# STA 247, Fall 2011 — Solution to Assignment #1

**Question 1:** Suppose that we flip a coin six times and roll a 6-sided die once. Suppose also that all outcomes of this experiment (consisting of an ordered sequence of results for the flips (heads or tails) and the number showing on the die after the roll (an integer from 1 to 6)) are equally likely.

Find the following probabilities:

1) The probability that all six flips are heads and the die shows the number 6.

   *The number of outcomes in the sample space is* $2 \times 2 \times 2 \times 2 \times 2 \times 2 \times 6 = 384$. *There is only one outcome for which all six flips are heads and the die shows 6. So the probability of this event is* $1/384$.

2) The probability that the first $R$ flips are heads, where $R$ is whatever number is showing on the die.

   *This event can happen in 6 ways, corresponding to the 6 possible values of* $R$. *When the die shows* $R$*, the first* $R$ *flips must be heads for an outcome to be in this event, and the* $6-R$ *flips after the first* $R$ *flips can be either heads or tails. There are therefore* $2^{6-R}$ *outcomes in this event for a particular value of* $R$. *Adding these up for all values of* $R$*, we find that there are*

   $$2^5 + 2^4 + 2^3 + 2^2 + 2^1 + 2^0 \;=\; 63$$

   *outcomes in this event. We found above that the number of outcomes in the sample space is 384, so the probability of this event is* $63/384$.

   *Alternatively, we can find the probability of this event using the Law of Total Probability: For each possible value of* $R$*, we multiply the probability of that value (always 1/6) and the probability of the first* $R$ *flips being all heads, and add these products for all possible values of* $R$*. The result is*

   $$(1/6) \times ((1/2) + (1/4) + (1/8) + (1/16) + (1/32) + (1/64)) \;=\; (1/6) \times (63/64) \;=\; 63/384$$

3) The probability that the number of flips that are heads times the number showing on the die is 18.

   *This event can occur only when the die shows 3 and the number of heads is 6 or when the die shows 6 and the number of heads is 3. These are disjoint possibilities, so we can just add their probabilities. There is only one way for the die to show 3 and all six flips to land heads, so the probability of that happening is 1/384. The number of outcomes in which the die shows 6 and the number of heads is 3 is*

   $$\binom{6}{3} \;=\; \frac{6!}{3!3!} \;=\; 20$$

   *so the probability of that happening is 20/384. The total probability of the product being 18 is therefore* $(1/384) + (20/384) \;=\; 21/384$.

4) The probability that the number of heads in the six flips is same as the number showing on the die.

*Call the event above $E$. Let $T$ be the event that all six flips land tails. We can use the Law of Total Probability to write*

$$P(E) \;=\; P(E|T)P(T) \;+\; P(E|T^c)P(T^c)$$

*Since there is only one way for all flips to land tails, and the total number of outcomes for the six coin flips is 64, we get that $P(T) = 1/64$, and hence $P(T^c) = 1 - P(T) = 63/64$.*

*If the flips are all tails, there will be zero heads, and it will be impossible for the die roll to match the number of heads (since the die never shows 0). So $P(E|T) = 0$.*

*If at least one flip is a head, the number of heads will be between 1 and 6, and whatever this number is, the chance of the die roll matching this number is 1/6, so $P(E|T^c) = 1/6$.*

*The answer is therefore*

$$P(E) \;=\; 0 \times (1/64) \;+\; (1/6) \times (63/64) \;=\; 63/384$$

*It is possible to solve this question in other ways. For example, one could use the Law of Total Probability the opposite way, in which for each possible value shown by the die (having probability 1/6) one would need to find the probability of a matching number of heads. That's harder than what was done in the solution above, however.*

**Question 2:** A school has a computer lab that is closed for the summer, when no students are there, and then is re-opened in the fall. Sometimes, a computer in the lab that worked before the lab closed is found to not work properly when it is powered up again in the fall.

These computer failures are always due to one or more of the following causes:

A) Failure of the power supply.

B) Failure of the disk drive.

C) Failure of the RAM.

D) Failure of the CPU.

From long experience, it is known that these failure events are independent, and that their probabilities of occurrence (for a single computer) are $P(A) = 0.04$, $P(B) = 0.03$, $P(C) = 0.02$, and $P(D) = 0.01$.

1) Find the probability that a computer will fail for any of these reasons — that is, find $P(F)$, where $F = A \cup B \cup C \cup D$.

*It's easier to find $P(F^c)$ than to find $P(F)$ directly. We can write $F^c$ as*

$$F^c \;=\; (A \cup B \cup C \cup D)^c \;=\; A^c \cap B^c \cap C^c \cap D^c$$

*Now we can use the fact that $A^c$, $B^c$, $C^c$, and $D^c$ are mutually independent, since $A$, $B$, $C$, and $D$ are. This lets us write*

$$\begin{aligned} F^c &= P(A^c)\,P(B^c)\,P(C^c)\,P(D^c) \;=\; (1-P(A))\,(1-P(B))\,(1-P(C))\,(1-P(D)) \\ &= 0.96 \times 0.97 \times 0.98 \times 0.99 \;=\; 0.90345 \end{aligned}$$

*The final answer is therefore $P(F) \;=\; 1 - P(F^c) \;=\; 1 - 0.90345 \;=\; 0.09655$.*

2) Find the probability that a non-working computer has failed for more than one reason — that is, find the conditional probability that more than one of $A$, $B$, $C$, and $D$ has occurred, given that $F = A \cup B \cup C \cup D$ has occurred.

*Let $E$ be the event that the computer failed for more than one reason. The easiest way to express $E$ is in terms of $E^c$ — the event that the computer failed for only one reason, or didn't fail at all:*

$$\begin{aligned} E^c \;=\; & (A^c \cap B^c \cap C^c \cap D^c) \;\cup\; (A \cap B^c \cap C^c \cap D^c) \;\cup\; (A^c \cap B \cap C^c \cap D^c) \\ & \cup\; (A^c \cap B^c \cap C \cap D^c) \;\cup\; (A^c \cap B^c \cap C^c \cap D) \end{aligned}$$

*We need to find $P(E|F) = P(E \cap F) / P(F) = P(E) / P(F)$. The last step here uses the fact that $E \cap F = E$, since if the computer has failed for more than one reason (event $E$), it certainly has failed (event $F$). We found $P(F)$ above. So we just need to find $P(E) = 1 - P(E^c)$. We can find $P(E^c)$ using the fact that the five terms in the union for $E^c$ above are disjoint, so $P(E^c)$ is the sum of the probabilities for these terms:*

$$\begin{aligned} P(E^c) \;=\; & P(A^c \cap B^c \cap C^c \cap D^c) \;+\; P(A \cap B^c \cap C^c \cap D^c) \;+\; P(A^c \cap B \cap C^c \cap D^c) \\ & +\; P(A^c \cap B^c \cap C \cap D^c) \;+\; P(A^c \cap B^c \cap C^c \cap D) \end{aligned}$$

*Using the fact that the failures are independent, we can then get*

$$\begin{aligned} P(E^c) \;=\; & 0.96 \times 0.97 \times 0.98 \times 0.99 \;+\; 0.04 \times 0.97 \times 0.98 \times 0.99 \;+\; 0.96 \times 0.03 \times 0.98 \times 0.99 \\ & +\; 0.96 \times 0.97 \times 0.02 \times 0.99 \;+\; 0.96 \times 0.97 \times 0.98 \times 0.01 \end{aligned}$$

*This expression gives $P(E^c) = 0.99660$, and hence $P(E) = 0.00340$. The final answer is therefore $P(E|F) = 0.00340 / 0.09655 = 0.0352$.*

**Question 3:** Suppose that Joe draws $k$ balls from an urn containing $n$ red balls and $n$ green balls, without replacing the balls after they are drawn. Similarly, Mary draws $k$ balls from an urn containing $m$ red balls and $m$ green balls, without replacing the balls after they are drawn. We want to compute the probability that Joe and Mary will draw the the same number of red balls.

1) Write an R function to compute this, which takes $n$, $m$, and $k$ as arguments. (These arguments must be positive integers, and $2n$ and $2m$ must be at least as big as $k$, but you don't have to check for this in your program.) This function should use one or more of the `sum`, `prod`, `factorial`, and `choose` functions. Note that `factorial` and `choose` can take vectors as arguments, and then return a vector of results. Note also that a vector that is a sequence can be created with an expression like `i:j`. Test your function on at least the following values for the arugments:

   $n = 20,\ m = 30,\ k = 1$
   $n = 20,\ m = 30,\ k = 2$
   $n = 200,\ m = 300,\ k = 2$
   $n = 2000,\ m = 3000,\ k = 2$
   $n = 50,\ m = 60,\ k = 17$

Comment on the results. Can you see why some of them are what they are (at least approximately) with simple calculations?

*Here is an R function for computing the probability that Joe and Mary draw the same number of red balls:*

```
prsame <- function (n, m, k)
{
  r <- max(0,k-n,k-m) : min(k,n,m)
  pr.1 <- choose(n,r) * choose(n,k-r) / choose(2*n,k)
  pr.2 <- choose(m,r) * choose(m,k-r) / choose(2*m,k)
  sum(pr.1*pr.2)
}
```

*It works by adding the probabilities of Joe and Mark drawing $r$ balls for all possible values of $r$. Note that Joe can't draw more than $n$ red balls, and Mary can't draw more than $m$ red balls. Also, if $k > n$, Joe must draw at least $k - n$ red balls, and if $k > m$, Mary must draw at least $k - m$ red balls. The variables **`pr.1`** and **`pr.2`** hold the vectors of probabilities for Joe and Mary to draw $r$ balls for each value of $r$ in the vector **`r`**. Since Joe and Mary drawn independently, we can multiply these probabilities to get the probabilities of Joe and Mary both drawing $r$ balls, and sum these over the possible values of $r$.*

*It also works to set **`r`** to **`0:k`**, since the **`choose`** function returns 0 for invalid arguments.*

*The output of this function for the set of arguments above is shown below:*

```
> prsame(n=20,m=30,k=1)
[1] 0.5
> prsame(n=20,m=30,k=2)
[1] 0.3804867
> prsame(n=200,m=300,k=2)
[1] 0.3755235
> prsame(n=2000,m=3000,k=2)
[1] 0.3750521
> prsame(n=50,m=60,k=17)
[1] 0.1474633
```

*The output with $k = 1$ is easy to understand, since when both Joe and Mary draw a single ball, the probability of it being red is 1/2. The probability of them both drawing a red or neither drawing a red is therefore $(1/2) \times (1/2) + (1/2) \times (1/2) = 1/2$. When $k = 2$, the probability that Joe and Mary draw the same number of red balls seems to be approaching 0.375 as $n$ and $m$ get bigger and bigger. This makes sense because when $n$ and $m$ are much bigger than $k$, drawing without replacement is almost the same as drawing with replacement, since even with replacement it's unlikely that the same ball will be drawn more than once. If Joe and Mary each draw two balls with replacement, the probability that they both draw zero red balls is $(1/4) \times (1/4) = 1/16$, the probability that they both drawn one red ball is $(1/2) \times (1/2) = 1/4$, and the proability that they both draw two red balls is $(1/4) \times (1/4) = 1/16$. Adding these up, we get a total probability of $3/8 = 0.375$.*

2) Try the function you wrote above with $n = 600$, $m = 500$, and $k = 400$. You should see a result of `NaN`, indicating that floating point overflow occurred as some point in the computation, so the final result was meaningless. Write a new version of the function that avoids this problem by working in terms of the logarithms of the values, until the final result needs to be computed. The (natural) logarithm is computed with the `log` function, and its inverse, the exponential function, is computed with `exp`. The `lfactorial` and `lchoose` functions compute the log of the factorial and "choose" functions. Test your new function on the same sets of arguments as above, for which it should produce the same (or very close to the same) answer, as well as on $n = 600$, $m = 500$, and $k = 400$ and on $n = 6000$, $m = 5000$, and $k = 4000$, for which it should not produce `NaN`.

*Here is a modified function that uses logarithms, until the final probabilities for each value of* $r$ *are computed:*

```
prsame.log <- function (n, m, k)
{
  r <- max(0,k-n,k-m) : min(k,n,m)
  pr.1 <- lchoose(n,r) + lchoose(n,k-r) - lchoose(2*n,k)
  pr.2 <- lchoose(m,r) + lchoose(m,k-r) - lchoose(2*m,k)
  sum(exp(pr.1+pr.2))
}
```

*Here are tests of the original and modified functions:*

```
> prsame(n=600,m=500,k=400)
[1] NaN
> prsame.log(n=600,m=500,k=400)
[1] 0.03542540
> prsame.log(n=6000,m=5000,k=4000)
[1] 0.01120863
```