STA 4503, Spring 2013 — Theory Assignment #2

Due at the start of class on March 13. Please hand it in on 8 1/2 by 11 inch paper, stapled in the upper left, with no other packaging.

This assignment is to be done by each student individually. You may discuss it in general terms with other students, but the work you hand in should be your own. In particular, you should not leave any discussion with someone else with any written notes (either on paper or in electronic form).

In this assignment, we consider Markov chains designed for use in sampling values of a state, x, assumed here to be discrete, according to probabilities $\pi(x)$.

Recall that the Metropolis-Hastings algorithm defines a transition with probabilities $T(x'|x)$ in terms of a distribution, $g(x^*|x)$, for a proposal, which is then accepted with probability

$$a(x, x^*) \;=\; \min\left[1,\; \frac{\pi(x^*)\, g(x|x^*)}{\pi(x)\, g(x^*|x)}\right]$$

The transition probabilities are therefore as follows, when $x' \neq x$:

$$T(x'|x) \;=\; g(x'|x)\, a(x, x')$$

Recall also that if T_1 and T_2 are two transitions that each leave π invariant, then the mixed transition, defined as

$$T_{\text{mixed}}(x'|x) \;=\; (1/2)\,(T_1(x'|x) \;+\; T_2(x'|x))$$

will also leave π invariant. The distribution π will also be left invariant by a transition that cycles between T_1 and T_2, which may be defined as follows, in terms of the intermediate state $x^\dagger$:

$$T_{\text{cycle}}(x'|x) \;=\; \sum_{x^\dagger} T_1(x^\dagger|x) T(x'|x^\dagger)$$

Finally, recall that transitions T are said to be irreducible if for any x and x', there is an n such that $T^n(x'|x) > 0$.

Suppose now that T_1 and T_2 are Metropolis-Hastings updates, based on the proposal distributions $g_1(x^*|x)$ and $g_2(x^*|x)$. Define the transition T_{merge} to be the Metropolis-Hastings transition with proposal distribution $g_{\text{merge}}(x^*|x) \;=\; (1/2)\,(g_1(x^*|x) \;+\; g_2(x^*|x))$.

a) Prove that if g_1 and g_2 are symmetrical — that is, $g_1(x^*|x) = g(x|x^*)$ and $g_2(x^*|x) = g_2(x|x^*)$ for all x and x^* — then T_{mixed} and T_{merge} are the same thing.

b) Show by giving an example that T_{mixed} and T_{merge} may not be the same if g_1 and g_2 are not symmetrical.

c) Prove that if either T_1 or T_2 are irreducible, then T_{mixed} is irreducible.

d) Show by giving an example that it is possible for T_1 and T_2 to both be irreducible Metropolis-Hastings transitions, but for T_{cycle} to not be irreducible. Hints: There's no requirement here for T_1 and T_2 to be aperiodic. If $T_1(x|x) > 0$ for all x or $T_2(x|x) > 0$ for all x, T_{cycle} will obviously be irreducible, so you should look for examples where that's not so.