

## CSC320— Visual Computing, Winter 2005

### Assignment 1: Fourier Transforms, Signal Interpolation, and Sampling

#### Solutions

See the Matlab files in a1\_05soln.zip for the solution source code.

#### 1. Discrete Fourier Transforms [15pts]:

(a) Let  $g(n)$  be the signal of length  $N$ , defined for  $0 \leq n < N$ , such that

$$g(n) = \begin{cases} 1 & \text{for } n = 0, \\ -1 & \text{for } n = 1, \\ 0 & \text{otherwise.} \end{cases} \quad (1)$$

Then, by the definition of the discrete Fourier transform,

$$\begin{aligned} \hat{g}(k) &\equiv \sum_{n=0}^{N-1} e^{-i\omega_k n} g(n) \\ &= e^0(1) + e^{-i\omega_k}(-1) = e^{-i\omega_k/2}(e^{i\omega_k/2} - e^{-i\omega_k/2}) \\ &= 2ie^{-i\omega_k/2} \sin(\omega_k/2), \text{ for } \omega_k = \frac{2\pi}{N}k. \end{aligned} \quad (2)$$

The desired signal  $f(n)$  can be obtained from  $g(n)$  with a (periodic) shift,  $f(n) = g(n - N/2)$ . By the Fourier shift property

$$\hat{f}(k) = \hat{g}(k)e^{-i\omega_k N/2} = 2ie^{-i\omega_k(N+1)/2} \sin(\omega_k/2). \quad (3)$$

(b) Let  $g(n)$  for  $0 \leq n < N$  be

$$g(n) = \begin{cases} 2 & \text{for } n = 0, \\ -1 & \text{for } n = 1 \text{ and } n = N - 1, \\ 0 & \text{otherwise.} \end{cases} \quad (4)$$

Then, by the definition of the discrete Fourier transform,

$$\begin{aligned} \hat{g}(k) &\equiv \sum_{n=0}^{N-1} e^{-i\omega_k n} g(n) \\ &= 2 - e^{-i\omega_k} - e^{-i\omega_k(N-1)} = 2 - e^{-i\omega_k} - e^{i\omega_k} \\ &= 2(1 - \cos(\omega_k)). \end{aligned} \quad (5)$$

In the second line above we used the fact that  $e^{-i\omega_k N} = e^{-i2\pi k} = 1$ , so  $e^{-i\omega_k(N-1)} = e^{i\omega_k}$ . The desired signal  $f(n)$  can be obtained from  $g(n)$  with a (periodic) shift,  $f(n) = g(n - N/2)$ . By the Fourier shift property

$$\hat{f}(k) = \hat{g}(k)e^{-i\omega_k N/2} = 2(-1)^k [1 - \cos(\omega_k)], \text{ for } \omega_k = \frac{2\pi}{N}k. \quad (6)$$

Here we have used the fact that  $e^{-i\omega_k N/2} = e^{-i\pi k} = (-1)^k$ .

(c) Define  $g(n)$  for  $0 \leq n < N$  to be

$$g(n) = \begin{cases} 1 & \text{for } 0 \leq n \leq 2, \text{ and } N - 2 \leq n \leq N - 1, \\ 0 & \text{otherwise.} \end{cases} \quad (7)$$

Then, by the definition of the discrete Fourier transform,

$$\begin{aligned}
\hat{g}(k) &\equiv \sum_{n=0}^{N-1} e^{-i\omega_k n} g(n) \\
&= 1 + e^{-i\omega_k} + e^{-i2\omega_k} + e^{-i(N-2)\omega_k} + e^{-i(N-1)\omega_k} \\
&= 1 + e^{-i\omega_k} + e^{-i2\omega_k} + e^{i2\omega_k} + e^{i\omega_k} \\
&= 1 + 2 \cos(\omega_k) + 2 \cos(2\omega_k), \text{ for } \omega_k = \frac{2\pi}{N}k.
\end{aligned} \tag{8}$$

Here again we have used the fact that  $e^{-i\omega_k N} = 1$ , so  $e^{-i\omega_k(N-1)} = e^{i\omega_k}$  and  $e^{-i\omega_k(N-2)} = e^{i2\omega_k}$ . The desired signal  $f(n)$  can be obtained from  $g(n)$  with a (periodic) shift,  $f(n) = g(n - N/2)$ . By the Fourier shift property

$$\hat{f}(k) = \hat{g}(k)e^{-i\omega_k N/2} = (-1)^k [1 + 2 \cos(\omega_k) + 2 \cos(2\omega_k)], \text{ for } \omega_k = \frac{2\pi}{N}k. \tag{9}$$

These results are compared to those from Matlab's `fft` function (and `fftshift` to center the transform on wave number  $k = 0$ ) in the script file `a1_05SolnQ1.m`. The plots for the above hand calculations are shown in green, and are superimposed on the plots of the results from Matlab's `fft` routine in blue. For  $N = 16$  the results agree. You are free to change to any other even value of  $N > 5$ . The plots for the amplitude spectrum, and for both the real and imaginary parts of the Fourier transforms are the same, up to double precision round-off errors with magnitudes around  $10^{-15}$  (for  $N = 16$ ). In the plots these rounding errors are only visible for the imaginary parts of the transforms in 1b and 1c, where the correct value is zero.

## 2. Fourier Series [15pts]:

(a) Let  $g(x)$  be the shifted and centered signal defined for  $x \in [-N/2, N/2]$ ,

$$g(x) = \begin{cases} 1 + \cos(\frac{\pi}{4}x) & \text{for } x \in [-4, 4], \\ 0 & \text{otherwise.} \end{cases} \tag{10}$$

Then, by the definition of the Fourier series coefficients,

$$\begin{aligned}
\hat{g}(k) &\equiv \frac{1}{N} \int_{-N/2}^{N/2} e^{-i\omega_k x} g(x) dx \text{ for } \omega_k = \frac{2\pi}{N}k, \\
&= \frac{1}{N} \int_{-4}^4 e^{-i\omega_k x} [1 + \cos(\frac{\pi}{4}x)] dx = \frac{1}{2N} \int_{-4}^4 e^{-i\omega_k x} [2 + e^{i\frac{\pi}{4}x} + e^{-i\frac{\pi}{4}x}] dx \\
&= \frac{1}{2N} \int_{-4}^4 [2e^{-i\omega_k x} + e^{i(\omega_k - \frac{\pi}{4})x} + e^{-i(\omega_k + \frac{\pi}{4})x}] dx \\
&= \frac{1}{2N} [16\text{sinc}(4\omega_k) + 8\text{sinc}(4\omega_k - \pi) + 8\text{sinc}(4\omega_k + \pi)] \\
&= \frac{4}{N} [2\text{sinc}(4\omega_k) + \text{sinc}(4\omega_k - \pi) + \text{sinc}(4\omega_k + \pi)].
\end{aligned} \tag{11}$$

In the second last line above we have used

$$\int_{-L}^L e^{-i\omega x} dx = \frac{i}{\omega} [e^{-i\omega L} - e^{i\omega L}] = \frac{-2i^2}{\omega} \sin(\omega L) = 2L\text{sinc}(\omega L). \tag{12}$$

The derivation of the integral in (12) holds for  $\omega \neq 0$  and, since  $\text{sinc}(0) = 1$ , the far right hand side is also the integral for  $\omega = 0$ .

The desired signal  $f(x)$  can be obtained from  $g(x)$  with a (periodic) shift,  $f(x) = g(x - N/2)$  (note a shift by  $N/2$  is the same as a shift by  $-N/2$  due to the periodicity). By the Fourier shift property

$$\hat{f}(k) = \hat{g}(k)e^{-i\omega_k N/2} = \hat{g}(k)e^{-i\pi k} = (-1)^k \hat{g}(k). \tag{13}$$

(b) Let  $g(x)$  be the shifted and centered signal defined for  $x \in [-N/2, N/2)$ ,

$$g(x) = \begin{cases} 1 + x/4 & \text{for } x \in [-4, 0], \\ 1 - x/4 & \text{for } x \in [0, 4], \\ 0 & \text{otherwise.} \end{cases} \quad (14)$$

Then, by the definition of the Fourier series coefficients,

$$\begin{aligned} \hat{g}(k) &\equiv \frac{1}{N} \int_{-N/2}^{N/2} e^{-i\omega_k x} g(x) dx \text{ for } \omega_k = \frac{2\pi}{N}k, \\ &= \frac{1}{N} \int_{-4}^0 e^{-i\omega_k x} (1 + x/4) dx + \frac{1}{N} \int_0^4 e^{-i\omega_k x} (1 - x/4) dx \\ &= \frac{1}{N} \int_0^4 (e^{i\omega_k x} + e^{-i\omega_k x})(1 - x/4) dx = \frac{2}{N} \int_0^4 (1 - x/4) \cos(\omega_k x) dx \\ &= \begin{cases} \frac{1}{2N\omega_k^2} (1 - \cos(\omega_k 4)) & \text{for } k \neq 0, \\ \frac{4}{N} & \text{for } k = 0. \end{cases} \end{aligned} \quad (15)$$

Here integration by parts can be used to derive the expression above for the value of the integral.

The desired signal  $f(x)$  can be obtained from  $g(x)$  with a (periodic) shift,  $f(x) = g(x - N/2)$  (note a shift by  $N/2$  is the same as a shift by  $-N/2$  due to the periodicity). By the Fourier shift property

$$\hat{f}(k) = \hat{g}(k) e^{-i\omega_k N/2} = \hat{g}(k) e^{-i\pi k} = (-1)^k \hat{g}(k). \quad (16)$$

(c) Let  $g(x)$  be the shifted and centered signal defined for  $x \in [-N/2, N/2)$ ,

$$g(x) = \begin{cases} 1 & \text{for } x \in [-4, 4], \\ 0 & \text{otherwise.} \end{cases} \quad (17)$$

Then, by the definition of the Fourier series coefficients,

$$\begin{aligned} \hat{g}(k) &\equiv \frac{1}{N} \int_{-N/2}^{N/2} e^{-i\omega_k x} g(x) dx \text{ for } \omega_k = \frac{2\pi}{N}k, \\ &= \frac{1}{N} \int_{-4}^4 e^{-i\omega_k x} dx \\ &= \frac{8}{N} \text{sinc}(\omega_k 4), \end{aligned} \quad (18)$$

where we have used equation (12) to do the integral.

The desired signal  $f(x)$  can be obtained from  $g(x)$  with a (periodic) shift,  $f(x) = g(x - N/2)$  (note a shift by  $N/2$  is the same as a shift by  $-N/2$  due to the periodicity). By the Fourier shift property

$$\hat{f}(k) = \hat{g}(k) e^{-i\omega_k N/2} = \hat{g}(k) e^{-i\pi k} = (-1)^k \hat{g}(k). \quad (19)$$

These results are used in the Matlab script file `a1_05SolnQ2.m` to approximate the original functions  $f(x)$  with the (truncated) Fourier series

$$f(x) \approx r_K(x) \equiv \sum_{k=-K}^K \hat{f}(k) e^{i\omega_k x}. \quad (20)$$

The default period for the signal is  $N = 16$  although any even  $N$  with  $N >= 8$  (i.e. the length of the support the functions  $f(x)$ ) works. For each  $K \in \{5, 10, 100\}$ , the script file plots  $r_K(x)$  in green (real part) and red (imaginary part), and these are both superimposed on the plot of the original function  $f(x)$  in blue. Notice the output from Matlab indicates the average error decreases as  $K$  increases. For  $N > 8$

the largest error standard deviation occurs for the step function in 2c, and the smallest standard deviation occurs with the smooth function in 2a. Note that the maximum absolute error decreases in cases 2a and 2b as  $K$  increases, but remains at 0.5 (roughly) for case 2c. The latter is caused by the discontinuity between function values of 0 and 1 at the edges of the step discontinuity in case 2c. These results illustrate the general theorem that  $r_K(x)$  converges to  $f(x)$  “almost everywhere” as  $K \rightarrow \infty$ .

3. **Up Sampling [10pts]:** Let  $S(n)$  be a discrete signal defined for integers  $n$  with  $0 \leq n < N$ . Let the up-sampling rate  $r > 1$  be an integer. As in class, define the raw up-sampled signal  $U(n)$  for  $0 \leq n < rN$  to be

$$U(n) = \begin{cases} S(n/r) & \text{for } n \bmod r = 0, \\ 0 & \text{otherwise.} \end{cases} \quad (21)$$

Then the discrete Fourier transform of  $U(n)$  is

$$\begin{aligned} \hat{U}(k) &\equiv \sum_{n=0}^{rN-1} e^{-i\omega_k n} U(n), \text{ for } \omega_k = \frac{2\pi}{rN}k \\ &= \sum_{m=0}^{N-1} e^{-i\omega_k r m} U(rm), \text{ the only nonzero values of } U, \\ &= \sum_{m=0}^{N-1} e^{-i\frac{2\pi}{rN} k r m} S(m) \\ &= \sum_{m=0}^{N-1} e^{-i\frac{2\pi}{N} k m} S(m) \\ &\equiv \hat{S}(k). \end{aligned} \quad (22)$$

The last equality follows directly from the definition of the discrete Fourier transform of the signal  $S(n)$ , which has length  $N$ . Since  $U(n)$  has length  $rN$ , the Fourier transform  $\hat{U}(k)$  is periodic with period  $rN$  (or some integer fraction of  $rN$ ). Similarly,  $S(n)$  has length  $N$ , so the Fourier transform  $\hat{S}(k)$  is periodic with period  $N$ . Therefore equation (22) shows that  $\hat{U}(k)$  is formed from  $r$  copies of the period- $N$  transform  $\hat{S}(k)$ .

4. **Image Translation [10pts]:** See the Matlab M-files `translateIm.m` and `testTranslateIm.m` in `a1_05soln.zip`. The implementation follows from the lecture notes on interpolation, with the filter kernels given by equation (5) and (6) of the Image Interpolation notes.

There is one further detail that is worth mentioning. This has to do with the behaviour of `rconv2sep` when given filter kernels with an even length. In particular, the filter kernel for bilinear interpolation with zero shift is  $f = [1, 0]$ . However, when this kernel is used in `rconv2sep` the output image is shifted by one pixel, up and to the left. The reason for this is that, for a filter kernel of length  $2n$ , `rconv2sep` centers the filter kernel at pixel  $n + 1$ . This does not correspond to the intended center of the filter kernel and, as a consequence, the output is shifted as described above. This discrete shift needs to be taken into account to obtain the correct image translation.

The M-file `testTranslateIm.m` evaluates `translateIm.m` on images where the grey-level varies linearly or quadratically with image position. Bilinear interpolation should give no error (within double precision arithmetic error) on linear variations, small error on quadratic brightness variations, while bicubic interpolation should give no error (within double precision arithmetic error) on both linear and quadratic brightness variations. This behaviour should hold away from the image boundaries. Near the image boundaries errors should arise due to the reflecting edge treatment used in the convolution in `rconv2sep`.

5. **Sampling and Aliasing [15pts]:**

- (a) See the Matlab script file `checkerAliasSoln.m`.  
 (b) Consider the Gaussian function used to blur the continuous image,

$$G(\vec{x}; \sigma) = \frac{1}{2\pi\sigma^2} e^{-(x_1^2 + x_2^2)/(2\sigma^2)}, \quad (23)$$

where  $\vec{x} \in [-N/2, N/2] \times [-N/2, N/2]$ , for some large value of  $N$ . This is a separable function of  $x_1$  and  $x_2$ , so the Fourier transform of  $G(\vec{x}; \sigma)$  is a product of the 1D transforms over  $x_1$  and  $x_2$ . For large enough  $N$ , so that  $G(\vec{x}; \sigma)$  is essentially zero at the borders of the image, we find the Fourier transform of  $G$  is another Gaussian (see `linSysTutorial.m`),

$$\hat{G}(\vec{\omega}; \sigma) = e^{-(\omega_1^2 + \omega_2^2)\sigma^2/2}. \quad (24)$$

Using the frequencies  $\omega_k = \frac{2\pi}{N}k$ , we could form a Fourier series representation of  $G(\vec{x}; \sigma)$ , just as we did in problem 2 above.

From the above expression for  $\hat{G}(\vec{\omega}; \sigma)$ , we see that  $G(\vec{x}; \sigma)$  is a low-pass kernel. The the maximum value of the amplitude spectrum for  $G$  is 1 and this occurs at frequency  $\vec{\omega} = \vec{0}$ . Whether or not a sampled signal is aliased depends on how well this Gaussian prefilter has reduced the amplitude of high frequency components of the original continuous signal, in particular, frequencies above the Nyquist frequency. The Nyquist frequency corresponding to sampling only at integer valued locations  $\vec{x}$  is  $\omega_c = \omega_{N/2} = \frac{2\pi}{N} \frac{N}{2} = \pi$ . The value of the Fourier transform of the blur kernel evaluated at the Nyquist frequency is then

$$\hat{G}([\pm\pi, 0]; \sigma) = \hat{G}([0, \pm\pi]; \sigma) = e^{-\pi^2\sigma^2/2}. \quad (25)$$

For  $\sigma = 2$  the corresponding value  $e^{-\pi^2\sigma^2/2}$  is very small (i.e.  $2.7 \times 10^{-9}$ ), indicating that the high frequency components of the original signal are massively attenuated. There will therefore be little aliasing. But frequencies that are lower than the Nyquist frequency are also attenuated significantly, indicating that the image has been excessively blurred.

For  $\sigma = 0.2$ ,  $e^{-\pi^2\sigma^2/2} \approx 0.82$  is relatively large, indicating that the low-pass blur kernel reduces the amplitude of frequencies at the Nyquist rate by only about 20%. Therefore it allows significant high frequency components to pass with relatively little attenuation, and aliasing can still be a problem.

Finally, for  $\sigma = 2/\pi$  the Nyquist rate corresponds to 2 standard deviations of the Gaussian function, and  $e^{-\pi^2\sigma^2/2}$  is relatively small (i.e. roughly 0.14). This implies the low-pass region is essentially within the Nyquist limits (thereby reducing the aliasing), but not too narrow to cause significant blurring of the image.