Relating Response Variables to Explanatory Variables

We are often interesting in how a *response variable* relates to an *explanatory* variable (also called a *predictor* variable).

When our interest is in making predictions:

The response variable is what we want to predict.

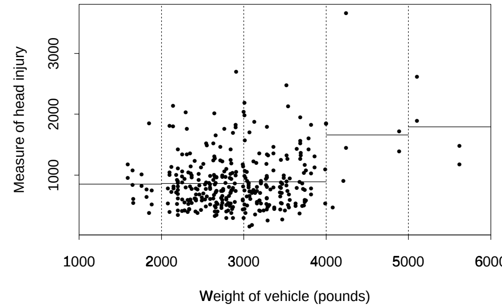
The predictor variable is something we can measure to help us predict

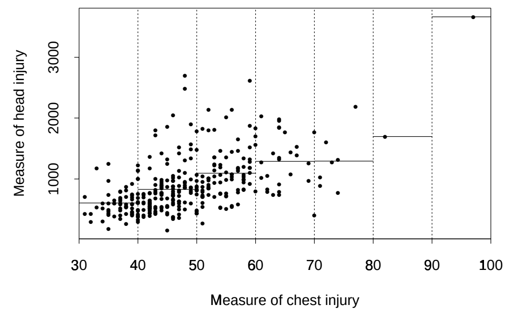
When our interest is in cause-and-effect explanation:

The response variable is what we think is the effect.

The explanatory variable is what we think is the cause.

Prediction by Averaging Nearby Points

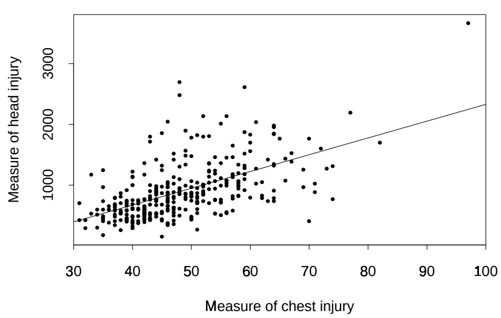




Linear Regression

Sometimes the response variable can be predicted from the explanatory variable by a straight line.

For the crashtest data:



The straight line gives the value to predict for the response variable for each value of the explanatory variable.

The "Least Squares" Regression Line

We can describe a line predicting y from x by the equation

$$\widehat{y} \;=\; a \;+\; b\,x$$

One criterion for the "best" line is the one that minimizes the sum of the squared prediction errors, that is

$$\sum_{i=1}^{n} (y_i - \widehat{y}_i)^2 \;=\; \sum_{i=1}^{n} (y_i - a - b\,x_i)^2$$

The values of a and b that minimize this are

$$a \;=\; \bar{y} - b\bar{x}, \qquad b \;=\; r\,\frac{s_y}{s_x} \;=\; \frac{s_{xy}}{s_x^2}$$

where $\bar{x}$ and $\bar{y}$ are the means, s_x and s_y are the standard deviations, s_{xy} is the covariance of x and y, and r is the correlation of x and y.

Derivation of the Least Squares Line

We need to find a and b that minimize

$$E \;=\; \sum_{i=1}^{n} \Big(y_i - a - b\,x_i\Big)^2$$

The minimum should be at a point where the derivative with respect to a is zero:

$$\frac{\partial E}{\partial a} \;=\; -2\sum_{i=1}^{n} \Big(y_i - a - b\,x_i\Big) \;=\; 0$$

This is equivalent to

$$\frac{1}{n}\sum_{i=1}^{n} \Big(y_i - a - b\,x_i\Big) \;=\; \bar{y} - a - b\,\bar{x} \;=\; 0$$

which implies $a = \bar{y} - b\,\bar{x}$.

One consequence: The regression line goes through the point $(\bar{x}, \bar{y})$.

Derivation of Least Squares (Continued)

At the minimum, the derivative with respect to b should also be zero:

$$\frac{\partial E}{\partial b} \;=\; -2\sum_{i=1}^{n} \Big(y_i - a - b\,x_i\Big)x_i \;=\; 0$$

Using $a = \bar{y} - b\bar{x}$, we get

$$-2\sum_{i=1}^{n} \Big(y_i - (\bar{y} - b\bar{x}) - b\,x_i\Big)x_i \;=\; 0$$

from which

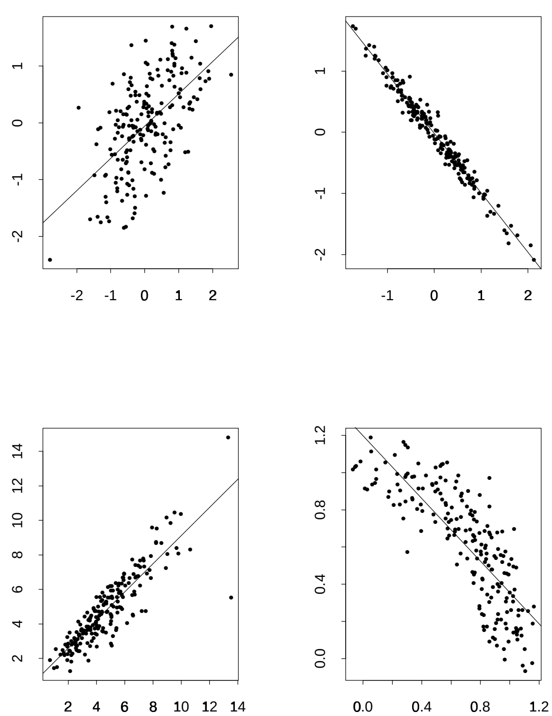
$$\frac{1}{n-1}\sum_{i=1}^{n} \Big((y_i - \bar{y}) - b(x_i - \bar{x})\Big)x_i \;=\; 0$$

Notice that $\sum(y_i - \bar{y}) = 0$, $\sum(x_i - \bar{x}) = 0$, and hence $\sum\Big((y_i - \bar{y}) - b(x_i - \bar{x})\Big)\bar{x} = 0$. So,

$$\frac{1}{n-1}\sum_{i=1}^{n} \Big((y_i - \bar{y}) - b(x_i - \bar{x})\Big)(x_i - \bar{x}) \;=\; 0$$

which is $s_{xy} - bs_x^2 = 0$, or $b = \dfrac{s_{xy}}{s_x^2}$.

Some Examples of Regression Lines

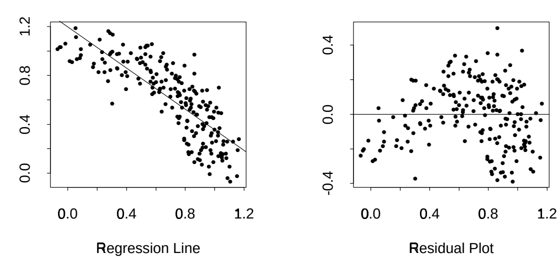


The Residuals for Observations

The *residual* is the difference between the observed response and the predicted response:

$$\text{residual for } y_i \;=\; y_i - \hat{y}_i \;=\; y_i - (a + bx_i)$$

The mean of the residuals from least squares regression is always zero, but there could be a pattern to them:



This pattern here (negative residual for large or small x, positive for x in the middle) shows that the relationship is not exactly linear.