STA 247 — Solutions to Assignment #3

1. We can find $P(Z=z)$ by adding up all the combinations of values for X and Y that can sum to z, as follows:

$$\begin{aligned}
P(Z=z) &= \sum_{x=0}^{z} P(X=x,\, Y=z-x) \\
&= \sum_{x=0}^{z} P(X=x)\, P(Y=z-x) \quad \text{since } X \text{ and } Y \text{ are independent} \\
&= \sum_{x=0}^{z} \frac{\lambda^x}{x!} e^{-\lambda} \frac{\lambda^{z-x}}{(z-x)!} e^{-\lambda} \quad \text{using the formula for Poisson probabilities} \\
&= \sum_{x=0}^{z} \frac{\lambda^z}{x!(z-x)!} e^{-2\lambda} \\
&= \lambda^z e^{-2\lambda} \sum_{x=0}^{z} \frac{1}{x!(z-x)!} \\
&= \frac{\lambda^z}{z!} e^{-2\lambda} \sum_{x=0}^{z} \frac{z!}{x!(z-x)!} \\
&= \frac{\lambda^z}{z!} e^{-2\lambda} \sum_{x=0}^{z} \binom{z}{x}
\end{aligned}$$

Now, note that the sum above is the total number of ways of choosing x things from a set of z things, for any value of x from 0 to z. This is just the number of ways of choosing *any* number of things from z things. In other words, it is the number of subsets of a set of size z. This is equal to 2^z, since each of the z things might either be in the subset or not be in the subset. Replacing the sum by 2^z, we get

$$P(Z=z) = \frac{\lambda^z}{z!} e^{-2\lambda} 2^z = \frac{(2\lambda)^z}{z!} e^{-2\lambda}$$

which matches the formula for the probabilities mass function of a Poisson random variable with mean 2λ.

2. We wish to find X, the sum of all the rolls. To help with this, define Y to be the number of rolls (including the final roll of a six). Y has the geometric distribution with $p=1/6$. Also, let Z_1, Z_2, etc. be the values of the die rolls.

To find $E(X)$, we will find $E(X|Y=y)$ for every possible value of y, and then apply the formula $E(X) = E(E(X|Y))$. Given $Y=y$, the rolls $Z_1, \ldots, Z_{y-1}$ are all uniformly distributed over $\{1,2,3,4,5\}$, since the value 6 for one of these earlier rolls is excluded (otherwise we would have stopped earlier), but their values can equally well be any of the other five values. From this, we see that

$$E(Z_i|Y=y) = (1+2+3+4+5)/5 = 3, \quad \text{for } i < y$$

It follows that $E(X|Y=y) \;=\; E(Z_1+\cdots+Z_{y-1}+6\,|\,Y=y) \;=\; 3(y-1)+6 \;=\; 3y+3$. We now use the fact that a geometric random variable has mean $1/p$ to get the answer:

$$E(X) \;=\; E(E(X|Y)) \;=\; E(3Y+3) \;=\; 3E(Y)+3 \;=\; 3(6)+3 \;=\; 21$$

Similarly, we will find $\text{Var}(X)$ (and hence $\text{STD}(X)$) using the formula

$$\text{Var}(X) \;=\; E(\text{Var}(X|Y)) + \text{Var}(E(X|Y))$$

To find $\text{Var}(X|Y=y)$, note that $Z_1,\ldots,Z_{y-1}$ are independent given $Y=y$ — if we know that $Y=y$, then we know that none of $Z_1\ldots,Z_{y-1}$ are six, but finding out the value for one of them will tell us nothing more about any of the others. It follows that

$$\text{Var}(X|Y=y) \;=\; \text{Var}(Z_1+\cdots+Z_{y-1}+6\,|\,Y=y) \;=\; \text{Var}(Z_1\,|\,Y=y)\,+\,\cdots\,+\,\text{Var}(Z_{y-1}\,|\,Y=y)$$

Given $Y=y$, the earlier Z_i are uniformly distributed over $\{1,2,3,4,5\}$, and we found above that they have mean 3, so

$$\text{Var}(Z_i\,|\,Y=y) \;=\; \frac{(1-3)^2+(2-3)^2+(3-3)^2+(4-3)^2+(5-3)^2}{5} \;=\; 2, \quad \text{for } i<y$$

Thus $\text{Var}(X\,|\,Y=y) \;=\; \text{Var}(Z_i\,|\,Y=y)(y-1) \;=\; 2(y-1) \;=\; 2y-2$. We can now find the answer, using the fact that the mean and variance of a geometric random variable are $1/p$ and $(1-p)/p^2$:

$$\begin{aligned}
\text{Var}(X) &= E(\text{Var}(X|Y)) + \text{Var}(E(X|Y)) \\
&= E(2Y-2) \;+\; \text{Var}(3Y-3) \\
&= 2E(Y)-2+3^2\text{Var}(Y) \\
&= 2/(1/6)-2+9(5/6)/(1/6)^2 \\
&= 12-2+270 \;=\; 280
\end{aligned}$$

The standard deviation of X is therefore $\sqrt{280}=16.73$.

3. The probability density function must integerate to one. The integral in regions where the function is zero will be zero. So c must have the value for which

$$1 \;=\; \int_1^3 \frac{c}{w^2}\,dw \;=\; \left[-\frac{c}{w}\right]_1^3 \;=\; -(c/3)+(c/1) \;=\; (2/3)c$$

So $c=3/2$.

The mean of W is found as follows:

$$\begin{aligned}
\int_1^3 w\,f(w)\,dw &= \int_1^3 w\,(3/2)/w^2\,dw \;=\; \int_1^3 (3/2)/w\,dw \;=\; \Big[(3/2)\log(w)\Big]_1^3 \\
&= (3/2)\log(3)-(3/2)\log(1) \;=\; (3/2)\log(3) \;=\; 1.6479
\end{aligned}$$

The median of W is the point a for which

$$\begin{aligned}
1/2 &= P(W\le a) \;=\; \int_1^a f(w)\,dw \\
&= \int_1^a \frac{3/2}{w^2}\,dw \;=\; \left[-\frac{3/2}{w}\right]_1^a \;=\; -(3/2)/a+(3/2)
\end{aligned}$$

The solution to this is $a=3/2$.

4. (a) The matrix of one-step transition probabilities is as follows (first row/column is for state 0, second for state 1):

$$P = \begin{bmatrix} 0.99 & 0.01 \\ 0.6 & 0.4 \end{bmatrix}$$

$P(E(i+3) = 1 \,|\, E(i) = 1)$ will be the bottom right entry in the third power of this matrix, which will have the three-step transition probabilities. Multiplying P by itself twice, we get

$$P^3 = \begin{bmatrix} 0.984579 & 0.015421 \\ 0.925260 & 0.074740 \end{bmatrix}$$

So $P(E(i+3) = 1 \,|\, E(i) = 1) = 0.074740$.

(b) Here is an R function to do the simulation:

```
# SIMULATE CHANNEL ERRORS AND FIND FRACTION OF UNCORRECTED BLOCKS.  Simulates
# a channel with a Markov model for burst errors, with probability p01 for
# a non-error to followed by an error, and probability p11 for an error to be
# followed by another error.  Determines whether each block of size block.size
# will have it's errors corrected, which will happen if there are no more than
# max.errors in the block.  Does this for n.blocks blocks, and returns the
# fraction of them that had uncorrectable errors.

sim.errors <- function (p01, p11, block.size, max.errs, n.blocks)
{
  uncorrected <- 0  # Number of uncorrected blocks

  err <- 0          # Whether previous bit was in error

  # Loop through the specified number of blocks.

  for (i in 1:n.blocks)
  {
    err.count <- 0  # Number of errors in this block

    # Loop through the bits in the block.

    for (j in 1:block.size)
    {
      # Simulate whether or not the next bit is received with an error.

      if (err==0)
      { err <- as.numeric(runif(1)<p01)
      }
      else
      { err <- as.numeric(runif(1)<p11)
      }

      # Add to the error count for this block.

      err.count <- err.count + err
    }
```

```
    # If there are too many errors, this block is uncorrectable.

    if (err.count>max.errs)
    { uncorrected <- uncorrected + 1
    }
  }

  # Return the fraction of blocks with uncorrected errors.

  uncorrected/n.blocks
}
```

Here is the output of ten runs of this function:

```
> sim.errors(0.01,0.4,100,4,1000)
[1] 0.083
> sim.errors(0.01,0.4,100,4,1000)
[1] 0.085
> sim.errors(0.01,0.4,100,4,1000)
[1] 0.076
> sim.errors(0.01,0.4,100,4,1000)
[1] 0.068
> sim.errors(0.01,0.4,100,4,1000)
[1] 0.077
> sim.errors(0.01,0.4,100,4,1000)
[1] 0.074
> sim.errors(0.01,0.4,100,4,1000)
[1] 0.078
> sim.errors(0.01,0.4,100,4,1000)
[1] 0.104
> sim.errors(0.01,0.4,100,4,1000)
[1] 0.088
> sim.errors(0.01,0.4,100,4,1000)
[1] 0.081
```