore, $B \subseteq \overline{A_\ell}$. Thus, by Talagrand's inequality,

$$\mathbf{Pr}(R \le \mu - \tau)\mathbf{Pr}(R \ge \mu) = \mathbf{Pr}(A)\mathbf{Pr}(B) \le \mathbf{Pr}(A)\mathbf{Pr}(\overline{A_\ell}) \le e^{-\ell^2/4} = e^{-\frac{\tau^2}{4\beta D}}.$$

■

Now, let μ be the median of R (The median of a random variable R is a number $\mathbf{Med}(R)$ such that $\mathbf{Pr}(R \le \mathbf{Med}(R)) \ge \frac{1}{2}$ and $\mathbf{Pr}(R < \mathbf{Med}(R)) \le \frac{1}{2}$). Since $\mathbf{Pr}(R \ge \mathbf{Med}(R)) \ge \frac{1}{2}$, the above claim implies

$$\mathbf{Pr}(R \le \mathbf{Med}(R) - \tau) \le 2e^{-\frac{\tau^2}{4\beta D}}.$$

Similarly, taking $\mu = \mathbf{Med}(R) + \tau$ implies

$$\mathbf{Pr}(R \ge \mathbf{Med}(R) + \tau) \le 2e^{-\frac{\tau^2}{4\beta D}}.$$

Therefore, for every real τ,

$$\mathbf{Pr}(|R - \mathbf{Med}(R)| \ge \tau) \le 4e^{-\frac{\tau^2}{4\beta D}}. \tag{3.1}$$

This equation means that R is concentrated around its median. Now we prove that the median and the expectation of R are not far away. We use the fact that for every positive random variable Y, $\mathbf{E}(Y) = \int_0^\infty \mathbf{Pr}(Y \ge y)dy$.

$$\begin{aligned} |\mathbf{E}(R) - \mathbf{Med}(R)| &= |\mathbf{E}(R - \mathbf{Med}(R))| \\ &\le \mathbf{E}(|R - \mathbf{Med}(R)|) \\ &= \int_0^\infty \mathbf{Pr}(|R - \mathbf{Med}(R)| \ge \tau)d\tau \\ &\le \int_0^\infty 4e^{-\frac{\tau^2}{4\beta D}}d\tau = 4\sqrt{\pi\beta D}. \end{aligned}$$

Therefore, if $|R(x) - \mathbf{E}(R)| \ge t$, then $|R(x) - \mathbf{Med}(R)| \ge t - 4\sqrt{\pi\beta D}$. Now, notice that if $t^2 < 70(\ln 4)\beta D$, then $4e^{-\frac{t^2}{70\beta D}} > 1$, and therefore the concentration inequality obviously holds. Therefore, we may assume without loss of generality that $t \ge \sqrt{70(\ln 4)\beta D} > 9.5\sqrt{\beta D}$ and hence $4\sqrt{\pi\beta D} < \frac{3}{4}t$. Thus, $|R(x) - \mathbf{E}(R)| \ge t$ implies that $|R(x) - \mathbf{Med}(R)| \ge \frac{t}{4}$. Therefore, by using Inequality 3.1 for $\tau = \frac{t}{4}$ we obtain

$$\mathbf{Pr}(|R - \mathbf{E}(R)| \ge t) \le \mathbf{Pr}(|R - \mathbf{Med}(R)| \ge \frac{t}{4}) \le 4e^{-\frac{(\frac{t}{4})^2}{4\beta D}} < 4e^{-\frac{t^2}{70\beta D}}.$$

■

The following two examples illustrate the use of the above theorem. The first example is an easy concentration proof. The second example is a more complicated probabilistic argument which makes use of two powerful tools: Talagrand's inequality and the Local Lemma.

Example 1: Number of nodes with degree at least two

Let G be an r-regular graph with n nodes and m edges. We construct a random subgraph H of G by picking each edge of G with probability $\frac{1}{2}$. Let R be the number of vertices of degree at least two in H. R is a random variable and using linearity of expectations, it is easy to see that

$$\mathbf{E}(R) = (1 - \frac{r+1}{2^r})n.$$

We want to use Theorem 3.18 to prove that R is concentrated. Let $\{e_1, e_2, \ldots, e_m\}$ be the set of edges of G, and let X_i be the random trial corresponding to picking (or not picking) the edge e_i (i.e., X_i is either zero or one, each with probability $\frac{1}{2}$. If $X_i = 0$, then e_i is not in H, and if $X_i = 1$, then e_i is in H). We prove that the conditions of Theorem 3.18 are satisfied with $\beta = 2$ and $D = 2n$.

For every vertex v, let R_v be the binary random variable which is one if the degree of v is at least two, and is zero otherwise. It is clear that $R = \sum_{v \in V} R_v$. Let $x = (x_1, x_2, \ldots, x_m)$ be an outcome of the trials. For every vertex v, if $R_v(x) = 1$, there are two edges e_i and e_j incident to v that are in H (i.e., $x_i = x_j = 1$). Take X_i and X_j as the witnesses of R_v. It is clear that these witnesses certify that $R_v(x) = 1$. Also, since every edge has two endpoints, for every i, X_i can be the witness of at most two R_v's. Furthermore, $\sum_{v \in V} |W_v| = 2R(x) \leq 2n$. Therefore, the conditions of Theorem 3.18 hold with $\beta = 2, D = 2n$. By using this theorem with $t = \sqrt{n} \ln n$ we obtain

$$\mathbf{Pr}(|R - \mathbf{E}(R)| > \sqrt{n} \ln n) < 4e^{-\Omega(\ln^2 n)} < \frac{1}{n},$$

if n is large. Therefore, for large n, R is between $(1 - \frac{r+1}{2^r})n - \sqrt{n} \ln n$ and $(1 - \frac{r+1}{2^r})n + \sqrt{n} \ln n$ with high probability.

Example 2: Colouring triangle-free graphs

We prove the following theorem using a probabilistic colouring procedure that has been used in many other probabilistic graph colouring arguments. This procedure is called *the naive colouring procedure* in [36, 37].

Theorem 3.19 *For sufficiently large Δ, the chromatic number of every triangle-free Δ-regular graph is at most $(1 - \frac{1}{2e^6})\Delta$.*

Proof: Let $N = \lfloor \frac{\Delta}{2} \rfloor$. We assign each vertex a colour chosen uniformly at random from $\{1, 2, \ldots, N\}$. After assigning the colours, if two neighbouring vertices receive the same colour, we uncolour *both* of them. As a result, some vertices lose their colours and others retain their colours. We prove that with positive probability, every vertex has at least $\frac{\Delta}{2e^6} + 1$ repeated colours in its neighbourhood. Therefore, if $\lfloor (1 - \frac{1}{2e^6})\Delta \rfloor$ colours are available, we can colour the vertices that have lost their colours by a greedy algorithm.

Now, we prove that with positive probability, every vertex has at least $\frac{\Delta}{2e^6} + 1$ repeated colours in its neighbourhood. Let T_v be the number of colours which are retained by *exactly* two neighbours of v. We use the following two lemmas (proofs of these lemmas are mentioned after the proof of the theorem).

Lemma 3.1 *For every vertex v, $\mathbf{E}(T_v) \geq \frac{2\Delta}{3e^6}$.*

Lemma 3.2 *For every vertex v, $\mathbf{Pr}(|T_v - \mathbf{E}(T_v)| > \sqrt{\Delta} \ln \Delta) < \frac{1}{4\Delta^4}$.*

These lemmas imply that for every vertex v, the probability that $T_v < \mathbf{E}(T_v) - \sqrt{\Delta} \ln \Delta$ is less than $\frac{1}{4\Delta^4}$. Therefore, since $\frac{2\Delta}{3e^6} - \sqrt{\Delta} \ln \Delta > \frac{\Delta}{2e^6} + 1$ for large Δ, for every vertex v, the probability that $T_v < \frac{\Delta}{2e^6} + 1$ is less than $\frac{1}{4\Delta^4}$. Let A_v be the event that $T_v < \frac{\Delta}{2e^6} + 1$. By the mutual independence principle, every event A_v is mutually independent of $\{A_u \mid u \text{ and } v \text{ are of distance more than } 4\}$. Therefore, each event is mutually independent of all but at most Δ^4 other events. Thus, we can use the Local Lemma with $d = \Delta^4$ and $p = \frac{1}{4\Delta^4}$, and the Local Lemma implies that with positive probability, $T_v \geq \frac{\Delta}{2e^6} + 1$ for *every* vertex v. Hence, there are at least $\frac{\Delta}{2e^6} + 1$ repeated colours in the neighbourhood of every vertex, and we can use the greedy algorithm to finish the colouring that is obtained from the probabilistic procedure. ∎

Now, we prove Lemmas 3.1 and 3.2.

Proof of Lemma 3.1: For every two neighbours u and w of v and colour c, the probability that both u and w retain the colour c, and no other vertex in the neighbourhood of v retains c is at least

$$\left(\frac{1}{N}\right)^2 \times \left(1 - \frac{1}{N}\right)^{3\Delta - 1} > \frac{4}{\Delta^2} \times \frac{2}{3e^6}.$$

Therefore, since there are $\binom{\Delta}{2}$ choices for u and w, and N choices for c, by the linearity of expectations, we have

$$\mathbf{E}(T_v) \geq \binom{\Delta}{2} \times \frac{\Delta}{2} \times \frac{4}{\Delta^2} \times \frac{2}{3e^6} = \frac{2\Delta}{3e^6}.$$ ∎

Proof of Lemma 3.2: We use Talagrand's inequality (Theorem 3.18) to prove this lemma. If you try to use Talagrand's inequality on the random variable T_v, you will soon realize that Theorem 3.18 does not yield the desired concentration for this variable. But we can define two random variables R and R' in such a way that $T_v = R - R'$, and we can prove the concentration of R and R' by Theorem 3.18.

Let R be the number of colours assigned to at least two neighbours of v, and R' be the number of colours that are either assigned to more than two neighbours of v, or are assigned to exactly two neighbours of v, but are not retained by both of them. Clearly, $T_v = R - R'$. We prove the concentration of R and R' separately.

First, we prove that R is concentrated. For every colour $c \in \{1, 2, \ldots, N\}$, let R_c be the binary random variable that is one iff the colour c is assigned to at least two neighbours of v. For every outcome x and every colour c, if $R_c(x) = 1$, we select the colour assignment trial corresponding to two neighbours of v which are assigned c as the witnesses of R_c. It is easy to see that the conditions of Theorem 3.18 hold with $\beta = 1, D = \Delta$. Therefore, by taking $t = \frac{1}{2}\sqrt{\Delta}\ln\Delta$ we obtain

$$\mathbf{Pr}(|R - \mathbf{E}(R)| > \frac{1}{2}\sqrt{\Delta}\ln\Delta) < 4e^{-\Omega(\ln^2\Delta)} < \frac{1}{8\Delta^4} \tag{3.2}$$

for large Δ.

Now, we prove that R' is concentrated. For every colour c, let R'_c be the binary random variable that is one iff either the colour c is assigned to at least three neighbours of v, or c is assigned to two neighbours of v, but is not retained by both of them. For every outcome x and every colour c, if $R'_c(x) = 1$, either we can select trials corresponding to three neighbours of v that are assigned c, or the trials corresponding to two neighbours of v that are assigned c, and a neighbour of one of them which is also assigned c, as the witnesses of R'_c. It is easy to see that the conditions of Theorem 3.18 hold with $\beta = 1, D = \frac{3}{2}\Delta$. Therefore, by taking $t = \frac{1}{2}\sqrt{\Delta}\ln\Delta$ we obtain

$$\mathbf{Pr}(|R' - \mathbf{E}(R')| > \frac{1}{2}\sqrt{\Delta}\ln\Delta) < 4e^{-\Omega(\ln^2\Delta)} < \frac{1}{8\Delta^4} \tag{3.3}$$

for large Δ. Inequalities 3.2 and 3.3 imply the lemma. ∎

Remark 1. There is a much simpler non-probabilistic proof for the above result. The only reason that we used the above proof is to illustrate the technique.

Remark 2. It is easy to generalize the above theorem to the case where G is sparse rather than triangle-free, i.e., there are a few edges in the neighbourhood of every vertex of G. Molloy and Reed [35] used this fact (by proving that the square of the line graph of every graph is sparse) to prove Theorem 2.1.

Remark 3. It turns out that the upper bound given in the above theorem for the chromatic number of triangle-free graphs is not best possible. Kim [28] proved that the slightly stronger assumption that G neither has a triangle, nor a cycle of length four (i.e., the girth of G is at least five), implies that $\chi(G) \leq (1 + o(1))\frac{\Delta}{\ln \Delta}$. Later, Johansson [24] proved that every triangle-free graph has chromatic number at most $O(\frac{\Delta}{\ln \Delta})$. In the next chapter, we will use the technique of Kim's proof to prove an upper bound for the strong chromatic index of C_4-free graphs.

3.5 The semirandom method

In the proof of Theorem 3.19, we used a colouring procedure to colour some vertices of the graph, and then coloured the remaining vertices using the greedy algorithm. One idea to improve this method is to apply another iteration of the probabilistic colouring procedure instead of the greedy algorithm. The more general idea is to apply several iterations (instead of a single iteration) of the random colouring procedure, and after each iteration prove that a certain intermediary property holds with positive probability. After a certain number of iterations, we stop and prove (using the intermediary properties) that we can complete the colouring using another method (such as the greedy method, or the method that is used in the next chapter).

The main step in solving a problem using this method is to define and prove the intermediary properties. Typically, we keep track of several parameters (such as the minimum number of colours available at each vertex, or the maximum number of uncoloured neighbours of each vertex), and the intermediary properties are lower or upper bound on these parameters. The proof of the intermediary properties are usually by induction. In each step, we compute the expected value of the parameters, and also prove that they are concentrated (similar to what we did in the proof of Theorem 3.19 for the parameter T_v). Then, usually the Local Lemma is used to prove that the intermediary property holds with positive probability.

The reason that we call this method the *semirandom* method is that in this method, we guide the random procedure by intermediary properties, and therefore the method is not quite random. This method is often referred to as the Rödl Nibble, because Rödl used this method to prove a conjecture of Erdős and Hanani (see [2]).

The proof of Theorem 4.1 in the next chapter is an example of this method.

Chapter 4

Strong chromatic index of C_4-free graphs

In this chapter, we use the probabilistic method to study the strong chromatic index of C_4-free graphs. A graph is called C_4-free if it does not have any (partial) subgraph isomorphic to C_4, i.e., does not have a cycle of length four. It turns out that C_4-free graphs have interesting properties with respect to strong edge colouring. In this chapter, we prove an upper bound for the strong chromatic index of these graphs. Since the upper bound is asymptotically less than Δ^2, it follows that Conjectures 2.1 and 2.2 are true for C_4-free graphs. Also, we show that our bound is asymptotically best possible, up to a constant multiple.

The following two theorems are our main results. We prove Theorem 4.1 in Section 4.1 and Theorem 4.2 in Section 4.4.

Theorem 4.1 *For every C_4-free graph G,*

$$s\chi'(G) \leq (2 + o(1))\frac{\Delta^2}{\ln \Delta}.$$

Theorem 4.2 *For every $g \geq 3$ and sufficiently large d there is a d-regular graph of girth at least g with strong chromatic index at least*

$$\left(\frac{1}{2} - o(1)\right)\frac{d^2}{\ln d}.$$

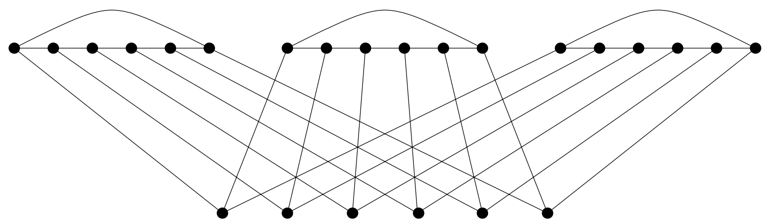

Figure 4.1: A 3-regular graph of girth at least five

4.1 Upper bound

In this section we prove that the strong chromatic index of every C_4-free graph is at most $(2+o(1))\frac{\Delta^2}{\ln \Delta}$. The following theorem implies that we can assume, without loss of generality, that the graph is regular.

Theorem 4.3 *For every C_4-free graph G of maximum degree Δ, there is a Δ-regular C_4-free graph H which contains G as a subgraph.*

Before proving Theorem 4.3, we prove the following lemma.

Lemma 4.1 *For every k, there is a k-regular graph G_k of girth at least five.*

Proof: It is easy to prove this lemma probabilistically (See Theorem 4.8), but here we present a constructive proof. We construct G_k recursively. G_1 is a single edge. Now, we construct G_k from G_{k-1}. Let n_k denote the number of vertices of G_k. For constructing G_k, we put k copies of G_{k-1} and add n_{k-1} new vertices. Each of the new vertices corresponds to a vertex in G_{k-1}. We connect every new vertex to its corresponding vertices in all copies of G_{k-1}. Let G_k be the resulting graph. For example, G_2 is a 6-cycle, and G_3 is the graph shown in Figure 4.1. It is easy to see that if G_{k-1} is a $(k-1)$-regular graph of girth at least five, G_k would be a k-regular graph of girth at least five.

An alternative way to prove this lemma is to consider the subgraph of $(2k-1)$-dimensional cube Q_{2k-1} induced by those vertices whose corresponding binary sequences are *balanced*, i.e., the number of zeros and ones differ by 1. It is not difficult to see that this subgraph is k-regular and its girth is six. ■

Proof of Theorem 4.3: For every vertex v of G, let the *deficiency* of v be $\Delta - \deg(v)$. We present a procedure to decrease the total deficiency of the graph, without creating any 4-cycle. It is clear that if we apply this procedure several times, eventually we will get a Δ-regular C_4-free supergraph of G.

If the total deficiency of G is one, it means that there is only one vertex v of degree $\Delta - 1$ in G, and the degree of every other vertex is Δ. We put two copies of G together and connect the vertices corresponding to v to each other. If the total deficiency is more than one, select two vertices x and y each having deficiency at least one (If there is only one vertex of deficiency at least two in the graph, let x and y be that vertex). Add a copy of the graph G_Δ constructed in Lemma 4.1 and remove one of its edges uv, and instead, add two edges xu and yv. It is easy to verify that this procedure decreases the deficiency of the graph by two. ■

The above theorem together with the equivalence of the strong edge-colouring and the vertex-colouring of the square of the line graph, implies that Theorem 4.1 is equivalent to the following which will be proved in this section.

Theorem 4.4 *For any $\epsilon > 0$, there is a Δ_0 such that for any graph G of maximum degree $\Delta > \Delta_0$ which is the square of the line graph of a regular C_4-free graph,*

$$\chi(G) \leq (2 + \epsilon)\frac{\Delta}{\ln \Delta}.$$

In fact, we will prove the more general result that under the conditions of the above theorem, the *list* chromatic number of G is at most $(2 + \epsilon)\frac{\Delta}{\ln \Delta}$.

The proof of this theorem is probabilistic and is similar to Kim's proof of the fact that the chromatic number of a graph of girth at least five is at most $(1 + o(1))\frac{\Delta}{\ln \Delta}$ (See [28]), as formulated in [37]. We use Kim's semirandom colouring procedure with a different analysis.

The proof of Theorem 4.4 is organized as follows. In Section 4.1.1 we prove two facts about the square of the line graph of C_4-free graphs. Section 4.1.2 describes the semirandom colouring procedure that is used in the proof, and the parameters that we keep track of. In Section 4.1.3 the sketch of the proof is presented. The five main lemmas are stated and Theorem 4.4 is proved using them. The proofs of two lemmas regarding the expected values and concentration are postponed to Section 4.1.4 and 4.1.5, respectively, and other proofs are presented in Section 4.1.3.

4.1.1 Properties of the graph

In this section, we prove two facts about the properties of the square of the line graph of C_4-free graphs. These facts will be used in Sections 4.1.4 and 4.1.5. In the following, for every vertex v, N_v denotes the set of neighbours of v.

Fact 1 *If G is the square of the line graph of a regular graph H, then for every vertex v in G, the size of any independent set in the subgraph induced by N_v is at most $O(\sqrt{\Delta})$, where Δ is the maximum degree in G.*

Proof: The vertex v in G corresponds to an edge xy in H. Let A be the set of vertices in H which are adjacent to x or y. It is clear that $|A| < 2\Delta(H) = O(\sqrt{\Delta})$, and every neighbour of v in G corresponds to one of the edges incident to an element of A. Furthermore, any two neighbours of v whose corresponding edges in H are incident to the same vertex of A are adjacent in G and therefore cannot be in an independent set together. Thus, the size of any independent set in $G[N_v]$ is less than or equal to $|A| = O(\sqrt{\Delta})$. ■

Now we prove the following lemma which implies the second fact.

Lemma 4.2 *Let H be a C_4-free graph, and e be any edge of H. For every edge e' of H which is not adjacent to e, there are at most 10Δ edges which are of distance at most two from both e and e'.*

Proof: Let u and v be the endpoints of e. We call the vertices in $N_u \cup N_v$ *heavy vertices*, and other vertices *light vertices*. We prove the following two claims:

Claim 4.1 *Every vertex other than u and v is adjacent to at most two heavy vertices.*

Proof: Assume, to the contrary, that there is a vertex $x \neq u, v$ which is adjacent to three heavy vertices w_1, w_2, w_3. Each heavy vertex is adjacent to either u or v. Therefore, two of the heavy vertices w_1, w_2, w_3 are adjacent to the same vertex. Assume that w_1 and w_2 are both adjacent to u. Now it is clear that uw_1xw_2 is a C_4 in H, contradicting our hypothesis. ■

Corollary 4.1 *There are at most four heavy vertices which are adjacent to an endpoint of e'.*

Claim 4.2 *Every light vertex is the endpoint of at most two edges of distance at most two from e.*

Proof: Let x be a light vertex that is an endpoint of three edges xw_1, xw_2, and xw_3 of distance at most two from e. Since x is not adjacent to u and v, and any edge of distance at most two from e must have at least one endpoint adjacent to either u or v, this implies that w_1, w_2, w_3 are adjacent to either u, or v, and in other words, they are heavy vertices. This contradicts Claim 4.1. ■

Claim 4.2 and Corollary 4.1 imply that if e' is not adjacent to e, its endpoints are adjacent to at most four heavy vertices and 2Δ light vertices. Each heavy vertex is an endpoint of at most Δ edges of distance at most two from e, and each light vertex is an endpoint of at most two edges of distance at most two from e. Also, there are at most 2Δ edges adjacent to e' which are of distance at most two from e. Therefore, the number of edges which are of distance at most two from both e and e' is at most $4 \times \Delta + 2\Delta \times 2 + 2\Delta = 10\Delta$. ■

Now, we can state the second fact.

Fact 2 *Let G be the square of the line graph of a regular C_4-free graph H, and v be a vertex of G. Then every vertex u of G, except at most $O(\sqrt{\Delta})$ neighbours of v, has at most $O(\sqrt{\Delta})$ common neighbours with v, where Δ is the maximum degree in G.*

Proof: Since G is the square of the line graph of H, and H is a regular graph, we have $\Delta(G) = \Theta(\Delta(H)^2)$. Now, it is clear that Lemma 4.2 implies this fact. ■

In subsequent sections, we will prove that Theorem 4.4 holds for every graph satisfying Fact 1 and Fact 2.

After proving Theorem 4.1, I came across the following recent theorem of N. Alon, M. Krivelevich, and B. Sudakov [1].

Theorem 4.5 *There exists a positive constant c such that the following holds. Let G be a graph with maximum degree Δ in which for every vertex v, there are at most Δ^2/f edges in the subgraph induced by N_v. Then the chromatic number of G is at most $c\frac{\Delta}{\log f}$.*

The proof of the above theorem makes use of Johansson's theorem [24]. Theorem 4.5 together with Fact 2 implies the following corollary.

Corollary 4.2 *The strong chromatic index of every C_4-free graph is at most a constant multiple of $\frac{\Delta^2}{\ln \Delta}$.*

Proof: We prove that for every vertex v, the number of edges in $G[N_v]$ is at most $O(\Delta\sqrt{\Delta})$. By Fact 2 the degree of every vertex in $G[N_v]$, except at most $O(\sqrt{\Delta})$ vertices is at most $O(\sqrt{\Delta})$. This implies that the number of edges in $G[N_v]$ is at most $O(\Delta\sqrt{\Delta})$. Thus, we can use Theorem 4.5 with $f = O(\sqrt{\Delta})$. ∎

However, the constant multiple in the above corollary is much more than 2, and also Theorem 4.5, and therefore Corollary 4.2 do not generalize to list colouring.

4.1.2 Random procedure

In this section, we describe the colouring procedure that we use to colour the vertices of any graph satisfying Facts 1 and 2 using $(2+\epsilon)\frac{\Delta}{\ln \Delta}$ colours. We apply several iterations of the wasteful colouring procedure described below. During the iterations, for each vertex v, we maintain a list L_v of available colours. Initially, $L_v = \{1, 2, \ldots, \lfloor (2+\epsilon)\frac{\Delta}{\ln \Delta} \rfloor\}$ for any vertex v (or the list of all colours acceptable for v, if we want to prove Theorem 4.4 for list chromatic number).

Wasteful Colouring Procedure (ith iteration)

1. For each uncoloured vertex v, activate v with probability $\frac{K}{\ln \Delta}$ (K is a small positive constant to be fixed later).

2. For each activated vertex v, assign to v a colour chosen uniformly at random from L_v.

3. For each activated vertex v, remove the colour assigned to v from the list of every neighbour of v.

4. Uncolour every vertex which receives the same colour as a neighbour.

5. Equalizing coin flip: For each vertex v and colour $c \in L_v$, remove c from L_v with probability $\mathrm{Eq}_i(v, c)$ (which is defined below).

During the execution of the above procedure we keep track of two parameters for each vertex: the size of the list and the number of neighbours of the vertex which have a certain colour in their list. Let $\ell_i(v)$ be the size of L_v at the beginning of iteration i. Also, let $N_i(v, c)$ denote the set of uncoloured neighbours u of v with $c \in L_u$ at the beginning of iteration i, and let $t_i(v, c)$ be the size of $N_i(v, c)$. Also, let $t'_i(v, c)$ be the number of neighbours u of v such that at the beginning of iteration i, u is still uncoloured, c has never been assigned to any neighbour of u outside $N_v \cup \{v\}$, and u has not lost c during equalizing coin flips. This parameter will be used instead of $t_i(v,c)$ in the computation of expectation and the proof of concentration, since $t_i(v,c)$ is not concentrated, but $t'_i(v,c)$ is.

We estimate the values of $\ell_i(v)$ and $t_i(v,c)$ by L_i and T_i, respectively. L_i, T_i, and Keep_i are defined using the following recurrences. Here $\gamma = (1+\frac{\epsilon}{20})/(1+\frac{\epsilon}{10})$ is a constant between zero and one.

- $L_1 = (2+\epsilon)\dfrac{\Delta}{\ln \Delta}, T_1 = \Delta$
- $\text{Keep}_i = \left(1 - \dfrac{K}{\ln \Delta} \times \dfrac{1}{L_i}\right)^{T_i}$
- $L_{i+1} = L_i \times \text{Keep}_i - L_i^{\gamma}$
- $T_{i+1} = T_i \left(1 - \dfrac{K}{\ln \Delta} \times \text{Keep}_i\right) \times \text{Keep}_i + T_i^{\gamma}$

We will prove by induction that with positive probability the following property holds for every i in some range which will be specified later.

Property P(i): For each uncoloured vertex v, and each colour $c \in L_v$ at the beginning of iteration i,

$$\ell_i(v) \geq L_i$$

$$t_i(v,c) \leq T_i.$$

Now, we specify the value of $\text{Eq}_i(v,c)$. Assume that $P(i)$ holds. If at the beginning of iteration i, $c \in L_v$, the probability that c remains in L_v after step 3 of iteration i, i.e., the probability that no neighbour of v is assigned c in iteration i is equal to

$$\text{Keep}_i(v,c) = \prod_{u \in N_i(v,c)} \left(1 - \frac{K}{\ln \Delta} \times \frac{1}{\ell_i(u)}\right) \geq \left(1 - \frac{K}{\ln \Delta} \times \frac{1}{L_i}\right)^{T_i} = \text{Keep}_i.$$

We define

$$\text{Eq}_i(v,c) = 1 - \frac{\text{Keep}_i}{\text{Keep}_i(v,c)}.$$

This definition ensures that for each vertex v and each colour $c \in L_v$, the probability that c remains in L_v in the ith iteration is precisely Keep_i.

4.1.3 Sketch of the proof

In this section, we sketch the proof of the fact that the wasteful colouring procedure described above actually yields a colouring of the graph. We state five lemmas and present the proofs of three of them, together with the proof of Theorem 4.4 in this section. The proof of the other two lemmas will be presented in the next two sections.

Assume that we have already carried out $i-1$ iterations of the wasteful colouring procedure and the property $P(i)$ holds. We prove that after iteration i, the property $P(i+1)$ holds with positive probability. Notice that iteration i is in fact a random experiment and $\ell_{i+1}(v)$ and $t'_{i+1}(v,c)$ are random variables. The following lemma gives the expected values of $\ell_{i+1}(v)$ and $t'_{i+1}(v,c)$. We postpone the proof of this lemma until Section 4.1.4.

Lemma 4.3 *If $P(i)$ holds and $L_i \geq \sqrt{\Delta}$, then for every uncoloured vertex v and colour $c \in L_v$*

(a) $\mathbf{E}(\ell_{i+1}(v)) = \ell_i(v) \times \mathrm{Keep}_i$,

(b) $\mathbf{E}(t'_{i+1}(v,c)) \leq T_i\left(1 - \frac{K}{\ln\Delta} \times \mathrm{Keep}_i\right) \times \mathrm{Keep}_i + O(\frac{\sqrt{\Delta}}{\ln\Delta} \times \frac{T_i}{L_i} + \sqrt{\Delta})$.

The following lemma proves that the random variables $\ell_{i+1}(v)$ and $t'_{i+1}(v,c)$ are concentrated around their expected values. We will prove this lemma in Section 4.1.5.

Lemma 4.4 *If $P(i)$ holds, and $T_i, L_i \geq \Delta^{\frac{1}{2}+\frac{\epsilon}{20}} \ln^3 \Delta$, then for any uncoloured vertex v and colour $c \in L_v$,*

(a) $\mathbf{Pr}(|\ell_{i+1}(v) - \mathbf{E}(\ell_{i+1}(v))| > L_i^{\gamma}) < \Delta^{-\ln\Delta}$,

(b) $\mathbf{Pr}(|t'_{i+1}(v,c) - \mathbf{E}(t'_{i+1}(v,c))| > \frac{1}{2}T_i^{\gamma}) < \Delta^{-\ln\Delta}$.

We prove the following lemma using the above lemmas and the Lovasz Local Lemma.

Lemma 4.5 *With positive probability, $P(i)$ holds for every i such that for all $1 \leq j \leq i$, $L_j, T_j \geq \Delta^{\frac{1}{2}+\frac{\epsilon}{20}} \ln^3 \Delta$.*

Proof: We use induction on i. The base of induction is trivial. Suppose that $P(i)$ holds. Let A_v be the event that $\ell_{i+1}(v) < L_{i+1}$ and $B_{v,c}$ be the event that $t'_{i+1}(v,c) > T_{i+1}$. By Lemma 4.3 and Lemma 4.4, the probability of A_v is less than $\Delta^{-\ln\Delta}$. Also, it is not difficult to see that $T_i/L_i = O(\ln\Delta)$, and the inequality $T_i \geq \Delta^{\frac{1}{2}+\frac{\epsilon}{20}} \ln^3\Delta$ implies that $T_i^\gamma > \Delta^{\frac{1}{2}+\frac{\epsilon}{40}} > \sqrt{\Delta}$. Thus, $\frac{\sqrt{\Delta}}{\ln\Delta} \times \frac{T_i}{L_i} + \sqrt{\Delta} = O(\sqrt{\Delta}) = o(T_i^\gamma)$. This allows us to replace the $O(\frac{\sqrt{\Delta}}{\ln\Delta} \times \frac{T_i}{L_i} + \sqrt{\Delta})$ term in the expectation result with $\frac{1}{2}T_i^\gamma$. Therefore, the probability of $B_{v,c}$ is less than $\Delta^{-\ln\Delta}$. On the other hand, by the mutual independence principle, it is easy to see that each of the events $\{A_v \mid v \in V(G)\} \cup \{B_{v,c} \mid v \in V(G), c \in L_v\}$ is mutually independent of all but a set of at most $\Delta^{O(1)}$ other events. Therefore, by the Local Lemma, with positive probability, for every uncoloured vertex v and colour $c \in L_v$, at the beginning of iteration $i+1$, $\ell_{i+1}(v) \geq L_{i+1}$ and $t'_{i+1}(v,c) \leq T_{i+1}$. Now, notice that if $c \in L_v$ at the beginning of iteration $i+1$, then $t_{i+1}(v,c) = t'_{i+1}(v,c)$. This observation completes the proof of the lemma. ■

The following lemma proves that after several iterations of the wasteful colouring procedure, we will reach a situation in which $\ell_i(v)$ is much more than $t_i(v,c)$. This will enable us to colour the remaining vertices by another method.

Lemma 4.6 *There is some i^* such that $\frac{T_{i^*}}{L_{i^*}} < \frac{1}{8}$ and $T_j, L_j > \Delta^{\frac{1}{2}+\frac{\epsilon}{20}} \ln^3\Delta$ for all $j \leq i^*$.*

Proof: Let $r_i := T_i/L_i$. First, we prove the following claim by induction on i.

Claim 4.3 *For every i such that $i \leq \ln^3\Delta$ and $r_j \geq \frac{1}{8}$ for any $j < i$,*

- $r_i < \dfrac{\ln\Delta}{2+\epsilon}\left(1 - \dfrac{2+\epsilon/3}{2+\epsilon/2} \times \dfrac{K}{\ln\Delta}\exp(-\dfrac{K}{2+\epsilon/2})\right)^{i-1}$,
- $L_i > \Delta^{\frac{1}{2}+\frac{\epsilon}{15}}$.

Proof: It is clear that $r_1 = \frac{\ln\Delta}{2+\epsilon}$ and $L_1 = (2+\epsilon)\frac{\Delta}{\ln\Delta} > \Delta^{\frac{1}{2}+\frac{\epsilon}{15}}$. Therefore the statement is true for $i = 1$. Now, we assume that the claim holds for i, and prove it for $i+1$. First, we estimate the value of Keep_i. Note that

$$\text{Keep}_i = (1 - \frac{K}{\ln\Delta} \times \frac{1}{L_i})^{T_i} > \left(e^{-\frac{2+\epsilon}{2+\epsilon/2} \times \frac{K}{\ln\Delta} \times \frac{1}{L_i}}\right)^{T_i} = \exp(-\frac{2+\epsilon}{2+\epsilon/2} \times \frac{K}{\ln\Delta} r_i).$$

Therefore, by the induction hypothesis

$$\text{Keep}_i > \exp\left(-\frac{K}{2+\epsilon/2}\left(1 - \frac{2+\epsilon/3}{2+\epsilon/2} \times \frac{K}{\ln\Delta}\exp(-\frac{K}{2+\epsilon/2})\right)^{i-1}\right). \quad (4.1)$$

In particular, Inequality 4.1 implies that

$$\text{Keep}_i > \exp(-\frac{K}{2+\epsilon/2}). \quad (4.2)$$

Now, we estimate the value of r_{i+1}:

$$\begin{aligned} r_{i+1} &= \frac{T_i\left(1-\frac{K}{\ln\Delta}\mathrm{Keep}_i\right)\times\mathrm{Keep}_i+T_i^\gamma}{L_i\mathrm{Keep}_i-L_i^\gamma} \\ &= \frac{r_i\left(1-\frac{K}{\ln\Delta}\mathrm{Keep}_i\right)+\frac{T_i^\gamma}{L_i\mathrm{Keep}_i}}{1-\frac{1}{L_i^{1-\gamma}\mathrm{Keep}_i}}. \end{aligned}$$

Since $L_i > \Delta^{\frac{1}{2}+\frac{\epsilon}{15}}$, $\frac{1}{L_i^{1-\gamma}\mathrm{Keep}_i} = o(1)$. Therefore the inequality $\frac{1}{1-x} < 1+2x$ which holds for $x < \frac{1}{2}$ implies that

$$\begin{aligned} r_{i+1} &< \left(r_i\left(1-\frac{K}{\ln\Delta}\mathrm{Keep}_i\right)+\frac{r_i^\gamma}{L_i^{1-\gamma}\mathrm{Keep}_i}\right)\times \\ &\quad \left(1+\frac{2}{L_i^{1-\gamma}\mathrm{Keep}_i}\right) \\ &< r_i\left(1-\frac{K}{\ln\Delta}\mathrm{Keep}_i\right)+\frac{r_i^\gamma}{L_i^{1-\gamma}\mathrm{Keep}_i} \\ &\quad +\frac{2r_i}{L_i^{1-\gamma}\mathrm{Keep}_i}+\frac{2r_i^\gamma}{L_i^{2(1-\gamma)}\mathrm{Keep}_i^2}. \end{aligned}$$

Using Inequality 4.2, the assumption $r_i \geq \frac{1}{8}$, and the induction hypothesis $L_i \geq \Delta^{\frac{1}{2}+\frac{\epsilon}{15}}$ we have:

$$\begin{aligned} r_{i+1} &< r_i\left(1-\frac{K}{\ln\Delta}\mathrm{Keep}_i\right)+\frac{O(1)r_i}{\Delta^{(\frac{1}{2}+\frac{\epsilon}{15})(1-\gamma)}} \\ &< r_i\left(1-\frac{2+\epsilon/3}{2+\epsilon/2}\times\frac{K}{\ln\Delta}\mathrm{Keep}_i\right), \end{aligned}$$

which together with the induction hypothesis and Inequality 4.2 implies that

$$r_{i+1} < \frac{\ln\Delta}{2+\epsilon}\left(1-\frac{2+\epsilon/3}{2+\epsilon/2}\times\frac{K}{\ln\Delta}\exp\left(-\frac{K}{2+\epsilon/2}\right)\right)^i.$$

By expanding the recurrence $L_{i+1} = L_i\mathrm{Keep}_i - L_i^\gamma$ we obtain

$$L_{i+1} = L_1\prod_{j=1}^{i}\mathrm{Keep}_j - L_1^\gamma\prod_{j=2}^{i}\mathrm{Keep}_j - L_2^\gamma\prod_{j=3}^{i}\mathrm{Keep}_j - \cdots - L_i^\gamma. \tag{4.3}$$

Let $L'_{i+1} := L_1\prod_{j=1}^{i}\mathrm{Keep}_j$. We find an upper bound for the kth negative term of the right-hand side of Equation 4.3.

$$\begin{aligned} L_k^\gamma\prod_{j=k+1}^{i}\mathrm{Keep}_j &< L_1^\gamma\prod_{j=1}^{k-1}\mathrm{Keep}_j^\gamma\prod_{j=k+1}^{i}\mathrm{Keep}_j \\ &< 2L_1^\gamma\prod_{j=1}^{i}\mathrm{Keep}_j^\gamma = 2{L'}_{i+1}^{\gamma}. \end{aligned}$$

Therefore,

$$L_{i+1} > L'_{i+1}(1 - \frac{2i}{L'^{1-\gamma}_{i+1}}) > L'_{i+1}(1 - \frac{2\ln^3\Delta}{L'^{1-\gamma}_{i+1}}). \tag{4.4}$$

Now, we use Inequality 4.1 to find a lower bound for L'_{i+1}.

$$\begin{aligned} L'_{i+1} &> (2+\epsilon)\frac{\Delta}{\ln\Delta}\exp\Bigg(-\frac{K}{2+\epsilon/2}\times \\ &\quad \sum_{j\geq 1}\left(1-\frac{2+\epsilon/3}{2+\epsilon/2}\times\frac{K}{\ln\Delta}\exp(-\frac{K}{2+\epsilon/2})\right)^{j-1}\Bigg) \\ &= (2+\epsilon)\frac{\Delta}{\ln\Delta}\exp\left(-\frac{\ln\Delta}{2+\epsilon/2}\times\frac{2+\epsilon/2}{2+\epsilon/3}\times\exp(\frac{K}{2+\epsilon/2})\right) \\ &= (2+\epsilon)\frac{\Delta}{\ln\Delta}\Delta^{-\frac{1}{2+\epsilon/3}e^{\frac{K}{2+\epsilon/2}}} \\ &= (2+\epsilon)\frac{1}{\ln\Delta}\Delta^{1-\frac{1}{2+\epsilon/3}e^{\frac{K}{2+\epsilon/2}}}. \end{aligned}$$

Now, we take $K < (2+\frac{\epsilon}{2})\ln(1+\frac{\epsilon}{42}-\frac{\epsilon^2}{42})$. This is a valid assumption, because for ϵ small, $1+\frac{\epsilon}{42}-\frac{\epsilon^2}{42}$ is more than one. Therefore, the exponent of Δ in the above equation is equal to

$$\begin{aligned} 1-\frac{1}{2+\epsilon/3}e^{\frac{K}{2+\epsilon/2}} &> 1-\frac{1}{2+\epsilon/3}e^{\ln(1+\frac{\epsilon}{42}-\frac{\epsilon^2}{42})} \\ &= 1-\frac{1+\frac{\epsilon}{42}-\frac{\epsilon^2}{42}}{2+\epsilon/3} \\ &= \frac{1}{2}+\frac{\epsilon}{14}. \end{aligned}$$

Thus

$$L'_{i+1} > 2\Delta^{\frac{1}{2}+\frac{\epsilon}{15}}. \tag{4.5}$$

The above equation together with Equation 4.4 implies that $L_{i+1} > \Delta^{\frac{1}{2}+\frac{\epsilon}{15}}$, as desired. ■

Let i^* be the smallest integer such that $r_{i^*} < \frac{1}{8}$ (or infinity if there is not such an integer). We prove that i^* satisfies the conditions of the lemma. First, we prove that $i^* \leq \ln^3\Delta$. Assume, by way of contradiction, that $i^* > \ln^3\Delta$. We use Claim 4.3 for $i = \lfloor \ln^3\Delta \rfloor$.

$$\begin{aligned} r_i &< \ln\Delta\left(1-\frac{2+\epsilon/3}{2+\epsilon/2}\times\frac{K}{\ln\Delta}\exp(-\frac{K}{2+\epsilon/2})\right)^{\ln^3\Delta} \\ &< \ln\Delta\times\exp\left(-\frac{2+\epsilon/3}{2+\epsilon/2}\times\frac{K}{\ln\Delta}\exp(-\frac{K}{2+\epsilon/2})\times\ln^3\Delta\right) \\ &= \ln\Delta\times e^{-\Omega(\ln^2\Delta)} < \frac{1}{8}. \end{aligned}$$

This contradicts with the choice of i^*. This contradiction shows that $i^* \leq \ln^3 \Delta$ and therefore, Claim 4.3 implies that $L_j > \Delta^{\frac{1}{2}+\frac{\epsilon}{15}} > \Delta^{\frac{1}{2}+\frac{\epsilon}{20}} \ln^3 \Delta$ for any $j \leq i^*$. Also, since $r_{i^*-1} \geq \frac{1}{8}$, we have

$$r_{i^*} > r_{i^*-1}\left(1 - \frac{K}{\ln \Delta}\mathrm{Keep}_i\right) > \frac{1}{16}.$$

Therefore, for any $j \leq i^*$, $T_j = L_j r_j > \frac{1}{16}\Delta^{\frac{1}{2}+\frac{\epsilon}{15}} > \Delta^{\frac{1}{2}+\frac{\epsilon}{20}} \ln^3 \Delta$. ■

Now we are in a position to prove Theorem 4.4.

Proof of Theorem 4.4: We carry out the wasteful colouring procedure for up to i^* iterations, where i^* is taken from Lemma 4.6. By Lemma 4.6 and Lemma 4.5, with positive probability, at the end of the i^*th iteration the size of each list is at least L_{i^*}, each colour $c \in L_v$ appears in the lists of at most T_{i^*} neighbours of v, and $\frac{T_{i^*}}{L_{i^*}} < \frac{1}{8}$. Therefore, we can use Theorem 3.13 to colour the uncoloured vertices. ■

4.1.4 The computation of the expected values

In this section we prove Lemma 4.3. Fact 2 will be used in the proof.

Proof of Lemma 4.3:

(a) This part is clear using the fact that for every $c \in L_v$ the probability that c remains in L_v after iteration i is equal to Keep_i.

(b) By Fact 2, there are at most $O(\sqrt{\Delta})$ vertices with more than $O(\sqrt{\Delta})$ common neighbours with v. Consider any vertex $u \in N_i(v, c)$ with at most $O(\sqrt{\Delta})$ common neighbours with v. We will prove that the probability that after iteration i, u remains uncoloured, no vertex in $N_u - N_v - \{v\}$ is assigned c, and u does not lose c during the equalizing coin flip is at most $\left(1 - \frac{K}{\ln \Delta} \times \mathrm{Keep}_i\right) \times \mathrm{Keep}_i + O(\frac{\sqrt{\Delta}}{\ln \Delta} \times \frac{1}{L_i})$. Note that this fact implies that the expectation of $t'_{i+1}(v, c)$ is at most T_i multiplied by $\left(1 - \frac{K}{\ln \Delta} \times \mathrm{Keep}_i\right) \times \mathrm{Keep}_i + O(\frac{\sqrt{\Delta}}{\ln \Delta} \times \frac{1}{L_i})$ plus $O(\sqrt{\Delta})$ (for the vertices which have more than $O(\sqrt{\Delta})$ common neighbours with v), which implies this lemma.

We consider two cases: either u is not activated (with probability $1 - \frac{K}{\ln \Delta}$), or it is activated (with probability $\frac{K}{\ln \Delta}$). If u is not activated, it will remain uncoloured. The two events "E_1: c is not assigned to any vertex in $N_u - N_v - \{v\}$" and "E_2: c is not eliminated from L_u during the equalizing coin flip" are independent. The probability of the first event is equal to

$$\mathbf{Pr}(E_1) = \prod_{w \in N_i(u,c) - N_v - \{v\}} \mathbf{Pr}(w \text{ is not assigned } c)$$

$$\begin{aligned} &= \prod_{w\in N_i(u,c)-N_v-\{v\}} \left(1-\frac{K}{\ln\Delta}\times\frac{1}{\ell_i(w)}\right) \\ &= \mathrm{Keep}_i(u,c)\Big/\left(1-\frac{K}{\ln\Delta}\times\frac{1}{L_i}\right)^{O(\sqrt{\Delta})}. \end{aligned}$$

The probability of the second event is equal to

$$\mathbf{Pr}(E_2)=\frac{\mathrm{Keep}_i}{\mathrm{Keep}_i(u,c)}.$$

Therefore, in this case, the desired probability is equal to

$$\mathbf{Pr}(E_1\wedge E_2)=\mathbf{Pr}(E_1)\times\mathbf{Pr}(E_2)\le \mathrm{Keep}_i\Big/\left(1-\frac{K}{\ln\Delta}\times\frac{1}{L_i}\right)^{O(\sqrt{\Delta})}.$$

In the second case (u is activated), consider the following events:
E_1: u is assigned a colour $d\neq c$.
E_2: A neighbour of u is assigned d.
E_3: No vertex in $N_u-N_v-\{v\}$ is assigned c.
E_4: c is not eliminated from L_u during the equalizing coin flip.

The desired probability is the probability that the above four events happen at the same time. It is clear that the events E_1, $E_2\wedge E_3$, and E_4 are independent, but E_2 and E_3 are dependent. The probabilities of E_1 and E_4 are:

$$\begin{aligned} \mathbf{Pr}(E_1) &= \frac{\ell_i(u)-1}{\ell_i(u)}<1 \\ \mathbf{Pr}(E_4) &= \frac{\mathrm{Keep}_i}{\mathrm{Keep}_i(u,c)}. \end{aligned}$$

Also, the probability of E_2 is:

$$\begin{aligned} \mathbf{Pr}(E_2) &= 1-\prod_{w\in N_i(u,d)}\mathbf{Pr}(w \text{ is not assigned } d) \\ &= 1-\prod_{w\in N_i(u,d)}\left(1-\frac{K}{\ln\Delta}\times\frac{1}{\ell_i(w)}\right) \\ &\le 1-\left(1-\frac{K}{\ln\Delta}\times\frac{1}{L_i}\right)^{T_i} \\ &= 1-\mathrm{Keep}_i. \end{aligned}$$

The probability of $E_2\wedge E_3$ is equal to $\mathbf{Pr}(E_2)$ multiplied by $\mathbf{Pr}(E_3|E_2)$. The latter probability is equal to the probability that no vertex in $N_u-N_v-\{v,x\}$ is assigned

c (x is the vertex that is assigned d). Therefore,

$$\begin{aligned}
\mathbf{Pr}(E_3|E_2) &= \prod_{w\in N_i(u,c)-N_v-\{v,x\}} \mathbf{Pr}(w \text{ is not assigned } c) \\
&= \prod_{w\in N_i(u,c)-N_v-\{v,x\}} \left(1-\frac{K}{\ln\Delta}\times\frac{1}{\ell_i(w)}\right) \\
&< \mathrm{Keep}_i(u,c) \Big/ \left(1-\frac{K}{\ln\Delta}\times\frac{1}{L_i}\right)^{O(\sqrt{\Delta})}.
\end{aligned}$$

Thus, the desired probability in this case is equal to

$$\begin{aligned}
\mathbf{Pr}(E_1\wedge E_2\wedge E_3\wedge E_4) &= \mathbf{Pr}(E_1)\times\mathbf{Pr}(E_4)\times\mathbf{Pr}(E_2)\times\mathbf{Pr}(E_3|E_2) \\
&< \mathrm{Keep}_i(1-\mathrm{Keep}_i) \Big/ \left(1-\frac{K}{\ln\Delta}\times\frac{1}{L_i}\right)^{O(\sqrt{\Delta})}.
\end{aligned}$$

Therefore, the probability that after iteration i, u remains uncoloured, no vertex in $N_u - N_v - \{v\}$ is assigned c, and u does not lose c during the equalizing coin flip is at most

$$\begin{aligned}
P &< (1-\frac{K}{\ln\Delta})\times\frac{\mathrm{Keep}_i}{(1-\frac{K}{\ln\Delta}\times\frac{1}{L_i})^{O(\sqrt{\Delta})}} + \frac{K}{\ln\Delta}\times\frac{\mathrm{Keep}_i(1-\mathrm{Keep}_i)}{(1-\frac{K}{\ln\Delta}\times\frac{1}{L_i})^{O(\sqrt{\Delta})}} \\
&< \mathrm{Keep}_i\left(1-\frac{K}{\ln\Delta}+\frac{K}{\ln\Delta}(1-\mathrm{Keep}_i)\right) \Big/ \left(1-\frac{K}{\ln\Delta}\times\frac{1}{L_i}\right)^{O(\sqrt{\Delta})} \\
&= \mathrm{Keep}_i\left(1-\frac{K}{\ln\Delta}\times\mathrm{Keep}_i\right) \Big/ \left(1-\frac{K}{\ln\Delta}\times\frac{1}{L_i}\right)^{O(\sqrt{\Delta})}.
\end{aligned}$$

Now, we use the inequality $\frac{1}{(1-x)^n} < 1+2nx$ which holds for every positive n and x such that $nx < \frac{1}{2}$. Since we know that $L_i \geq \sqrt{\Delta}$, this inequality implies that

$$\begin{aligned}
P &< \mathrm{Keep}_i\left(1-\frac{K}{\ln\Delta}\times\mathrm{Keep}_i\right)\times\left(1+\frac{O(\sqrt{\Delta})}{\ln\Delta}\times\frac{1}{L_i}\right) \\
&< \mathrm{Keep}_i\left(1-\frac{K}{\ln\Delta}\times\mathrm{Keep}_i\right) + O(\frac{\sqrt{\Delta}}{\ln\Delta}\times\frac{1}{L_i}).
\end{aligned}$$

which completes the proof of the lemma.

■

4.1.5 The concentration lemma

In this section we prove Lemma 4.4. We will use Theorem 3.18 for several random variables with parameters $\beta = O(\sqrt{\Delta})$, $D = O(M)$, and $t = \Theta(M^\gamma)$, where M is L_i in the first part, and T_i in the second part. By substituting these parameters in Theorem 3.18 we get

$$\mathbf{Pr}(|R - \mathbf{E}(R)| > t) \leq 4e^{-\Omega(\frac{M^{2\gamma-1}}{\sqrt{\Delta}})}.$$

Since $\gamma = \frac{1+\epsilon/20}{1+\epsilon/10}$ and $M \geq \Delta^{\frac{1}{2}+\frac{\epsilon}{20}} \ln^3 \Delta$, we have

$$\frac{M^{2\gamma-1}}{\sqrt{\Delta}} \geq \frac{\Delta^{(\frac{1}{2}+\frac{\epsilon}{20})(\frac{1}{1+\epsilon/10})}(\ln \Delta)^{3(\frac{1}{1+\epsilon/10})}}{\sqrt{\Delta}} = (\ln \Delta)^{\frac{3}{1+\epsilon/10}} > (\ln \Delta)^{2.5}.$$

for ϵ small. Thus

$$\mathbf{Pr}(|R - \mathbf{E}(R)| > t) \leq 4e^{-(\ln \Delta)^{2.5}} = o(\Delta^{-\ln \Delta}).$$

The random trials will be random colour assignments and equalizing coin flips. In each iteration, for every uncoloured vertex v, there is a random colour assignment trial which assigns a colour of L_v to v with probability $\frac{K}{\ln \Delta} \times \frac{1}{|L_v|}$, and does not assign any colour to v with probability $1 - \frac{K}{\ln \Delta}$. Notice that we do not have two separate random trials for activating a vertex and assigning a colour to it, because if we consider two random trials for each vertex, the trials would not be independent, so we cannot use Theorem 3.18.

Proof of Lemma 4.4:

(a) Instead of proving that $\ell_{i+1}(v)$, i.e., the number of colours which remain in L_v after the ith iteration, is concentrated, we prove that $\ell_i(v) - \ell_{i+1}(v)$, i.e., the number of colours which are removed from L_v in the ith iteration, is concentrated. Let R be the random variable $\ell_i(v) - \ell_{i+1}(v)$.

For every colour $c \in L_v$, let R_c be the binary random variable that is zero if c remains in L_v after iteration i, and one otherwise. Clearly, $R = \sum_{c \in L_v} R_c$. Furthermore, for every outcome x and every colour $c \in L_v$, if $R_c(x) = 1$, there is either a colour assignment trial which assigns c to a neighbour of v, or an equalizing trial which removes c from L_v. Let this trial be the witness of R_c. It is easy to see that these witnesses satisfy the conditions of Theorem 3.18 with $\beta = 1, D = L_i$ (since $\sum_{c \in L_v} |W_c| = R(x) \leq L_i$). Therefore the concentration inequality yields

$$\mathbf{Pr}(|R - \mathbf{E}(R)| > L_i^{\gamma}) < \Delta^{-\ln \Delta}.$$

Therefore,

$$\mathbf{Pr}(|\ell_{i+1}(v) - \mathbf{E}(\ell_{i+1}(v))| > L_i^{\gamma}) < \Delta^{-\ln \Delta}.$$

(b) Let T be the set of uncoloured neighbours of v whose lists contain c at the beginning of iteration i. We define two random variables R and R'. R is the number of vertices in T which remain uncoloured after the ith iteration, and R' is the number of vertices u in T which remain uncoloured, but either one of the vertices in $N_u - N_v - \{v\}$ is assigned c in iteration i, or u loses c by an equalizing coin flip in iteration i. Clearly,

$$t'_{i+1}(v, c) = R - R'. \tag{4.6}$$

For proving that $t'_{i+1}(v, c)$ is highly concentrated, we prove that each of R and R' is highly concentrated.

First, we prove that R is highly concentrated. For every vertex $u \in T$, let R_u be the binary random variable which is one if u remains uncoloured after iteration i, and zero otherwise. Clearly, $R = \sum_{u \in T} R_u$. Also, for every outcome x of the trials and every $u \in T$, if $R_u(x) = 1$, either u is not assigned any colour, or it is assigned a colour the same as a neighbour. In the former case, let the random colour assignment to u be the witness of R_u. In the latter case, we choose two witnesses for R_u. The first witness is the random colour assignment to u, and the second witness is the colour assignment to one of the neighbours of u which is assigned the same colour as the colour assigned to u. Call every such vertex a *potential witness* of u. It is clear that no matter how we choose the second witness among potential witnesses, the first and the third conditions of Theorem 3.18 hold with $D = 2T_i$ (since $\sum_{u \in T} |W_u| \leq 2R(x) \leq 2T_i$). Now, we describe a way to choose the second witnesses such that the second condition also holds with $\beta = O(\sqrt{\Delta})$.

For every $u \in T$ which is assigned the same colour as a neighbour, if u does not have any potential witness in N_v, choose an arbitrary potential witness as its second witness. For those which have at least one potential witness in N_v, partition them according to the colour assigned to them. We will get several disjoint subgraphs of N_v, such that in each subgraph, every neighbour of a vertex is one of its potential witnesses. Therefore, in each subgraph, we need to choose for each vertex one of its neighbours as its second witness. We use the following simple lemma to choose these neighbours.

Lemma 4.7 *Let G be a graph with no isolated vertex. If α is the size of the maximum independent set in G, then it is possible to choose for each vertex one of its neighbours in such a way that no vertex is chosen more than $\alpha + 1$ times.*

Proof: Consider a maximum matching M in G, and for each vertex u of the matching, choose the vertex that is matched with u. Also, for every vertex that is not in the matching, choose an arbitrary neighbour. Since M is maximum, $G - V(M)$ is an independent set and therefore cannot have more than α vertices. Thus, every vertex is chosen at most $\alpha + 1$ times. ■

Therefore by Fact 1 and Fact 2, it is clear that no vertex is chosen as the second witness of more than $O(\sqrt{\Delta})$ vertices. Therefore, the second condition of Theorem 3.18 holds with $\beta = O(\sqrt{\Delta})$. Thus, Theorem 3.18 implies

$$\mathbf{Pr}(|R - \mathbf{E}(R)| > \frac{1}{4}T_i^{\gamma}) < \frac{1}{2}\Delta^{-\ln\Delta}. \tag{4.7}$$

Now, we prove the concentration of R'. For every vertex $u \in T$, let R'_u be the binary random variable which is one iff u is uncoloured after iteration i and either a neighbour of u outside $N_v \cup \{v\}$ is assigned c in iteration i, or u loses c by an equalizing coin flip in iteration i. Clearly, $R' = \sum_{u \in T} R'_u$. For every outcome x of the trials and every $u \in T$, if $R'_u(x) = 1$, then either a neighbour of u outside $N_v \cup \{v\}$ is assigned c in iteration i, or u loses c by an equalizing coin flip in iteration i. In the former case, let the first witness of R'_u be the random colour assignment to an arbitrary neighbour of u outside $N_v \cup \{v\}$ which is assigned c, and in the latter case, let the first witness of R'_u be the equalizing trial which removes c from L_u. Also, since $R'_u(x) = 1$, u must remain uncoloured after iteration i. Consider the witnesses that we found above for R_u as the other witnesses of R'_u. (These witnesses certify that u remains uncoloured). It is clear that these witnesses satisfy the conditions of Theorem 3.18 with $\beta = O(\sqrt{\Delta})$ (because of Fact 2) and $D = 3T_i$ (since $\sum_{u \in T} |W_u| \leq 3R'(x) \leq 3T_i$). Therefore, by Theorem 3.18 we obtain

$$\mathbf{Pr}(|R' - \mathbf{E}(R')| > \frac{1}{4}T_i^{\gamma}) < \frac{1}{2}\Delta^{-\ln\Delta}. \tag{4.8}$$

Therefore, by Equations 4.6, 4.7, and 4.8 we have

$$\mathbf{Pr}(|t'_{i+1}(v,c) - \mathbf{E}(t'_{i+1}(v,c))| > \frac{1}{2}T_i^{\gamma}) < \Delta^{-\ln\Delta}.$$

which completes the proof.

■

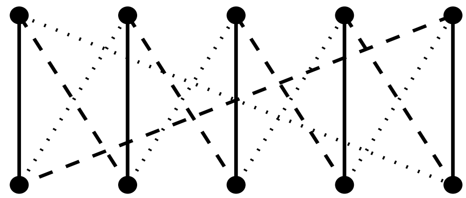

Figure 4.2: A graph G such that $\chi(G') = 5$

4.2 A simpler proof for a weaker theorem

In this section, we present a simple proof for the following weaker version of Theorem 4.1.

Theorem 4.6 *There is a constant c such that for every graph G of girth at least seven, the strong chromatic index of G is at most $c\frac{\Delta^2}{\ln \Delta}$.*

Proof: By Vizing's theorem, the edges of G can be coloured with $\Delta + 1$ colours in such a way that no two adjacent edges receive the same colour. Consider such a colouring $\mathcal{C}_1$, and construct a new graph G' as follows: The vertices of G' are the edges of G, and two vertices of G' are adjacent if and only if the corresponding edges in G are of distance two, and have received the same colour in $\mathcal{C}_1$. It is easy to see that since G does not have any cycle of length at most six, G' does not contain any triangle. Therefore, by Johansson's theorem [24], if Δ' is the maximum degree of G', there is a colouring $\mathcal{C}_2$ for G' with $O(\frac{\Delta}{\ln \Delta})$ colours. Furthermore, it is easy to see that Δ' is at most 2Δ.

Now, we can colour every edge e of G with a pair $(\mathcal{C}_1(e), \mathcal{C}_2(e))$. This colouring is clearly a strong edge-colouring, and the number of colours used is $(\Delta + 1) \times O(\frac{\Delta}{\ln \Delta}) < c\frac{\Delta^2}{\ln \Delta}$ for some constant c. ■

The idea of the above proof is interesting and may be useful for proving Conjecture 2.1 or Conjecture 2.2. However, the example in Figure 4.2 shows that it is not that simple. Figure 4.2 shows a graph G of maximum degree $\Delta = 3$ and an edge-colouring $\mathcal{C}_1$ of G with three colours. But the graph G' constructed from G and $\mathcal{C}_1$ contains a clique of size five, and therefore cannot be coloured with less than five colours. This example can be generalized for arbitrary Δ.

This example shows that we cannot take an arbitrary edge-colouring $\mathcal{C}_1$ of G and try to find another colouring $\mathcal{C}_2$ with $\Delta + o(\Delta)$ colours so that $(\mathcal{C}_1, \mathcal{C}_2)$ becomes a strong edge-colouring of G. However, it might be possible to impose some restrictions on $\mathcal{C}_1$, so that we can find such a $\mathcal{C}_2$.

4.3 A generalization

It is easy to modify the proof of Theorem 4.1 to prove the following generalization.

Theorem 4.7 *For every constant r, the strong chromatic index of every $K_{2,r}$-free graph is at most*

$$(2+o(1))\frac{\Delta^2}{\ln \Delta}.$$

Since in the proof of Theorem 4.1 we do not use any property of the square of C_4-free graphs except Facts 1 and 2, if we prove that Facts 1 and 2 also hold for the squares of $K_{2,r}$-free graphs, Theorem 4.7 follows. Fact 1 does not make any use of C_4-freeness of the graph. Also, it is easy to see that the proof of Fact 2 holds for the squares of $K_{2,r}$-free graphs. Therefore, we can use the proof of Theorem 4.1 to prove Theorem 4.7.

Also, we conjecture that if $K_{2,r}$ in Theorem 4.7 is replaced by $K_{r,r}$, the theorem still holds.

4.4 Lower bound

In this section we mention the proof of Theorem 4.2. The proof is a joint work with Mike Molloy.

Theorem 4.2 *For every $g \geq 3$ and sufficiently large d there is a d-regular graph of girth at least g with strong chromatic index at least*

$$\left(\frac{1}{2}-o(1)\right)\frac{d^2}{\ln d}.$$

Proof: Let n denote the number of vertices of the graph. Let S consist of n groups $A_1, A_2, \ldots, A_n$ of vertices, each group consisting of d vertices. For each random pairing P of the vertices of S, we can construct a d-regular (random) graph G_P (which is not necessarily simple) as follows: For each i ($1 \leq i \leq n$), there is a vertex v_i in G_P corresponding to the group A_i in S, and for each pair $\{x, y\}$ in P, where $x \in A_i$ and $y \in A_j$, there is an edge between the two vertices v_i and v_j. We use the following theorem of [53].

Theorem 4.8 *For every fixed g and d, when n tends to infinity, the probability that G_P has no cycle of length $< g$ tends to a nonzero constant depending only on g and d.*

Therefore, we have the following corollary.

Corollary 4.3 *If the probability that G_P has some property $\mathcal{P}$ tends to zero as n tends to infinity, then there is a d-regular graph of girth at least g which does not have the property $\mathcal{P}$.*

Now, let the property $\mathcal{P}$ be the property of having an induced matching of size $k = \lceil \frac{cn \ln d}{d} \rceil$, where c is a constant which will be fixed later. We count the number of pairings P for which G_P has an induced matching of size k. There are $\binom{n}{2k}$ ways to select the $2k$ groups corresponding to the endpoints of the induced matching. The number of pairings of these groups is equal to $(2k-1)(2k-3)\cdots \times 3 \times 1$, which we denote by $M(2k)$. We can select one vertex from each of these $2k$ groups in d^{2k} different ways. These vertices will be paired according to the pairing of the groups. This constitutes a matching of size k in G_P. In order to make this matching an *induced* matching, each of the other $2k(d-1)$ vertices of the $2k$ groups must be paired with one of the vertices of the other $n-2k$ groups. This can be done in $\binom{nd-2kd}{2kd-2k} \times (2kd-2k)!$ ways. The number of remaining vertices is $nd-4kd+2k$, and the number of pairings of these vertices is $M(nd-4kd+2k)$. Therefore, the total number of pairings P for which G_P has an induced matching of size k is at most

$$\binom{n}{2k} \times M(2k) \times d^{2k} \times \binom{nd-2kd}{2kd-2k} \times (2kd-2k)! \times M(nd-4kd+2k)$$

and the probability that G_P has an induced matching of size k is at most the above value divided by the total number of pairings which is equal to $M(nd)$. We use Stirling's approximation for $n!$ to estimate this probability. Stirling's formula implies

$$\left(\frac{n}{e}\right)^n < n! < n\left(\frac{n}{e}\right)^n$$

and therefore

$$\binom{n}{m} < n\frac{n^n}{m^m(n-m)^{n-m}}$$

and

$$\frac{1}{k}\left(\frac{2k}{e}\right)^k < M(2k) = \frac{(2k)!}{2^k \times k!} < 2k\left(\frac{2k}{e}\right)^k.$$

So, the desired probability is

$$\begin{aligned}
\Pr &< n\frac{n^n}{(2k)^{2k}(n-2k)^{n-2k}} \times (2k)\frac{(2k)^k}{e^k} \times d^{2k} \times \\
&(dn-2dk)\frac{(dn-2dk)^{dn-2dk}}{(2dk-2k)^{2dk-2k}(dn-4dk+2k)^{dn-4dk+2k}} \times \\
&(2dk-2k)\frac{(2dk-2k)^{2dk-2k}}{e^{2dk-2k}} \times \\
&(dn-4dk+2k)\frac{(dn-4dk+2k)^{\frac{dn}{2}-2dk+k}}{e^{\frac{dn}{2}-2dk+k}} \times \left(\frac{dn}{2}\right)\frac{e^{dn/2}}{(dn)^{dn/2}} \\
&= O(n^6)\frac{n^n d^{2k+dn-2dk}(n-2k)^{dn-2dk}}{(2k)^k(n-2k)^{n-2k}(dn-4dk+2k)^{\frac{dn}{2}-2dk+k}d^{\frac{dn}{2}}n^{\frac{dn}{2}}} \\
&= O(n^6)\frac{\left(1-\frac{2k}{n}\right)^{dn-2dk-n+2k}}{\left(\frac{2k}{dn}\right)^k\left(1-\frac{4k}{n}+\frac{2k}{dn}\right)^{\frac{dn}{2}-2dk+k}} \\
&= O(n^6)\left(\frac{\left(1-2c\frac{\ln d}{d}\right)^{d-2c\ln d-1+2c\frac{\ln d}{d}}}{\left(2c\frac{\ln d}{d^2}\right)^{c\frac{\ln d}{d}}\left(1-4c\frac{\ln d}{d}+2c\frac{\ln d}{d^2}\right)^{\frac{d}{2}-2c\ln d+c\frac{\ln d}{d}}}\right)^n.
\end{aligned}$$

Let A be the value inside the parentheses. Therefore, $\Pr < O(n^6)A^n$. Furthermore, the variable n does not appear in A. Thus, if $A<1$, then the limit of Pr when n tends to infinity is zero. Now, we prove that for every $c>1$ and sufficiently large d, A is less than one. We use the inequalities $1-x<e^{-x-\frac{x^2}{2}}$ and $1-x>e^{-x-(1+\epsilon)\frac{x^2}{2}}$ which hold for every positive ϵ and sufficiently small positive number x. Using these inequalities we have

$$\begin{aligned}
A &< \exp\Bigg(-\left(2c\frac{\ln d}{d}+2c^2\frac{\ln^2 d}{d^2}\right)\left(d-2c\ln d-1+2c\frac{\ln d}{d}\right) \\
&\quad - c\frac{\ln d}{d}\ln\left(\frac{2c\ln d}{d^2}\right)+\left(\frac{d}{2}-2c\ln d+c\frac{\ln d}{d}\right)\times \\
&\quad \left(\left(4c\frac{\ln d}{d}-2c\frac{\ln d}{d^2}\right)+\frac{(1+\epsilon)}{2}\left(4c\frac{\ln d}{d}-2c\frac{\ln d}{d^2}\right)^2\right)\Bigg) \\
&= \exp\Bigg(-2c\ln d-2c^2\frac{\ln^2 d}{d}+4c^2\frac{\ln^2 d}{d}+o(\frac{\ln^2 d}{d}) \\
&\quad + o(\frac{\ln^2 d}{d})+2c\frac{\ln^2 d}{d}+\left(\frac{d}{2}-2c\ln d+c\frac{\ln d}{d}\right)\times \\
&\quad \left(4c\frac{\ln d}{d}+8(1+\epsilon)\frac{ln^2 d}{d^2}+o(\frac{ln^2 d}{d^2})\right)\Bigg)
\end{aligned}$$

$$
\begin{aligned}
&= \exp\Bigg(-2c\ln d + (2c^2+2c)\frac{\ln^2 d}{d} + o(\frac{\ln^2 d}{d}) \\
&\qquad + 2c\ln d + 4(1+\epsilon)c^2\frac{\ln^2 d}{d} - 8c^2\frac{\ln^2 d}{d} + o(\frac{\ln^2 d}{d}) \Bigg) \\
&= \exp\left(2c\,(1-(1-2\epsilon)c)\,\frac{\ln^2 d}{d} + o(\frac{\ln^2 d}{d})\right).
\end{aligned}
$$

So, for any $c > 1$, we can fix ϵ such that the value of the exponent in the above expression becomes negative when d gets large. Therefore, for any $c > 1$, if d is sufficiently large, the limit of Pr when n tends to infinity tends to zero.

Therefore, Corollary 4.3 implies that for every positive integer g and every $c > 1$, if d is sufficiently large, there are d-regular graphs of girth at least g which do not have any induced matching of size $k = \lceil \frac{cn\ln d}{d} \rceil$. But it is easy to see that every d-regular graph with strong chromatic index at most $\frac{1}{2c}\left(\frac{d^2}{\ln d}\right)$ must have an induced matching of size k. Therefore, for every g and any constant $c' < \frac{1}{2}$, if d is sufficiently large, there are d-regular graphs of girth at least g and strong chromatic index at least $\frac{c'd^2}{\ln d}$. ■

Chapter 5

Algorithmic questions

In this chapter, we consider several algorithmic questions regarding the strong edge-colouring problem. In Section 5.1, we will prove that this problem is **NP**-complete even in a very restricted setting. In Section 5.2, a closely related problem, namely the problem of finding the largest antimatching in the given graph is studied, and some **NP**-completeness results and a polynomial time algorithm for a subproblem are derived.

Another related problem, namely the problem of finding the largest induced matching, is studied in [9] (See Section 2.6.2), and it is proved that the problem is **NP**-complete for bipartite graphs. In fact, a slightly modified version of the argument in [9] implies that for every g, the maximum induced matching problem is **NP**-complete for bipartite graphs of girth at least g.

5.1 The strong edge-colouring problem

In this section, we prove that the strong edge-colouring problem is **NP**-complete. The problem is defined as follows:

STRONG EDGE-COLOURING
INSTANCE: A graph G and a positive integer K.
QUESTION: Does G have a strong edge-colouring with K colours?

A simple proof for the **NP**-completeness of the above problem is presented in Section 5.1.1. In Section 5.1.2 we will use a more complicated argument to prove that this problem is **NP**-complete even in a very restricted setting.

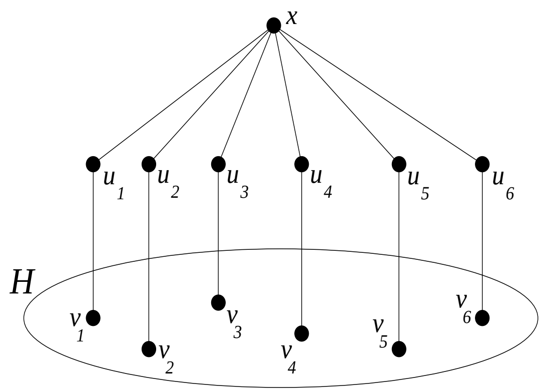

Figure 5.1: Reduction from vertex-colouring to strong edge-colouring

5.1.1 Basic NP-completeness result

In this section, we prove the following theorem.

Theorem 5.1 STRONG EDGE-COLOURING *is* **NP***-complete.*

Proof: It is clear that STRONG EDGE-COLOURING is in **NP**. In [18, 19], it is proved that the following problem is **NP**-complete.

3-COLOURABILITY OF GRAPHS WITH MAXIMUM DEGREE 4
INSTANCE: A graph H having no vertex degree exceeding four.
QUESTION: Is H 3-colourable?

We show a reduction from the above problem to STRONG EDGE-COLOURING. Let H be the given graph with maximum degree four. We construct an instance (G, K) of STRONG EDGE-COLOURING such that H is 3-colourable if and only if G has a strong edge-colouring with K colours.

We can assume, without loss of generality, that the number of vertices of H is more than 32, since for smaller graphs, we can use exhaustive search to check whether the graph is 3-colourable or not. Let $\{v_1, v_2, \dots, v_n\}$ be the set of vertices of H. We add $n+1$ new vertices $x, u_1, u_2, \dots, u_n$ to H, and connect x to u_i, and u_i to v_i for every $i = 1, \dots, n$. Let G be the resulting graph and $K = n+3$ (See Figure 5.1). It is clear that this construction can be done in polynomial time. We claim that G has a strong edge-colouring with K colours if and only if H is 3-colourable.

First, assume that G has a strong edge-colouring with K colours. It is clear that the edges $xu_1, xu_2, \dots, xu_n$ must be coloured with n different colours. Also, for every i, the edge u_iv_i must be coloured with a colour different from those n colours. Therefore, since $K = n + 3$, only three colours remain for colouring the edges $u_1v_1, u_2v_2, \ldots, u_nv_n$. Now, we colour the vertices of H as follows: We colour the vertex v_i with the colour of the edge u_iv_i in the strong edge-colouring of G. It is clear that only three colours are used, and also by the definition of strong edge-colouring, the colour of any two adjacent vertices must be different. Thus, H is 3-colourable.

Conversely, assume that H has a vertex-colouring with three colours. We construct a strong edge-colouring for G with $n + 3$ colours as follows: For every i, we colour the edge u_iv_i with the colour of v_i in the vertex-colouring of H. We colour the edges $u_1v_1, u_2v_2, \ldots, u_nv_n$ with n different colours other than the colours that were used in colouring H. The only edges that remain are the edges between v_i's. But it is easy to see that for every such edge v_iv_j the number of other edges in G of distance at most two from v_iv_j is at most 34, and therefore, since we have $n+3 \geq 35$ colours available, we can use a greedy algorithm for colouring the remaining edges, so that each edge receives a colour different from the colour of every other edge of distance at most two. Therefore, G has a strong edge-colouring with K colours. ■

Notice that the above proof is also a simple proof for the fact that distance-2 colouring (i.e., colouring the square of a graph) is **NP**-complete. This fact is proved in [33] using a more complicated reduction from the satisfiability problem.

5.1.2 NP-completeness of a restricted problem

For $K \leq 3$, the problem of strong K-edge-colourability is trivial (It is not difficult to see that the only strong 3-edge-colourable graphs are disjoint unions of cycles, paths, and $K_{1,3}$'s). The following theorem implies that the problem is **NP**-complete for any other value of K.

Theorem 5.2 *For every fixed g, $K \geq 4$,* STRONG EDGE-COLOURING *is* **NP***-complete for bipartite graphs with girth at least g.*

Proof: It is clear that the problem is in **NP**. We prove that it is **NP**-complete by showing a reduction from GRAPH K-COLOURABILITY to this problem. It is proved [27, 19] that GRAPH K-COLOURABILITY is **NP**-complete for every fixed $K \geq 3$. For every graph H, we construct another graph G such that H is K-colourable if and only if G has a strong edge-colouring using K colours.

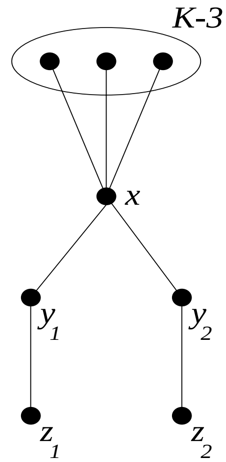

Figure 5.2: The graph T

Before starting to construct G, consider the graph T shown in Figure 5.2. This graph consists of a vertex x adjacent to a set of $K-1$ other vertices, including two vertices y_1 and y_2, which are adjacent to two other vertices z_1 and z_2, respectively. The important property of this graph, which will be used throughout the proof, is that in every strong edge-colouring of T using K colours the colours of the two edges y_1z_1 and y_2z_2 are the same. It is not difficult to observe that T has this property.

Now, for every $K \geq 4, d \geq 2$, we construct a graph $G_{K,d}$. $G_{K,d}$ contains a path $v_0v_1 \ldots v_{3(d-1)}$. For every $i = 0 \ldots d-1$, the vertex v_{3i} is connected to another vertex u_i, and u_i is connected to another vertex w_i. Also, for every $i = 0, \ldots, d-1$, the vertex v_{3i} is connected to a set A_i of new vertices (A_i's are pairwise disjoint). A_0 and A_{d-1} each contain $K-3$ vertices, and A_i (for $i = 1, \ldots, d-2$) contains $K-4$ vertices. There is no other vertex or edge in $G_{K,d}$. For example, Figure 5.3 shows the graph $G_{6,4}$. The vertices $w_0, w_1, \ldots, w_{d-1}$ are called the *heads* of $G_{K,d}$.

Using the property of the graph T, it is easy to see that in every strong edge-colouring of $G_{K,d}$ using K colours, the colours of the edges which are indicated by thick lines in Figure 5.3 are the same.

Now, we are ready to construct the graph G from the graph H. We assume, without loss of generality, that the degree of every vertex in H is at least two. Corresponding to each vertex v of degree d in H, we put a copy C_v of $G_{K,d}$ in G. Each head of C_v corresponds to one of the edges incident to v. If two vertices u and v in H are joined by an edge e, we combine the heads corresponding to e in C_u and C_v into a single vertex. Let G be the resulting graph. We claim that G has a strong edge-colouring with K colours if and

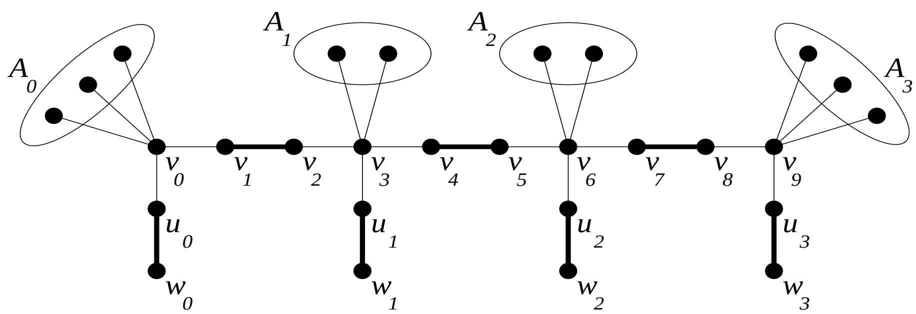

Figure 5.3: The graph $G_{6,4}$

only if H is K-colourable.

Assume that G has a strong edge-colouring with K colours. By the property of $G_{K,d}$, we know that for every vertex $v \in V(H)$ the colours of the thick edges of C_v are the same. Colour the vertex v in H with the colour of the thick edges of C_v. Since for every two adjacent vertices u and v in H, two heads of C_u and C_v are combined into one vertex, the colour of the thick edges of C_u and C_v, and therefore the colour of the vertices u and v, cannot be the same. Thus, the colouring is a proper vertex colouring of H using K colours.

Conversely, assume that H has a vertex-colouring using K colours. We construct a strong edge-colouring of G using K colours. For every v, we colour the thick edges of C_v with the colour of v in H. Then, we colour the remaining edges of C_v using the following lemma.

Lemma 5.1 *Let T' be the graph obtained from T (Figure 5.2) by adding two new vertices t_1 and t_2 and connecting them to z_1 and z_2, respectively (See Figure 5.4). If the edges y_1z_1, y_2z_2, z_1t_1, and z_2t_2 are coloured in such a way that the colours of y_1z_1 and y_2z_2 are the same and every two edges of distance at most two are coloured with different colours, then it is possible to complete this partial colouring to a strong edge-colouring of T'.*

Proof: It is sufficient to consider two cases: In the first case, z_1t_1 and z_2t_2 are coloured with the same colour, and in the second case, they are coloured with different colours. In each case, it is easy to complete the colouring. ∎

It is not difficult to see that using the above lemma, we can colour every edge in G, and obtain a strong edge-colouring of G using K colours.

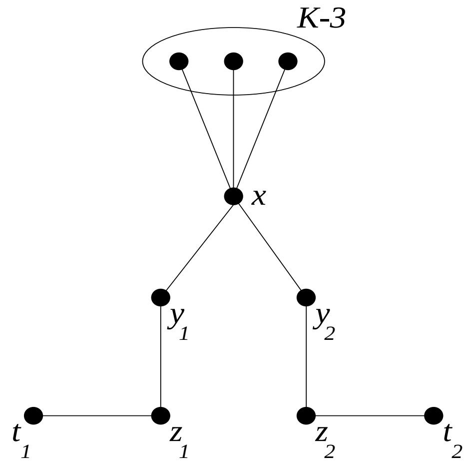

Figure 5.4: The graph T'

Therefore, we have proved that for every fixed $K \geq 4$, GRAPH K-COLOURABILITY reduces to STRONG EDGE-COLOURING. The only thing that remains is to modify the reduction in such a way that the resulting graph G becomes bipartite and of girth at least g.

In order to do this, it is sufficient to consider the graph $G'_{K,d} = G_{K,2dg}$, and let the *heads* of $G'_{K,d}$ be the vertices $w_0, w_{2g}, w_{4g}, \ldots, w_{2g(d-1)}$ (i.e., w_{2gi} for $i = 0, \ldots, d-1$), where $w_0, w_1, \ldots, w_{2gd-1}$ are the heads of $G_{K,2dg}$. It is easy to see that if instead of $G_{K,d}$, we use $G'_{K,d}$ in the above reduction, the resulting graph G will be bipartite and has girth more than g. Furthermore, G has a strong edge-colouring using K colours if and only if H is K-colourable. ■

Notice that in particular the above theorem implies that STRONG EDGE-COLOURING is **NP**-complete for C_4-free graphs.

5.2 The maximum antimatching problem

A trivial lower bound for the strong chromatic index of a graph is the size of the largest antimatching (i.e., the largest set of edges such that any two edges in the set are of distance at most two) in the graph. Therefore, it is natural to study the following problem.

MAXIMUM ANTIMATCHING
INSTANCE: A graph G and a positive integer K.

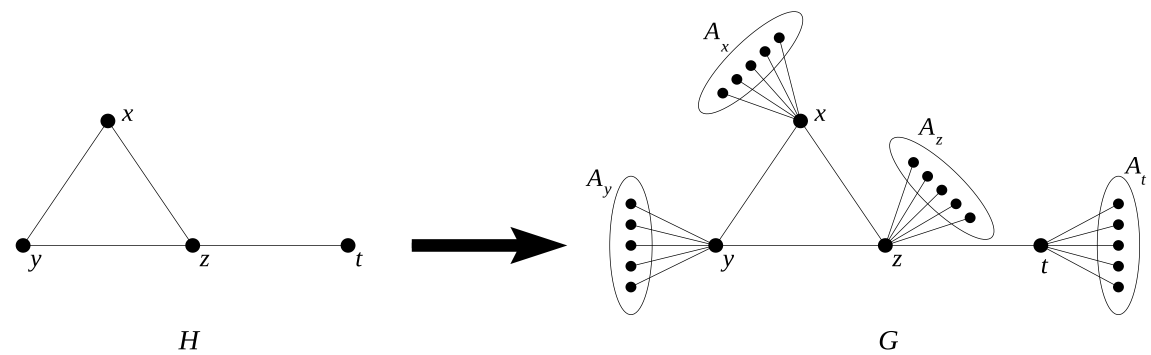

Figure 5.5: Reduction from CLIQUE to MAXIMUM ANTIMATCHING

QUESTION: Does G have an antimatching with at least K edges?

In Sections 5.2.1 and 5.2.2, we prove the **NP**-completeness of this problem for general graphs and bipartite multigraphs. In Section 5.2.3, the problem is studied for C_4-free graphs and a polynomial time algorithm is derived.

5.2.1 General graphs

In this section, we prove the following theorem.

Theorem 5.3 MAXIMUM ANTIMATCHING *is* **NP**-*complete.*

Proof: It is clear that the problem is in **NP**. We prove that the following **NP**-complete problem (See [27, 19]) can be reduced to MAXIMUM ANTIMATCHING.

CLIQUE
INSTANCE: A graph H and a positive integer B.
QUESTION: Does H contain a clique of size B or more?

For every instance (H, B) of CLIQUE we construct an instance (G, K) of MAXIMUM ANTIMATCHING such that G has an antimatching of size K if and only if H has a clique of size B. Let m denote the number of edges of H. Corresponding to every vertex v of H, we add a set A_v of $m+1$ new vertices and connect them to v (See Figure 5.5). Let G be the resulting graph and $K = B(m+1) + \binom{B}{2}$. Also, let E_v denote the set of edges between v and A_v.

Assume that H has a clique of size B. Clearly, the set of edges of G incident to at least one of the vertices of the clique is an antimatching. This set contains at least

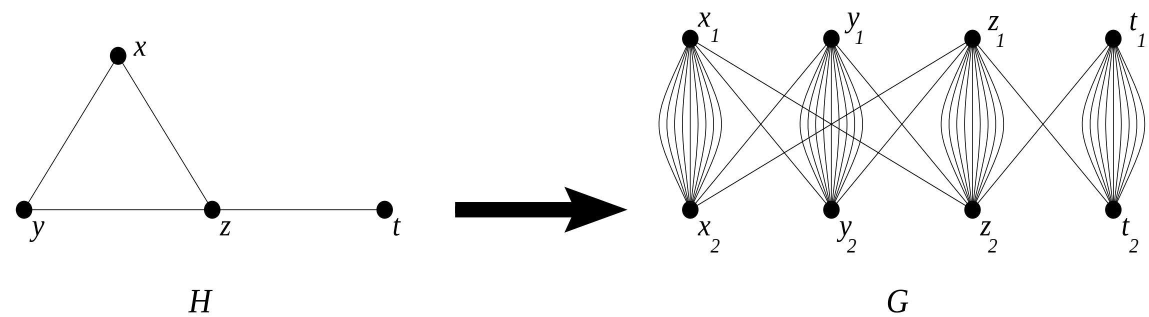

Figure 5.6: Reduction from CLIQUE to bipartite MAXIMUM ANTIMATCHING

$\binom{B}{2} + B(m+1) = K$ edges ($\binom{B}{2}$ edges between the vertices of the clique, and $B(m+1)$ edges in E_v's).

Conversely, assume that G has an antimatching of size K. At most m edges of this antimatching are edges of H, and the others are edges from E_v's. Therefore, the antimatching contains at least

$$K - m > B(m+1) - m > (B-1)(m+1)$$

edges from E_v's. Therefore, there are at least B different vertices v such that the antimatching has at least one edge from E_v. This is impossible unless these B vertices form a clique in H. Thus, H has a clique of size B. ■

5.2.2 Bipartite graphs

Perhaps the most important special case of the antimatching problem is MAXIMUM ANTIMATCHING restricted to bipartite graphs. The following theorem shows that this special case is also **NP**-complete if we allow multi-edges. The problem of the computational complexity of MAXIMUM ANTIMATCHING restricted to *simple* bipartite graphs is still open.

Theorem 5.4 MAXIMUM ANTIMATCHING *is* **NP**-*complete for bipartite multigraphs.*

Proof: We use a technique similar to the technique of the proof of Theorem 5.3 to reduce CLIQUE to this problem. For every instance (H, B) of CLIQUE, we construct a graph G as follows: Corresponding to each vertex v of H, we put two vertices v_1 and v_2 in G and connect them using $2m+1$ edges. Let E_v denote the set of these edges. For every two adjacent vertices u and v in H, we connect u_1 to v_2 and u_2 to v_1. Let $K = (2m+1)B$.

We claim that H has a clique of size B if and only if G has an antimatching of size K. It is clear that if H has a clique C of size B, the set $\cup_{v \in C} E_v$ is an antimatching of size K in G. Conversely, if G has an antimatching of size K, at least $K - 2m$ of the edges of the antimatching are in $\cup_{v \in V(H)} E_v$. Therefore, since $K - 2m > (2m+1)(B-1)$, for at least B different vertices v, the antimatching has at least one edge from E_v. The set of such vertices form a clique of size at least B in H. ■

5.2.3 C_4-free graphs

We saw in Section 5.1.2 that STRONG EDGE-COLOURING remains **NP**-complete even if we impose the restriction that the graph has an arbitrarily large girth. But the status is different for MAXIMUM ANTIMATCHING. We will see in this section that this problem can be solved in polynomial time if we restrict the input to be C_4-free. The reason is that for C_4-free graphs, the structure of antimatchings can be characterized by the following theorem.

Theorem 5.5 *Let G be a C_4-free graph and for every vertex v of G, E_v be the set of edges incident to v. Then for every antimatching A in G, at least one of the following hold:*
(a) There is an edge uv such that $A - (E_u \cup E_v)$ contains at most 13 edges.
(b) There is a triangle xyz such that $A \subseteq E_x \cup E_y \cup E_z$.

It is easy to apply the above theorem to prove the following result.

Theorem 5.6 *There is a polynomial time algorithm that for every given C_4-free graph G, finds the largest antimatching in G.*

Proof: By Theorem 5.5 there are only a polynomial number of cases for a maximum antimatching in G. Therefore we can check all possibilities exhaustively as follows.

For every pair (uv, S), where uv is an edge of G and S is a set of at most 13 edges of $E(G) - (E_u \cup E_v)$, we check whether the set S is an antimatching in G. If it is an antimatching, for every edge $e \in E_u \cup E_v$, we check whether the set $S \cup \{e\}$ is an antimatching. Let S' be the set of edges $e' \in E_u \cup E_v$ such that $S \cup \{e'\}$ is an antimatching. Since every two edges in $E_u \cup E_v$ are of distance at most two, the set $S \cup S'$ is an antimatching in G. Also, it is clear that every antimatching A such that $A - (E_u \cup E_v) = S$ is a subset of $S \cup S'$. Therefore, $S \cup S'$ is the largest such antimatching.

Since there are at most $O(n^{15})$ pairs (uv, S), it is possible to check every such pair, and find the largest antimatching $A_1 = S \cup S'$ which can be obtained by the above process. Clearly, A_1 is the largest antimatching for which case (a) occurs in Theorem 5.5.

Also, for every triangle xyz in G, it is clear that $E_x \cup E_y \cup E_z$ is an antimatching. Therefore, we can check every triangle in G, and find the largest antimatching $A_2 = E_x \cup E_y \cup E_z$ for which case (b) occurs in Theorem 5.5.

Therefore, by Theorem 5.5, the largest antimatching in G is either A_1 or A_2, whichever is larger; and this antimatching can be obtained in polynomial time. ■

Before proving Theorem 5.5, we prove the following simple lemmas.

Lemma 5.2 *For every C_4-free graph G and every vertex u in G, every vertex in G other than u is adjacent to at most one vertex in N_u.*

Proof: If a vertex x is adjacent to two vertices u_1 and u_2 in N_u, then xu_1uu_2 is a 4-cycle in G, contradicting with C_4-freeness of G. ■

Lemma 5.3 *Let G be a C_4-free graph and u and v be two adjacent vertices of G. If H is the induced subgraph $G[N_u \cup N_v - \{u, v\}]$, then the degree of every vertex in H is at most one. Furthermore, there is no edge in H between a vertex in $N_u - \{v\}$ and a vertex in $N_v - \{u\}$.*

Proof: If there is an edge between a vertex x in N_u and a vertex y in N_v, $uxyv$ would be a 4-cycle. Therefore, there is no edge between N_u and N_v. Now, assume, on the contrary, that a vertex x in H has degree more than one in H. Assume, without loss of generality, that $x \in N_u$. Since there is no edge between N_u and N_v, the neighbours of x in H are in N_u. Therefore, x is adjacent to more than one vertex in N_u. This contradicts with Lemma 5.2. This contradiction shows that the degree of every vertex in H is at most one. ■

Lemma 5.4 *Let G be a C_4-free graph and x be a vertex of G. If at least three edges of an antimatching A are incident to x, then every edge in A has at least one endpoint adjacent to x.*

Proof: Let xy_1, xy_2, xy_3 be three edges of A incident to x. Assume, on the contrary, that there is an edge ab in A such that a and b are not adjacent to x. Therefore, since

For every $i \in \{1, 2, 3\}$, xy_i and ab must be of distance at most two, y_i must be adjacent to either a or b. Therefore, for some $i_1, i_2 \in \{1, 2, 3\}$, y_{i_1} and y_{i_2} are both adjacent to the same vertex, say a. Thus, $ay_{i_1}xy_{i_2}$ is a C_4 in G, contradicting our assumption. ■

Proof of Theorem 5.5: Let A be an arbitrary antimatching in G, and uv be an edge in A. Let $A' = A - (E_u \cup E_v)$. Since $uv \in A$, every edge in A' must have at least one endpoint in $N_u \cup N_v$. Let A'_1 be the set of edges in A' which have exactly one endpoint in $N_u \cup N_v$, and A'_2 be the set of edges in A' whose both endpoints are in $N_u \cup N_v$. Also, let S be the set of vertices in $N_u \cup N_v$ which are endpoints of an edge in A'_1. We prove the following claim.

Claim 5.1 $|S \cap N_u| \leq 3$.

Proof: Assume, on the contrary, that there are four vertices x_1, x_2, x_3, x_4 in N_u, which are endpoints of the edges x_1y_1, x_2y_2, x_3y_3, and x_4y_4 in A'_1. By Lemma 5.2, y_1, y_2, y_3, y_4 are distinct. Also, by Lemma 5.3, there are at most two edges amongst x_1, x_2, x_3, x_4. Assume, without loss of generality, that there is no edge between x_1, x_2, x_3, x_4 except possibly between x_1 and x_2, and between x_3 and x_4. Therefore, for every $i \in \{1, 2\}$ and $j \in \{3, 4\}$, there is no edge between x_i and x_j. Also, by Lemma 5.2 there is no edge between y_i and x_j or between y_j and x_i. Therefore, since x_iy_i and x_jy_j are both in the antimatching, there is an edge between y_i and y_j. Therefore, there is a 4-cycle $y_1y_3y_2y_4$ in G, contradicting our assumption. ■

Now, we consider three cases:

(i) No vertex in S is an endpoint of more than two edges of A'_1:

By Claim 5.1, $|S \cap N_u| \leq 3$. Similarly, $|S \cap N_v| \leq 3$. Therefore, $|S| \leq 6$. Thus, $|A'_1| \leq 2 \times 6 = 12$. Also, we prove that A'_2 contains at most one edge. Assume, on the contrary that there are two edges e_1, e_2 in A'_2. By Lemma 5.3, the degree of the endpoints of e_1 and e_2 in $G[N_u \cup N_v]$ is one, and therefore there is no edge between the endpoints of e_1 and the endpoints of e_2. Therefore e_1 and e_2 cannot be in an antimatching simultaneously. Therefore, $|A'_2| \leq 1$. Hence, $|A'| \leq 13$. This means that (a) holds in this case.

(ii) Exactly one vertex in S is an endpoint of more than two edges of A'_1:

Let x be the vertex in S which is an endpoint of at least three edges xy_1, xy_2, xy_3 of A'_1. We consider the following two subcases:

- $x \in N_u \cap N_v$: In this case, we prove that every edge in A' must be incident to x. Assume, on the contrary, that there is an edge ab in A' which is not incident to x. One of the endpoints of ab must be in $N_u \cup N_v$. Assume, without loss of generality, that a is in N_u. By Lemma 5.2, neither a nor b is adjacent to x. This contradicts with Lemma 5.4. Thus, every edge in A must be incident to at least one of the vertices u, v, or x. Hence, (b) holds in this case.

- $x \notin N_u \cap N_v$: Without loss of generality, assume that $x \in N_u - N_v$. We prove that for every vertex $y \in N_v - N_u$, the edge vy is not in A. Assume, on the contrary, that $vy \in A$. By Lemma 5.3, x and y are not adjacent. Also, we know that x and y_1 are not adjacent to v. Therefore, since vy is of distance at most two of xy_1, there must be an edge between y and y_1. Similarly, there is an edge between y and y_2. Therefore, xy_1yy_2 is a 4-cycle in G. This contradiction shows that the edges between v and $N_v - N_u$ are not in A.

  Now, consider $A'' = A - (E_u \cup E_x)$. By Lemma 5.4 every edge in A'' has at least one endpoint in $N_u \cup N_x$ (In fact, Lemma 5.4 implies that it has at least one endpoint in N_x). Let A''_1 be the set of edges in A'' with exactly one endpoint in $N_u \cup N_x$, and A''_2 be the set of edges in A'' whose endpoints are both in $N_u \cup N_x$. Since x is the only vertex which is an endpoint of more than two edges of A', every vertex in $N_u - \{x\}$ is an endpoint of at most two edges of A''_1. Also, by Lemma 5.2, every vertex in N_x is an endpoint of at most two edges of A''_1. Therefore, we can use the argument of case (i) to show that A''_1 contains at most 12 edges and A''_2 contains at most one edge. Therefore, $A - (E_u \cup E_x)$ contains at most 13 edges. Thus, in this case (a) holds.

(iii) At least two vertices in S are an endpoint of more than two edges of A'_1:
Let x, x' be two vertices in S which are each endpoints of at least three edges of A'_1. Since by C_4-freeness, x and x' cannot have more than one common neighbour, there are at least two edges xy_1 and xy_2 in A such that y_1 and y_2 are not adjacent to x'. Similarly, there are at least two edges $x'y'_1$ and $x'y'_2$ in A such that y'_1 and y'_2 are not adjacent to x. Therefore, for every $i, j \in \{1, 2\}$, since xy_i and $x'y'_j$ must be of distance at most two, either x is adjacent to x', or y_i and y'_j are adjacent. Therefore, if x is not adjacent to x', $y_1y'_1y_2y'_2$ will be a C_4 in G. Thus, x must be adjacent to x'. Therefore, by Lemma 5.3, either x and x' must be both in N_u, or they must be both in N_v. Assume, without loss of generality, that they are both in N_u. We prove that every edge in A is incident to u, x, or x'. Assume, on the contrary, that ab is an edge in A which is not incident to u, x, or x'. By Lemma

5.4, at least one of the vertices a, b is adjacent to x, and similarly, at least one of them is adjacent to x'. If one of them is adjacent to x, and the other is adjacent to x', the vertices a, b, x, x' would form a 4-cycle. Therefore, either a or b is adjacent to both x and x'. Without loss of generality, assume that a is adjacent to both x and x'. Therefore, $axux'$ is a 4-cycle. This contradiction shows that every edge in A is incident to u, x, or x'. Therefore, (b) holds in this case.

■

Remark 1. The constant 13 in Theorem 5.5 is not optimal. Using a straightforward (but lengthy) argument, it is possible to replace this constant by 7. This improves the running time of the algorithm greatly.

Remark 2. We can use the above theorem to prove that the size of the largest antimatching in a C_4-free graph is at most $\max\{3\Delta - 1, 2\Delta + 12\}$.

Chapter 6

Concluding remarks

In this thesis, we presented some new results and a survey of previous results about the strong edge-colouring problem. There are many open problems about strong edge colouring. The most important open problems are Conjecture 2.1 of Erdős and Nešetřil and Conjecture 2.2 of Faudree et al. These conjectures are still widely open. In the present research, we tried to use the probabilistic method to prove this conjecture asymptotically. We applied a method that Kim [28] used to prove an upper bound on the chromatic number of graphs of girth at least five, but as we saw in Chapter 4, we need to assume that the graph is C_4-free in order to be able to prove the concentration. There is a similar situation in Kim's result. He needs to assume that the graph is C_4-free in order to be able to prove the concentration. This suggests that one possible approach is to try to use Johansson's method. Johansson [24] used another probabilistic argument to relax the condition that the graph does not have any C_4, and prove an upper bound on the chromatic number of triangle-free graphs. Also, in [25], he proved an upper bound on the chromatic number of K_r-free graphs. Although the square of the line graph of a general graph is neither triangle-free nor K_r-free, it might be helpful to use a variant of the probabilistic procedures that are used in Johansson's proofs. However, Johansson's method is very complicated.

There are many other open problems regarding strong edge-colourings. Some open problems were mentioned in Section 2.7. In the following sections, we will see several other open problems.

6.1 Strong chromatic index of C_5-free graphs

We conjecture that the following generalization of Conjecture 2.2 is also true.

Conjecture 6.1 *The strong chromatic index of every C_5-free graph is at most Δ^2.*

The following theorem supports the above conjecture.

Theorem 6.1 *Every C_5-free graph G which does not have an induced matching of size two has at most Δ^2 edges.*

Proof: The theorem is clearly true for $\Delta = 1$. Therefore, assume $\Delta > 1$. Let v be a vertex of maximum degree in G, and S be the set of vertices adjacent to v. We will show that every edge in G has at least one endpoint in S.

Assume, by way of contradiction, that xy is an edge such that $x \notin S$ and $y \notin S$. First, we prove that both x and y are adjacent to at least one of the vertices in S. If y is not adjacent to any vertex in S, then x must be adjacent to every vertex in S, for otherwise there is a vertex u in S such that uv and xy form an induced matching of size two. Thus, x is adjacent to all the vertices in S, and therefore its degree is at least one more than the degree of v, which contradicts with the assumption that v is a vertex of maximum degree. This contradiction shows that both x and y are adjacent to at least one vertex in S. If x is adjacent to $u_1 \in S$ and y is adjacent to $u_2 \in S$, and $u_1 \neq u_2$, then vu_1xyu_2 is a 5-cycle in G. Therefore both x and y are adjacent to the same vertex $u \in S$ and are not adjacent to any other vertex of S. So, if $u' \neq u$ is another vertex in S, then $u'v$ and xy form an induced matching of size two, which is a contradiction.

Thus, every edge in G has at least one endpoint in S. This, together with the assumption that the degree of every vertex in G (and therefore $|S|$) is at most Δ, implies that there are at most Δ^2 edges in G. ■

6.2 List strong edge-colouring

It is natural to define the *list strong edge-colouring* of a graph G as a list colouring of the square of the line graph of G, and the *list strong chromatic index* of G as the choice number of the square of the line graph of G. We conjecture that the list strong chromatic index of every graph is equal to its chromatic index. In particular, we conjecture that Conjectures

2.1, 2.2, and 2.3 also hold for the list strong chromatic index. This generalization may be useful in the proof of these conjectures.

This conjecture is a special case of a recent conjecture by D. R. Woodall [29]. The conjecture states that the choice number of every graph which is the square of another graph is equal to its chromatic number. This conjecture seems to be a very difficult conjecture.

Notice that in Section 4.1, indeed we have proved an upper bound on the list strong chromatic index of C_4-free graphs. Therefore, our result implies that the generalized version of Conjecture 2.1 is true for C_4-free graphs with maximum degree large.

6.3 r-strong edge-colouring

An r-strong edge-colouring of a graph G is an assignment of colours to the edges of G in such a way that every two edges of distance at most r receive different colours. In other words, an r-strong edge-colouring of G is a distance-r colouring of the line graph of G, or a vertex colouring of $L(G)^r$. The case $r = 2$ corresponds to the strong edge-colouring. This is perhaps the most natural generalization of the strong edge-colouring. Finding the maximum number of colours necessary in an r-strong edge-colouring of a graph of maximum degree Δ is an open problem which is not studied yet.

Erdős and Nešetřil in [13] asked a simpler question: Determine the smallest integer $h_r(\Delta)$ so that every graph G of maximum degree Δ and with $h_r(\Delta)$ edges contains two edges e_1 and e_2 so that the shortest path joining one of the endpoints of e_1 to one of the endpoints of e_2 has length at least r. In [13], it is also remarked that the above problem seems to be interesting only if there is a nice expression for $h_r(\Delta)$.

6.4 Algorithmic open problems

There are many open problems regarding algorithmic aspects of the strong edge colouring. Determining the computational complexity of many special cases of the problems Strong Edge-Colouring and Maximum Antimatching and also Maximum Induced Matching is open. For example, it may be interesting to consider the restriction of the above problems to special classes of graphs such as k-trees, partial k-trees, planar graphs, interval graphs, etc. Also, determining the computational complexity of

Maximum Antimatching for simple bipartite graphs is an open problem.

Also, in the proof of Theorem 5.2, the maximum degree of the graph G that is constructed is $K-1$. One may ask the following question: For every fixed K, what is the minimum value of Δ such that Strong Edge-Colouring is **NP**-complete for graphs of maximum degree Δ? This problem is probably very difficult, since the similar problem about the ordinary vertex colouring is very difficult and has remained open.

Another question is to find a constructive proof of Theorem 4.2. Finding constructive proofs for theorems which are proved probabilistically is usually a difficult task. For Theorem 4.2, the best known example is a C_4-free graph with strong chromatic index at least $\Delta\sqrt{\Delta}$. This example is constructed using projective planes and it is presented in [15] in order to show that there are graphs for which $am(G)$ and $s\chi'(G)$ are of different orders of magnitude.

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