STA 247 — Answers to Quiz #1, 2001-09-30, 3:10pm – 35 minutes long

No books, no notes, and no calculators may be used.

Write your answers in ink in the space provided, cross out any errors.

Show enough of your work to indicate how you obtained your answer. Just a number with no indication of the method used to obtain it (or with an incorrect method) will receive zero marks.

You must give actual numerical answers (decimals such as 0.15 or simple fractions such as 3/13), not just a formula. If getting a numerical answer involves arithmetic on number bigger than 1000, you've either made a mistake, or you should think of an easier way to solve the problem.

Questions Q1, Q2, Q3, and Q4 involve the following situation: Twenty students are attending a lecture, of which 3 are first-year students, 12 are second-year students, 4 are third-year students, and 1 is a fourth-year student. The fire alarm goes off, so all the students leave the room. The doorway is small enough that they have to leave one at a time. In the confusion, the students move around randomly, so the order in which they leave is equally likely to be any of the possible orders.

Q1 (20 marks): Let E be the event that the first two students who leave the room are first-year students. Find $P(E)$.

We can find $P(E)$ *as* $\#(E)\,/\,\#(S)$.

If we choose a sample space in which we look at only the first two students to leave (the others are irrelevant) and pay attention to their order, then $\#(S) = 20\cdot 19$, *and* $\#(E) = 3\cdot 2$, *so* $P(E) = 3/190$.

If we use a sample space in which the outcomes specify the order in which all 20 students leave, then $\#(S) = 20!$, *and* $\#(E) = (3\cdot 2)\cdot 18!$. *Since* $20!\,/\,18! = 20\cdot 19$, *we again get* $P(E) = 3/190$.

It's also possible to use a sample space in which the order of the first two students to leave is ignored. In that case $\#(S) = 20\cdot 19\,/\,2$ *and* $\#(E) = 3\cdot 2\,/\,2$, *with the answer again being* $P(E) = 3/190$.

Q2 (20 marks): Let A be the event that the first student who leaves the room is a second-year student, and let B be the event that exactly one of the first two students who leave is a first-year student. Find $P(A|B)$.

By definition $P(A|B) = P(A\cap B)/P(B)$. *Note that* $A\cap B$ *is the event that the first student to leave is a second-year student and the second student to leave is a first-year student, since that's the only way that both* A *and* B *can happen.*

We can use a sample space just like that for drawing two balls from an urn with 20 balls, without replacement, paying attention to order. With this sample space, $\#(A\cap B) = 12\cdot 3$, *so* $P(A\cap B) = 12\cdot 3\,/\,\#(S)$, *and* $\#(B) = 3\cdot 17 + 17\cdot 3$, *so that* $P(B) = 2\cdot 3\cdot 17\,/\,\#(S)$. *The answer is therefore* $(12\cdot 3)\,/\,(2\cdot 3\cdot 17) = 6/17$.

Q3 (20 marks): What is the probability that the first student who leaves the room and the last student who leaves the room are in the same year? (In other words, that the first and last are both first-year students, or both second-year students, or both third-year students, or both fourth-year students.)

We can split this event into a union of disjoint events, so that the total probability of the event is the sum of the probabilities of these disjoint events:

$$\begin{aligned} P(\textit{first and last in same year}) &= P(\textit{first and last both 1st year}) + P(\textit{first and last both 2nd year}) \\ &\quad + P(\textit{first and last both 3rd year}) + P(\textit{first and last both 4th year}) \\ &= (3/20)\,(2/19) \; + (12/20)\,(11/19) \; + (4/20)\,(3/19) \; + (1/20)\,(0/19) \\ &= 15/38 \end{aligned}$$

Q4 (25 marks): Joe and Mary are two students in the lecture room. What is the probability that as the students leave the room, Joe and Mary will leave one after the other (either Joe immediately follows Mary, or Mary immediately follows Joe).

This can be solved in several ways.

Here is one way. We don't care whether Joe immediately follows Mary, or Mary immediately follows Joe. So we can look at the sample space in which the outcomes are the two positions in which Mary and Joe leave the room (each from 1 to 20, with order not mattering). For example, one outcome would be that they are the 2nd and 4th persons to leave the room. The number of ways of choosing these two positions from the 20 positions is $C(20,2) = 20 \cdot 19\,/\,2$. *The number of such outcomes in which Mary and Joe leave one after the other is 19 (from 1st and 2nd positions to 19th and 20th positions). The probability of Mary and Joe leaveing one after the other is therefore* $19\,/\,(20 \cdot 19\,/\,2) = 1/10$.

Another way to solve this problem is to first ask how likely Joe and Mary are to be next to each other if the students are arranged randomly in a circle, and then multiply this by the probability that if they are next to each other, they will remain next to each other if the circle is broken at a random point to give a line of students who leave the room in order.

Q5 (15 marks): Prove that for any events A and B, if $P(A) > 0$ then $P(A \cap B|A) \geq P(A \cap B)$. Justify the steps in your proof.

$$\begin{aligned} P(A \cap B|A) &= \frac{P(A \cap (A \cap B))}{P(A)} && \textit{by definition} \\ &= \frac{P(A \cap B)}{P(A)} && \textit{since } A \cap A = A \\ &\geq P(A \cap B) && \textit{since } 0 < P(A) \leq 1 \end{aligned}$$