

## Factor Analysis — A Probabilistic Model Related to PCA

PCA doesn't provide a probabilistic model of the data. If we use  $m = 10$  principal components for data with  $p = 1000$  variables, it's not clear what we're saying about the distribution of this data.

A latent variable model called *factor analysis* is similar, and does treat the data probabilistically.

We assume that each data item,  $x = (x_1, \dots, x_p)$  is generated using  $m$  latent variables  $z_1, \dots, z_m$ . the relationship of  $x$  to  $z$  is assumed to be linear.

The  $z_i$  are independent of each other. They all have Gaussian distributions with mean 0 and variance 1. (This is just a convention — any mean and variance would do as well.)

The observed data,  $x$ , are obtained by

$$x = \mu + \Lambda z + \epsilon$$

where  $\mu$  is a vector of means for the  $p$  components of  $x$ ,  $\Lambda$  is a  $p \times m$  matrix, and  $\epsilon$  is a vector of  $p$  "residuals", assumed to be independent, and to come from Gaussian distributions with mean zero. The variance of  $\epsilon_j$  is  $\sigma_j^2$ .

## The Distribution Defined by a Factor Analysis Model

Since the factor analysis model expresses  $x$  as a linear combination of independent Gaussian variables, the distribution of  $x$  will be multivariate Gaussian. The mean vector will be  $\mu$ . The covariance matrix will be

$$E\left((x - \mu)(x - \mu)^T\right) = E\left((\Lambda z)(\Lambda z)^T + \epsilon\epsilon^T + (\Lambda z)\epsilon^T + \epsilon(\Lambda z)^T\right)$$

Because  $\epsilon$  and  $z$  are independent, and have means of zero, the last two terms have expectation zero, so the covariance is

$$E\left((\Lambda z)(\Lambda z)^T + \epsilon\epsilon^T\right) = \Lambda E(zz^T)\Lambda^T + E(\epsilon\epsilon^T) = \Lambda\Lambda^T + \Sigma$$

where  $\Sigma$  is the diagonal matrix containign the residual variances,  $\sigma_j^2$ .

This form of covariance matrix has  $mp + p$  free parameters, as opposed to  $p(p + 1)/2$  for a unrestricted covariance matrix. So when  $m$  is small, factor analysis is a restricted Gaussian model.

## Fitting Factor Analysis Models

We can estimate the parameters of a factor analysis model ( $\Lambda$  and the  $\sigma_j$ ) by maximum likelihood.

This is a moderately difficult optimization problem. There are local maxima, so trying multiple initial values may be a good idea.

When there is more than one latent factor ( $m > 1$ ), the result is non-unique, since the latent space can be rotated (with a corresponding change to  $\Lambda$ ) without affecting the probability distribution of the observed data.

Sometimes, one or more of the  $\sigma_j$  are estimated to be zero. This is maybe not too realistic.

## Factor Analysis in R

The `factanal` procedure in R does maximum likelihood factor analysis. An example with simulated data, using  $m = 1$ :

```
> n = 1000           # number of training cases
> z = rnorm(n)       # simulate values for the latent factor
> x = cbind(         # simulate observed data
+ 4*3*z+rnorm(n,0,0.1),
+ 1-2*z+rnorm(n,0,0.3),
+ 4*z+rnorm(n,0,1))
>
> f = factanal(x,1) # find maximum likelihood estimate
>
> f$loadings *      # look at lambda, correcting for factanal
+ apply(x,2,sd)    # having standardized variables

Loadings:
      Factor1
[1,]  3.036
[2,] -2.031
[3,]  4.080

      Factor1
SS loadings  29.994
Proportion Var  9.998
>
> sqrt(f$uniquenesses * # look at noise standard deviations
+ apply(x,2,var))
[1] 0.2152241 0.2874030 0.9887391
```

## Factor Analysis and PCA

If we constrain all the  $\sigma_j$  to be equal, the results of maximum likelihood factor analysis are essentially the same as PCA. The mapping  $x = \Lambda z$  defines an embedding of an  $m$ -dimensional manifold in  $p$ -dimensional space, which corresponds to the hyperplane spanned by the first  $m$  principal components.

But if the  $\sigma_j$  can be different, factor analysis can produce much different results from PCA:

- Unlike PCA, maximum likelihood factor analysis is not sensitive to the units used, or other scaling of the variables.
- Lots of noise in a variable (unrelated to anything else) will not affect the result of factor analysis except to increase  $\sigma_j$  for that variable. In contrast, a noisy variable may dominate the first principal component (at least if the variable is not rescaled to make the noise smaller).
- In general, the first  $m$  principal components are chosen to capture as much *variance* as possible, but the  $m$  latent variables in a factor analysis model are chosen to explain as much *covariance* as possible.