CSC 310, Spring 2004 — Assignment #3

Due at **start** of tutorial on March 26. Worth 6% of the course grade.

Note that this assignment is to be done by each student individually. You may discuss it in general terms with other students, but the work you hand in should be your own.

Question 1 (60 marks): This question concerns a Z Channel, in which channel inputs of 0 are always received correctly, but channel inputs of 1 are received as 0 with probability f.

Consider the following three codes for use with the fourth extension of this channel:

$$\mathcal{C}_1 = \{0001,\ 0010,\ 0100,\ 1000\}$$
$$\mathcal{C}_2 = \{0101,\ 0110,\ 1001,\ 1010\}$$
$$\mathcal{C}_3 = \{1110,\ 1101,\ 1011,\ 0111\}$$

Since all these codes have four codewords, they could be used to send a block of two message bits.

a) For each of these three codes, derive the maximum likelihood decoder (ie, the decoder which, given the four received bits, finds which codeword was most likely to have been transmitted, assuming that all codewords are sent with equal probability).

   You should show your derivation of the decoders, and make clear exactly what the decoding is for all possible received vectors, either by a table or by some more compact explanation. If the decoding is not unique (ie, there are two or more equally good decodings for some received vector), you should make clear what all the possible maximum likelihood decodings are. It is conceivable that the maximum likelihood decoders might depend on the value of f (assumed to be in $(0, 1)$), in which case you should explain the dependence.

b) For each of the three codes, find the probability that the maximum likelihood decoding is erroneous (ie, produces a codeword other than the one actually sent), assuming that the four codewords are equally likely to be sent. Note that when the decoder can't tell for sure which codeword was sent, it can still guess, and will sometime guess right. The decoder error probabilities may depend on the value of f.

c) Discuss whether or not the results you found for part (b), and in particular, which code is better, are what you would expect in view of the results on mutual information and capacity for the Z channel that were presented in lectures and are found in the textbook.

Question 2 (40 marks): Suppose that $\mathcal{C}_H$ is a code for the Hth extension of a binary channel, and the $\mathcal{C}_V$ is a code for the Vth extension of a binary channel. The *product* of H and V is a code for the HVth extension of the channel. A codeword of this product code can be visualized as a two-dimensional array with V rows and H columns, in which all the rows are codewords of $\mathcal{C}_H$ and all the columns are codewords of $\mathcal{C}_V$. The product code consists of all such V by H arrays.

a) Give an example of two non-empty codes $\mathcal{C}_H$ and $\mathcal{C}_V$ for which the product code is empty (ie, for which it is not possible to construct a V by H array with the rows being codewords of H and the columns being codewords of V).

b) Give a one-sentence proof that if $\mathcal{C}_H$ and $\mathcal{C}_V$ are linear codes, their product code is non-empty.

c) Suppose that C_V is the repetition code of length three, and that C_H is a single-parity-check code of length two or more. Give an time-efficient decoding algorithm for this product code that is guaranteed to correctly decode whenever the received block has no more than two errors. Discuss whether or not it is possible to correctly decode some or all received vectors that have three errors, giving examples to show that it is sometimes possible, and/or sometimes not possible to guarantee correct decoding with three errors.