University of Toronto
St. George Campus

Computer Science 238
Fall 2001

Midterm Exam

18 October, 2001

DO NOT OPEN THIS BOOKLET UNTIL YOU ARE TOLD TO DO SO

Last Name:

First Name:

Student Number:

Rules:

1. Legibly write your name on this page and every page that you wish to be marked or returned. The pages of this exam will be separated during marking. *Unnamed pages will not be marked.*
2. There are 3 problems of equal weight, of which you must choose 2. Indicate below which problem you did NOT choose. *It will not be marked, regardless of whether work appears for it. If you do not indicate, it will be assumed that you did not choose problem 1.*
3. Total time is 50 minutes.
4. The exam is closed book, and no aids of any kind are allowed.
5. Write your answers directly on the exam in the space provided, or on the blank pages provided at the end. For each blank page that you use, write the number of the problem that you are solving on that page. Do not write the answer to more than one problem on the same blank page.
6. For rough work, you may use the back of any page. *These will not be marked.*
7. The instruction, "prove inductively," means that you must prove the given proposition by induction, complete induction, or structural induction. No credit will be given for (even very clever) non-inductive proofs.

Complete this line: I chose not to solve Problem __________.
Do not write below this line.

1:

2:

3:

Total:

Name:
1. (15 marks)
(a) (5 marks) State the general divide-and-conquer recurrence theorem.
(b) (5 marks) State a recurrence for the worst case time complexity of the following algorithm:

```
max(f,l,A)
Precondition: A is an array of integers, 1 <= f <= l <= length(A)
Postcondition: Returns the maximum value in A[f..l]
  if (f==l) then
    return A[f]
  else
    p := (f+l-1) div 2;
    m1 := max2(f,p,A);
    m2 := max2(p+1,l,A);
    if (m1 > m2) then
      return m1
    else
      return m2
end
```

c) (5 marks) Use the general divide-and-conquer recurrence theorem to find a tight asymptotic bound for the recurrence you gave in (b).

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2. (15 marks)

(a) (10 marks) Let A be the smallest set of binary strings such that:

- $10 \in A$, and
- if $a \in A$, then $a00 \in A$.

Let $\text{eval} : A \to \mathbb{N}$ be the function that maps each binary string, a, to the number that has a as a binary representation.

Prove inductively that, for all $a \in A$, $\log_2(\text{eval}(a))$ is odd.

(b) (5 marks) Indicate (without proof) whether the following statements are true:

1. $n^2 + 2n + 3 \in \mathcal{O}(n^2)$
2. $\log_2 n \in \Theta(\log_3 n)$
3. $2^n \in \Theta(3^n)$
4. if $f_1 \in \mathcal{O}(g_1)$ and $f_2 \in \mathcal{O}(g_2)$, then $f_1 + f_2 \in \mathcal{O}(g_1 + g_2)$
5. if $f_1 \in \mathcal{O}(g_1)$ and $f_2 \in \mathcal{O}(g_2)$, then $f_1 + f_2 \in \mathcal{O}(\max(g_1, g_2))$

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3. (15 marks) Consider the following program and loop invariant:

Precondition: $a, b \in \mathbb{N}$, $a > 0$, $b > 1$.
Postcondition: `idiv(a,b)` returns a `div` b, i.e., it returns a q such that $qb \leq a < (q+1)b$.

```
idiv(a,b)
  l := 0; n := a;
  while (n >= b) do
    n := n - b;
    l := l + 1
  end while;
  return l
end
```

Loop Invariant Lemma: Given $\geq i$ iterations and the precondition, it is true that:

1. $b * l_i + n_i \leq a < b * l_i + n_i + 1$, and
2. $n_i \geq 0$.

(a) (10 marks) Prove inductively that the loop invariant lemma is true.
(b) (5 marks) Prove that `idiv(a,b)` is partially correct, using this loop invariant.

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