STA 247 — Answers to problem set #3

Question 1: Suppose that a technical support person takes 6 minutes on average to deal with one customer's problem, that the standard deviation of the time required to deal with one customer is 2 minutes, and that it never takes more than 15 minutes to deal with one customer. Suppose that each customer's problem is independent of the problems of other customers. Find a good approximation to the probability that the technical support person will take more than 175 minutes to deal with the problems of 25 customers.

Since the time to deal with a customer is bounded by 0 and 15 minutes, its mean must exist and its variance must be finite. The Central Limit Theorem therefore applies to the total time for dealing with many customers, when these times are independent.

The mean time for dealing with 25 customers will be $6 \times 25 = 150$ *minutes. The standard deviation of the time to deal with 25 customers will be* $2 \times \sqrt{25} = 10$ *minutes. The distribution of the total time,* X*, to deal with 25 customers will therefore be approximately* $N(150, 10^2)$*. The probability that dealing with these customers takes more than 175 minutes is* $P(X > 175) = 1 - P(X \leq 175)$*, with* $P(X \leq 175)$ *being approximately the CDF of the* $N(150, 10^2)$ *distribution at 175. We can get this using R as follows:*

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> 1-pnorm(175,150,10)
[1] 0.006209665
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If we define $Z = (X - 150)/10$*, the distribution of* Z *will be approximately* $N(0, 1)$*, and we will have* $P(X > 175) = P(Z > 2.5) = 1 - P(Z \leq 2.5)$*. We could find this with R as follows:*

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> 1-pnorm(2.5)
[1] 0.006209665
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The answer is the same as above. When using R, there's no good reason to transform X *to* Z*, but this is useful if you have to use an old-fashioned table printed on paper that gives values for the CDF of the* $N(0, 1)$ *distribution, but not for the infinite number of other normal distributions.*

Question 2: A cosmic ray detector installed on a mountain in Alberta records the arrival time and energy of every cosmic ray that it detects, and sends the data on these events via satellite to a computer in Toronto. To reduce the number of satellite transmissions needed, the detector waits until it has data on 400 new cosmic ray events before sending the data for these events to Toronto as a single packet. The detector then forgets these events, and waits until 400 more events have occurred before sending the next packet.

Suppose that the time in seconds, T, from when one cosmic ray is detected to when the next cosmic ray is detected has exponential distribution with rate 2, so the probability density function for T is $2e^{-2t}$. Suppose also that times between detections are independent.

a) What are the mean and standard deviation of the distribution for the time between when one packet is sent and the next packet is sent?

The mean of the exp(2) distribution is 1/2 and its variance is $1/2^2 = 1/4$. *The mean of the sum of 400 independent variables with this distribution is* $(1/2) \times 400 = 200$ *and its standard deviation is* $(1/4) \times \sqrt{400} = 5$.

b) Find a good approximation to the probability that the time between one packet and the next packet is greater than 215 seconds. Explain why the approximation you use should be good.

Since the times between cosmic rays are independent and the variance of these times is finite, the Central Limit Theorem applies. The distribution of the time, K*, from one packet being sent to the next packet being sent is therefore approximately* $N(200, 5^2)$. *We can find* $P(K > 215)$ *approximately as follows:*

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> 1-pnorm(215,200,5)
[1] 0.001349898
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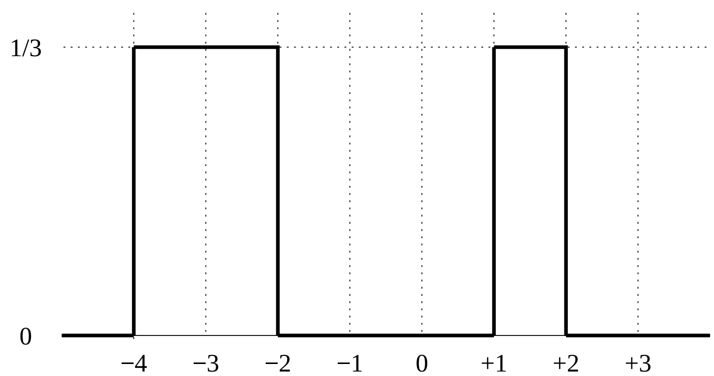
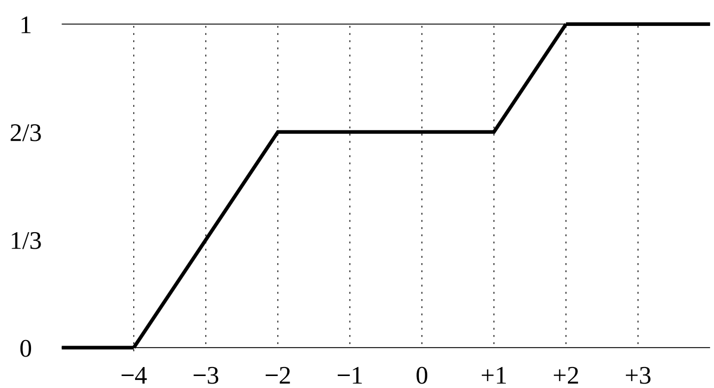
Question 3: Here is a graph of the probability density function for a random variable X:



a) Draw a graph of the cumulative distribution function for this random variable.



b) Compute $P(X \leq 0)$.

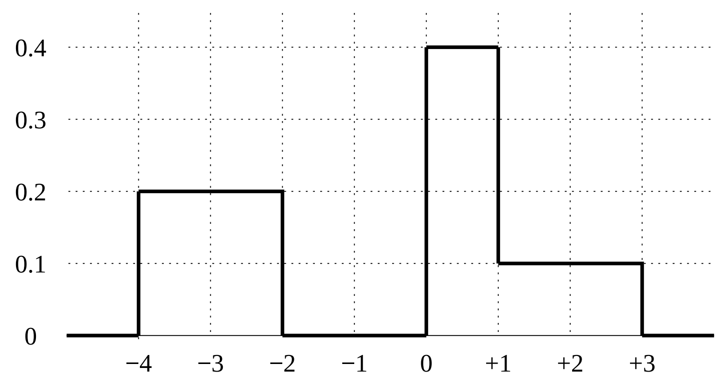
$$P(X \leq 0) \;=\; F_X(0) \;=\; 2/3$$

c) Compute $E(X)$.

$$E(X) \;=\; \int_{-\infty}^{+\infty} x\, f_X(x)\, dx \;=\; \int_{-4}^{-2} x\,(1/3)\, dx \;+\; \int_{1}^{2} x\,(1/3)\, dx \;=\; (-2) \,+\, (1/2) \;=\; -3/2$$

Question 4: Here is a graph of the cumulative distribution function for a random variable X:

a) Draw a graph of the probability density function for this random variable.



b) Find $P(-3 \le X \le 1)$.

$$P(P(-3 \le X \le 1) \;=\; F_X(1) - F_X(-3) \;=\; 0.8 - 0.2 = 0.6$$

c) Compute $E(X)$.

$$E(X) \;=\; \int_{-\infty}^{+\infty} x\, f_X(x)\, dx \;=\; \int_{-4}^{-2} x\, 0.2\, dx \;+\; \int_{0}^{1} x\, 0.4\, dx \;+\; \int_{1}^{3} x\, 0.1\, dx \;=\; -0.6$$

Question 5: Let X and Y be independent random variables. Suppose that X has the geometric distribution with parameter p_X and Y has the geometric distribution with parameter p_Y. Let Z be the minimum of X and Y. Prove that Z has a geometric distribution, and find the parameter, p_Z, of this distribution.

If we view X as the number of "flips" of a coin, X, before a head, where a head has probability p_X, and we view Y as the number of "flips" of a different coin, Y, before a head, where a head has probability p_Y, then we can view Z as the number of pairs of flips, one of coin X and one of coin Y, until either of these coins lands heads. The probability that either coin lands heads is $1-(1-p_X)(1-p_Y) = p_X+p_Y-p_Xp_Y$. The distribution of Z is therefore geometric($p_X+p_Y-p_Xp_Y$).

Question 6: Suppose that the life time of a light bulb has an exponential distribution with mean 50 hours. You wish to study for 5 hours in a room light by a lamp holding such a light bulb.

a) Suppose you put a new light bulb in the lamp when you start studying. What is the probability that the light bulb will last at least as long as you are studying (5 hours)?

   The $\exp(1/50)$ *distribution has mean 50, so this is the distribution of life time, T, of the bulb. The CDF of T is* $1-\exp(-t/50)$. *The probability that the bulb lasts at least 5 hours is* $P(T \geq 5) = 1-P(T < 5) = 1-P(T \leq 5) = 1-(1-\exp(-5/50)) = \exp(-1/10) = 0.905$.

b) Suppose you know that a new light bulb was put in the lamp 12 hours before you start studying, and that the lamp has been on since then. What is the probability that the light bulb will last at least as long as you are studying (5 more hours)?

   This is $P(T \geq 17 \,|\, T \geq 12) = (1–P(T \leq 17)) \,/\, (1–P(T \leq 12) = \exp(-17/50)/\exp(-12/50) = \exp(-1/10) = 0.0905$, *the same as for part (a). That the answers are the same is an illustration of the "memoryless" property of the exponential distribution.*