Jan Krajíček , Pavel Pudlák , and Gaisi Takeuti. *Bounded arithmetic and the polynomial time hierarchy. Annals of pure and applied logic*, vol. 52 (1991), pp. 143-153.

Samuel R. Buss. *Relating the bounded arithmetic and polynomial time hierarchies.* Ibid., vol 75 (1995), pp. 67-77.

Domenico Zambella. *Notes on polynomially bounded arithmetic. The journal of symbolic logic,* vol. 61 (1996), pp. 942-966.

The polynomial hierarchy PH extends the fundamental complexity classes P $= \Sigma_0^p$ and NP $= \Sigma_1^p$ to classes Σ_i^p and Π_i^p $(i = 0, 1, 2, ...)$ whose members are defined by polynomial-time relations preceded by appropriate strings of bounded quantifiers. Buss, in his 1985 thesis, introduced a related hierarchy $S_2^1 \subseteq T_2^1 \subseteq S_2^2 \subseteq T_2^2 \subseteq ...$ of arithmetic theories. Here Σ_i^b and Π_i^b $(i > 0)$ are classes of bounded formulas whose members define respectively the Σ_i^p and the Π_i^p relations in PH. The theory T_2^i has the induction scheme for Σ_i^b formulas, while S_2^i has the apparently weaker length induction scheme (induction on the binary length of numbers) for Σ_i^b formulas. The functions definable in S_2^1 by Σ_1^b formulas are precisely the polynomial-time functions, and more generally the functions definable in S_2^{i+1} by Σ_{i+1}^b formulas are those polynomial-time reducible to Σ_i^p relations.

The fundamental question in complexity theory is whether P = NP, or equivalently whether PH collapses to P. By extension, an important open question is whether PH collapses to one of its members Σ_i^p. The corresponding question for Buss's hierarchy is whether the union class S_2 collapses to one of its members S_2^i, which is equivalent to the question of whether S_2 is finitely axiomatizable. As far as we know, PH could collapse without S_2 collapsing, but it is easy to see that if the collapse of PH is provable in S_2, then the S_2 hierarchy collapses.

The three papers under review collectively establish the converse: If the S_2 hierarchy collapses then S_2 proves that PH collapses. Specifically, the first paper, [KPT], proves that if $T_2^i = S_2^{i+1}$ then PH collapses to $\Sigma_{i+2}^p = \Pi_{i+2}^p$. The paper leaves open the question as to whether the collapse is provable in S_2. However it establishes an interesting result, now known as the KPT witnessing theorem, which is used by each of the other two papers (independently of each other) to establish that S_2 proves a weaker form of the collapse. Specifically, the other two papers [B] and [Z] prove that if $T_2^i = S_2^{i+1}$, then $T_2^i = S_2$ and T_2^i proves that PH collapses to $\Sigma_{i+3}^p = \Pi_{i+3}^p$ (and in fact to the Boolean hierarchy over Σ_{i+2}^p).

Although the proof of the main result in [Z] is less clearly presented than in [B], a nice aspect of [Z] is that S_2 is replaced by an equivalent second-order theory BA, which has a very pleasing syntax and seems more elegant than either S_2 or earlier equivalent second-order theories. [Z] gives model-theoretic proofs of some basic results of bounded arithmetic in the framework of BA, providing alternatives to the model-theoretic proofs in the framework of S_2 in the books by Hájek/Pudlák and by Krajíček .

The base case of the collapse results, in all three papers, is different from the formulations above, and is especially interesting. Let QPV (a.k.a. PV_1) be the first-order theory with function symbols for all polynomial-time functions, and with defining axioms for these functions and induction for open formulas. The papers (collectively) prove that if S_2^1 is conservative over QPV, then PH collapses to P/poly (relations computable in polynomial-time with polynomial "advice") and provably collapses to NP/poly. It follows from the methods of Karp and Lipton that PH collapses to $\Sigma_2^p = \Pi_2^p$ and provably collapses to $\Sigma_3^p = \Pi_3^p$.

We now outline the proof of this base case (which is similar to the general case), mainly following [B]. The relevant form of the KPT witnessing theorem states that if QPV proves $\forall x \exists y \forall z B(x, y, z)$ for an open formula B, then there is a finite sequence of polynomial-time functions $f_1, ..., f_k$ such that QPV proves the finite disjunction $\bigvee_{i=1}^k B(a, f_i(a, b_1, ..., b_{i-1}), b_i)$. (The proof is an easy application of the Herbrand theorem, using the fact that QPV is a universal theory.) Now the assertion

"Every sequence $\phi_1, ..., \phi_n$ of propositional formulas has an initial sequence of maximal length ℓ of satisfiable formulas" has the form $\forall x \exists y \forall z B(x, y, z)$, and is provable in S_2^1 by length induction on a Σ_1^b formula expressing the condition that the first ℓ formulas of the sequence coded by x are satisfiable. Thus if we assume that S_2^1 is conservative over QPV, then we can apply the KPT witnessing theorem, which after some work, shows the following: There is a finite sequence $g_1, ..., g_k$ of polynomial-time functions such that QPV proves that for every sequence $\phi_1, ..., \phi_k$ of formulas with respective satisfying assignments $w_1, ..., w_k$ there is an $i, 1 \leq i \leq k$, such that $g_i(\phi_1, ..., \phi_k, w_1, ..., w_{i-1})$ codes a satisfying assignment for ϕ_i.

At this point we follow [KPT] to explain how to find the advice string W_n for each n, where W_n allows us to efficiently find a satisfying assignment for an arbitrary satisfiable formula of length n. Let V be any set of satisfiable formulas of length n. For each subset $\{\phi_1, ..., \phi_k\}$ of V of size k, one of the functions g_i above provides a satisfying assignment to ϕ_i using only $\overline{\phi}$ and $w_1, ..., w_{i-1}$ as arguments. Hence we can assign to each cardinality k subset X of V a cardinality $k-1$ subset $Y = X - \{\phi\}$ of V such that some g_i provides a satisfying assignment for ϕ, from Y together with satisfying assignments to the formulas in Y. The ratio of domain size to range size of this map is at least $(|V| - k + 1)/k$, and hence some cardinality $k-1$ subset Y of V, together with a list W_Y of satisfying assignment for each formula in Y, provides satisfying assignments to at least $(|V| - k + 1)/k$ formulas ϕ in V, using the functions g_i. This Y and W_Y provide part of the advice W_n. We now remove all these formulas ϕ from V which can be satisfied using this advice, forming a smaller set V', and repeat this process, finding a fruitful cardinality $k-1$ subset Y' of V'. If initially V is the set of all satisfiable formulas of length n, then after a few iterations of this process we have found a small collection of advice pieces (Y, W_Y) which, together with a small collection $\{...(\phi, w)...\}$ of satisfying assignments to take care of a few remaining formulas, provides the required advice W_n.

Unfortunately the above counting argument does not appear to be formalizable in S_2, so [B] and [Z] give different arguments. Following [B], we show how to define advice W_n', which can be used to put the satisfiability problem in co-NP/poly, provably in S_2^1. Again we use the functions $g_1, ..., g_k$ described two paragraphs above. The advice W_n' is a maximal length sequence $(\phi_{\ell+1}, ..., \phi_k)$ of formulas of length n such that for all lists $(\phi_1, ..., \phi_\ell)$ and $(w_1, ..., w_\ell)$, such that assignment w_i satisfies formula ϕ_i of length n, $1 \leq i \leq \ell$, there exists $j, 1 \leq j \leq \ell$ such that $g_j(\phi_1, ..., \phi_k, w_1, ..., w_{j-1})$ codes a satisfying assignment for ϕ_j. (We know $\ell = k$ and the empty list of formulas satisfies this condition, but we want $\ell \geq 1$ to be as small as possible.) Then an arbitrary formula ϕ_ℓ of length n is satisfiable iff for all lists $(\phi_1, ..., \phi_{\ell-1})$ and satisfying assignments $(w_1, ..., w_{\ell-1})$, there exists $j, 1 \leq j \leq \ell$ such that $g_j(\phi_1, ..., \phi_k, w_1, ..., w_{j-1})$ codes a satisfying assignment for ϕ_j. (The "if" direction follows from the minimality of ℓ.) Hence we have expressed length n satisfiability with a Π_1^b formula involving the advice W_n', which shows SAT is in co-NP/poly, and hence PH collapses to NP/poly = co-NP/poly, as desired.

STEPHEN COOK