CSC320— Visual Computing, Winter 2005

Assignment 2: Image Morphs

Solutions

See the Matlab files in a2_05soln.zip for the solution source code.

1. **Fourier Transform of Comb Functions [10pts]:** Let $f(n) = \text{Comb}(n; n_s)$. Define $\omega_k = \frac{2\pi}{N}k$ and $m = N/n_s$, which is assumed to be an integer. Then, by the definition of the DFT,

$$\begin{aligned}
\hat{f}(k) &\equiv \sum_{n=0}^{N-1} f(n)e^{-i\omega_k n} = \sum_{n=0}^{N-1} \text{Comb}(n; n_s)e^{-i\omega_k n} \\
&= \sum_{j=0}^{N/n_s-1} e^{-i\omega_k n_s j} \text{ since Comb}(n; n_s) \text{ is 1 only for } n = jn_s \\
&= \sum_{j=0}^{m-1} e^{-i\frac{2\pi}{N}n_s jk} \\
&= \sum_{j=0}^{m-1} e^{-i2\pi\frac{k}{m}j}. \qquad (1)
\end{aligned}$$

A similar argument to the one given in section 1.3 of the Fourier notes shows that the last sum above is equal to $m\text{Comb}(k; m)$. To show this, there are two cases to consider, depending on whether k/m is an integer or not.

First, suppose k/m is an integer. Then every term in the sum in equation (1) is $e^{-i2\pi\frac{k}{m}j} = 1$. So the sum is the number of terms, namely $m = N/n_s$.

Alternatively, suppose k/m is not an integer. Let z be the sum in (1). Then,

$$\begin{aligned}
ze^{-i2\pi\frac{k}{m}} &= \sum_{j=0}^{m-1} e^{-i2\pi\frac{k}{m}j}e^{-i2\pi\frac{k}{m}} = \sum_{j=0}^{m-1} e^{-i2\pi\frac{k}{m}(j+1)} = \sum_{j=1}^{m} e^{-i2\pi\frac{k}{m}j} \\
&= e^{-i2\pi\frac{k}{m}m} + \sum_{j=1}^{m-1} e^{-i2\pi\frac{k}{m}j} = e^{-i2\pi k} + \sum_{j=1}^{m-1} e^{-i2\pi\frac{k}{m}j} \\
&= 1 + \sum_{j=1}^{m-1} e^{-i2\pi\frac{k}{m}j} = \sum_{j=0}^{m-1} e^{-i2\pi\frac{k}{m}j} = z, \text{ by equation 1.}
\end{aligned}$$

Therefore we have shown that $ze^{-i2\pi\frac{k}{m}} = z$. That is, $z(e^{-i2\pi\frac{k}{m}} - 1) = 0$. Since we are assuming that $\frac{k}{m}$ is not an integer, $e^{-i2\pi\frac{k}{m}} \neq 1$, and therefore it follows that z must be zero.

Therefore, we have shown that

$$\hat{f}(k) = \begin{cases} m & \text{for } k/m \text{ an integer,} \\ 0 & \text{otherwise.} \end{cases} \qquad (2)$$

That is, $\hat{f}(k) = m\text{Comb}(k; m) = \frac{N}{n_s}\text{Comb}(k; N/n_s)$.

2. **Convolutions of Fourier Transforms [10pts]:** Let $s(n) = f(n)g(n)$. Then

$$\mathcal{F}(s) = \hat{s}(k) = \sum_{n=0}^{N-1} f(n)g(n)e^{-i\omega_k n}, \qquad (3)$$

for $\omega_k = \frac{2\pi}{N}k$.

On the other hand, let $\hat{f}(k)$ and $\hat{g}(k)$ be the Fourier transforms of $f(n)$ and $g(n)$, respectively. By the definition of convolution of N-periodic functions we have

$$\begin{aligned}
(\hat{f} * \hat{g})(k) &= \sum_{j=0}^{N-1} \hat{f}(j)\hat{g}(k-j), \\
&= \sum_{j=0}^{N-1}\sum_{n=0}^{N-1} f(n)e^{-i\frac{2\pi}{N}jn} \sum_{m=0}^{N-1} g(m)e^{-i\frac{2\pi}{N}(k-j)m} \text{ by the definition of the DFT,} \\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} f(n)g(m) \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}(jn+(k-j)m)}, \\
&= \sum_{n=0}^{N-1}\sum_{m=0}^{N-1} f(n)g(m)e^{-i\frac{2\pi}{N}km} \sum_{j=0}^{N-1} e^{-i\frac{2\pi}{N}j(n-m)}. \qquad (4)
\end{aligned}$$

Notice that since $0 \leq n, m \leq N-1$ it must be the case that $(n-m)/N$ is not an integer unless $n = m$. When $n = m$ it follows that the last sum in (4) is equal to N (since each term is simply 1 and there are N terms). Alternatively, when $n \neq m$ a similar argument to the one in Question 1 shows that this sum is 0. Therefore, equation (4) simiplifies to

$$(\hat{f} * \hat{g})(k) = N \sum_{n=0}^{N-1} f(n)g(n)e^{-i\frac{2\pi}{N}kn} = N\hat{s}(k) \quad \text{from eqn (3),} \qquad (5)$$

which is what we were asked to show.

3. **Warp for Image Morphing [10pts]:** See morphSoln.zip.

4. **Perform the Warp by Looping Over Pixels [10pts]:** Aliasing will occur in situations where the output image involves a significant shrinkage of a textured region in one of the original images. In such a case this textured region will be downsampled (but not blurred) in the warped image. An example of such a region is Reagan's upper lip, which is shrunk significantly for small values of s. [2pts] (Remaining [8pts] for the implementation.)

5. **Perform the Warps Using Interp2 [10pts]:** See morphSoln.zip.