UNIVERSITY OF TORONTO
Faculty of Arts and Science

APRIL/MAY EXAMINATIONS 2001

CSC 238H1 S

Duration - 3 hours

No Aids Allowed

Last Name:

First Name:

Student Number:

Rules:

1. This exam paper contains 13 pages. Legibly write your last name and student number on every page, as the pages of this exam will be separated during marking. *Unnamed pages will not be marked.*
2. There are 6 problems of equal weight, of which you must choose 5. Indicate below which problem you did NOT choose. *It will not be marked, regardless of whether work appears for it. If you do not indicate, it will be assumed that you did not choose problem 1.*
3. Write your answers directly on the exam in the space provided, or on the blank pages provided at the end. For each blank page that you use, write the number of the problem that you are solving on that page. Do not write the answer to more than one problem on the same blank page.
4. For rough work, you may use the back of any page. *These will not be marked.*
5. The instruction, "prove inductively," means that you must prove the given proposition by induction, complete induction, or structural induction. No credit will be given for (even very clever) non-inductive proofs.

Complete this line: I chose not to solve Problem ___________.
Do not write below this line.

1:

2:

3:

4:

5:

6:

Total:

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1. (20 marks) Consider the following program and specification:

Precondition: $a, b \in \mathbb{N}$, $a > 0$, $b > 1$.
Postcondition: `Ilog(a,b)` returns $\lfloor \log_b a \rfloor$.

```
Ilog(a,b)
  l := 0; n := a;
  while (n >= b) do
    n := n div b;
    l := l + 1
  end while;
  return l
end
```

and the following two lemmata:

- div lemma: if $q = a$ **div** b, where $a, b, q \in \mathbb{N}$, then $qb \leq a < (q+1)b$. You may use this lemma without proving it.
- loop-invariant lemma: if there are at least i iterations, then:
  1. $l_i \in \mathbb{N}$,
  2. $b^{l_i} n_i \leq a < b^{l_i}(n_i + 1)$, and
  3. $n_i \in \mathbb{N}$, and $n_i \geq 1$.

(a) (8 marks) Prove the loop-invariant lemma.
(b) (8 marks) Using the loop-invariant lemma, prove the partial correctness of this program with respect to its specification.
(c) (4 marks) Prove that this program terminates, given the precondition.

(Continue your answer to Problem 1 here)

2. (20 marks)

(a) (10 marks) For each of the following claims, state whether it is true or false. No justification is necessary.

1. (2 marks) $n^2 \in \mathcal{O}(n^3)$
2. (2 marks) $n^3 \in \mathcal{O}(n^2)$
3. (2 marks) $2^n \in \mathcal{O}(3^n)$
4. (2 marks) $\log_2 n \in \mathcal{O}(\log_3 n)$
5. (2 marks) $(n^2 - n) \in \Omega(n^2)$

(b) (10 marks) For which $k \in \mathbb{N}$ does $f_k \in \Theta(2^n)$? Prove your answer.

$$f_k(n) = \begin{cases} k & n = 0 \\ 1 & n = 1 \\ f_k(n-1) + 2f_k(n-2) & n > 1 \end{cases}$$

3. (20 marks) In this problem, you will prove that $\{\oplus, \rightarrow\}$ is a complete set of connectives, where $\oplus$ represents exclusive OR, using the knowledge that $\{\neg, \vee\}$ is a complete set of connectives.

(a) (2 marks) Draw the truth tables for $\oplus$ and $\rightarrow$.

(b) (9 marks) Find a propositional formula that (1) is logically equivalent to $\neg x$ and (2) uses only x, $\oplus$ and $\rightarrow$. We call this an *implementation* of negation in terms of $\oplus$ and $\rightarrow$. Justify your answer with truth tables.

(c) (9 marks) Using your answer to (b), find an implementation of disjunction, $x \vee y$, using only x, y, $\oplus$ and $\rightarrow$. Again, justify with truth tables.

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4. (20 marks) Consider the relational database for a library that we discussed in lecture:

Call no.	Title	Author	SIN	Name	Address	SIN	Call no.	Due
1	Iliad	Homer	1	Penn	PT 276	1	1	2001/4/1
2	Odyssey	Homer	2	Hirst	PT 283A	2	1	2001/3/1
3	Odyssey	Homer	3	Reiter	PT 283D	2	2	2001/4/1
4	Oliver Twist	Dickens				3	1	2001/2/1
						3	2	2001/3/1
						3	3	2001/4/1
	`Book/3`			`Sub/3`			`Borrowed/3`	

(a) (5 marks) Relative to the entries above, indicate what answer the following query will return, and briefly describe what it asks in English:

$$\forall b.\forall t.\exists s.\exists d.(\texttt{Book}(b,t,\text{Homer}) \rightarrow \texttt{Sub}(s,n,a) \wedge \texttt{Borrowed}(s,b,d))$$

(5 marks each) For each of the following queries, state whether it is logically equivalent to the query in (a) above or not. No justification is necessary.

(b) $\forall b.\forall t.(\texttt{Book}(b,t,\text{Homer}) \rightarrow \exists s.\exists d.(\texttt{Sub}(s,n,a) \wedge \texttt{Borrowed}(s,b,d)))$

(c) $\forall b.\exists s.\exists d.\forall t.(\texttt{Book}(b,t,\text{Homer}) \rightarrow \texttt{Sub}(s,n,a) \wedge \texttt{Borrowed}(s,b,d))$

(d) $\exists s.\exists d.\forall b.\forall t.(\texttt{Book}(b,t,\text{Homer}) \rightarrow \texttt{Sub}(s,n,a) \wedge \texttt{Borrowed}(s,b,d))$

(e) $\exists b.\exists b'.\exists t.\exists s.\exists d.\exists d'.(\texttt{Book}(b,t,\text{Homer}) \wedge \texttt{Book}(b',t,\text{Homer}) \wedge \neg(b \approx b') \wedge$
$\texttt{Borrowed}(s,b,d) \wedge \texttt{Borrowed}(s,b',d') \wedge \texttt{Sub}(s,n,a))$

5. (20 marks) Let Σ be an alphabet that does not contain the symbols $*$, $+$, $\emptyset$, or ϵ. Assuming the existence of the following functions:

- **alph** : $\mathcal{RE} \longrightarrow \mathbb{N}$, the number of alphabetic symbols, $a \in \Sigma$, in a regular expression,
- **conn** : $\mathcal{RE} \longrightarrow \mathbb{N}$, the number of plus signs ($+$), Kleene stars ($*$), nulls ($\emptyset$) and epsilons (ϵ) in a regular expression (*not* parentheses),

prove inductively that for every regular expression $R \in \mathcal{RE}$, there is an equivalent FSA with less than or equal to $\mathbf{conn}(R) + 2\mathbf{alph}(R)$ states (Hint: the proof of equivalence between regular expressions and FSAs constructed such an automaton).

6. (20 marks) Let $\Sigma_P = \{(,)\}$, the left and right parenthesis symbols. The set of *balanced* strings of parentheses, Bal, is defined to be the smallest set such that:

- $\epsilon \in \text{Bal}$, and
- for all $b \in \text{Bal}$, $(b) \in \text{Bal}$, and
- for all $b_1 \in \text{Bal}$ and $b_2 \in \text{Bal}$, $b_1 b_2 \in \text{Bal}$.

Clearly $\text{Bal} \subseteq \Sigma_P^*$ and can thus be viewed as a formal string language over the alphabet of parenthesis symbols.

Prove that Bal is not regular.

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