Computer Science 236 Fall
Evening Section

Sept. 23, 2010
University of Toronto

Assignment 1
Due Thursday, 14th October, 2010 at **6:10pm in tutorial**

On the cover page of your assignment, you must write and sign the following statement: *"I have read and understood the policy on collaboration on homework stated on the course web page."* Without this signed statement your homework will not be marked.

1. (5 marks) Prove by induction that $\forall n \geq 0, n \in \mathbb{N}$

$$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}.$$

SOLN:

Let $S(n)$ be "$\sum_{i=1}^{n} i(i+1)(i+2) = \frac{n(n+1)(n+2)(n+3)}{4}$. WTS $S(n), \forall n \in \mathbb{N}$.

BASE CASE: Let $n = 0$ Then 0=0.

INDUCTIVE HYPOTHESIS: Assume for arbitrary $k \in \mathbb{N}$ that $S(k)$ holds.

INDUCTIVE STEP: Prove $S(k+1)$ is true.

$$\begin{aligned}
\sum_{i=1}^{k+1} i(i+1)(i+2) &= \sum_{i=1}^{k} i(i+1)(i+2) + (k+1)(k+2)(k+3) \\
&= \frac{k(k+1)(k+2)(k+3)}{4} + (k+1)(k+2)(k+3) \text{ by inductive hypothesis} \\
&= \frac{(k+4)(k+1)(k+2)(k+3)}{4} \text{ factoring } (k+1)(k+2)(k+3) \\
&= \frac{(k+1)(k+2)(k+3)(k+4)}{4}.
\end{aligned}$$

2. (10 marks) Let $T(n,m) = T(n,m-1) + T(n-1,m)$ and $T(s,2) = T(2,s) = s$ for all $s \in \mathbb{N}$.

Use induction to prove that $T(n,m) \leq \binom{n+m-1}{n-1}$ for all $n, m \in \mathbb{N}$ such that $n, m \geq 2$.

[Hint: The trick to this question is coming up with the *correct* inductive hypothesis.]

SOLN:

Let $S(x)$ be that $T(n,m) \leq \binom{n+m-1}{n-1}$, where $x = n+m$ and $n, m \geq 2$. WTS that $S(x)$ holds for all $x \in \mathbb{N}$ such that $x \geq 4$.

BASE CASE: The base case is when $x = 4$. Then $n = m = 2$, and so:

$$T(2,2) = 2 = \binom{2+2-2}{2-1}.$$

INDUCTIVE HYPOTHESIS: Assume $S(x)$. WTS $S(x+1)$.

INDUCTIVE STEP: If either $n=2$ or $m=2$, then we get:

$$T(2,s) = s = \binom{s+2-2}{s-1} \text{ or } T(s,2) = s = \binom{2+s-2}{s-1}$$

Otherwise, $T(n,m) = T(n,m-1) + T(n-1,m)$ by definition. By the inductive hypothesis, we know that

$$T(n,m-1) \leq \binom{n+m-3}{n-1}$$

and

$$T(n-1,m) \leq \binom{n+m-3}{n-2}.$$

Therefore

$$\begin{aligned}
T(n,m) &= T(n,m-1) + T(n-1,m) \leq \binom{n+m-3}{n-1} + \binom{n+m-3}{n-2} \\
&= \frac{(n+m-3)!}{(n-1)!(m-2)!} + \frac{(n+m-3)!}{(n-2)!(m-1)!} = \frac{(n+m-3)!}{(n-2)!(m-2)!}(\frac{1}{n-1} + \frac{1}{m-1}) \\
&= \frac{(n+m-2)!}{(n-1!)(m-1)!} \\
&= \binom{n+m-1}{n-1}.
\end{aligned}$$

3. (10 marks) Recall the Fibonacci numbers,

$$fib(n) = \begin{cases} fib(n-1) + fib(n-2) & n \geq 2 \\ 1 & n = 1 \\ 0 & n = 0. \end{cases}$$

Prove that the Fibonacci numbers satisfy the following identities

$$\begin{aligned}
fib(2n-1) &= (fib(n))^2 + (fib(n-1))^2 \\
fib(2n) &= (fib(n))^2 + 2fib(n)fib(n-1)
\end{aligned}$$

for all natural numbers $n \geq 1$.

SOLN:

Let $S(n)$ be the statement:

$$\text{“} \begin{aligned}
fib(2n-1) &= (fib(n))^2 + (fib(n-1))^2 \\
fib(2n) &= (fib(n))^2 + 2fib(n)fib(n-1)
\end{aligned} \text{”}$$

WTS that $S(n)$ is true for all natural numbers $n \geq 1$.

BASE CASE: n=1.

Then $fib(1) = 1$ and $fib(2) = 1$ by definition. It is easy to see that $fib(2n-1) = 1^2 + 0^2 = 1$ and $fib(2n) = 1^2 + 0 = 1$. Hence the base case holds.

INDUCTIVE HYPOTHESIS: Assume that $S(k)$ holds for all $1 \leq k < n, k \in \mathbb{N}$ where n is an arbitrary natural number.

INDUCTIVE STEP: We will prove $S(n)$.

$$\begin{aligned} fib(2n-1) &= (fib(n))^2 + (fib(n-1))^2 \\ fib(2n) &= (fib(n))^2 + 2fib(n)fib(n-1). \end{aligned}$$

Notice that we can rewrite $fib(2n-1)$ as

$fib(2n-1) = fib(2n-2) + fib(2n-3) = fib(2(n-1)) + fib(2(n-1)-1)$ since $n \geq 2$.

By the inductive hypothesis,

$$\begin{aligned} fib(2(n-1)) &= (fib(n-1))^2 + 2fib(n-1)fib(n-2) \\ &= fib(n-1)(fib(n-1) + 2fib(n-2)) \\ &= fib(n-1)(fib(n) + fib(n-2)) and \\ \\ fib(2(n-1)-1) &= (fib(n-1))^2 + (fib(n-2))^2. \end{aligned}$$

Therefore,

$$\begin{aligned} fib(2n-1) &= fib(n-1)(fib(n) + fib(n-2)) + (fib(n-1))^2 + (fib(n-2))^2 \\ &= fib(n-1)fib(n) + fib(n-1)fib(n-2) + (fib(n-1))^2 + (fib(n-2))^2 \\ &= fib(n-1)fib(n) + fib(n-2)(fib(n-1) + fib(n-2)) + (fib(n-1))^2 \\ &= fib(n-1)fib(n) + fib(n-2)fib(n) + (fib(n-1))^2 \\ &= fib(n)(fib(n-1) + fib(n-2)) + (fib(n-1))^2 \\ &= (fib(n))^2 + (fib(n-1))^2. \end{aligned}$$

In a similar fashion, we can prove the second identity:

$fib(2n) = fib(2n-1) + fib(2(n-1)) = fib(2n-3) + 2fib(2(n-1))$.

By the inductive hypothesis

$fib(2(n-1)-1) = (fib(n-1))^2 + (fib(n-2))^2$ and

$fib(2(n-1)) = fib(n-1)(fib(n) + fib(n-2))$ (see previous part).

Therefore,

$$
\begin{aligned}
fib(2n) &= (fib(n-1))^2 + (fib(n-2))^2 + 2[fib(n-1)(fib(n) + fib(n-2)) \\
&= 2fib(n)fib(n-1) + (fib(n-1))^2 + 2fib(n-1)fib(n-2) + (fib(n-2))^2 \\
&= 2fib(n)fib(n-1) + (fib(n-1) + fib(n-2))^2 \\
&= (fib(n))^2 + 2fib(n)fib(n-1).
\end{aligned}
$$

4. (10 marks) Consider a chocolate bar of dimensions $2 \times n$. How many different ways are there to split the bar up into 2×1 size pieces? For example, a 2×2 bar has 2 ways, a 2×3 bar has 3 ways, a 2×4 bar has 5 ways, etc.

   Prove your answer using induction for all natural numbers $n > 1$.

   SOLN:

   Proceeding left-to-right through the chocolate bar, you can either break off one 2×1 piece, and then proceed with a chocolate bar of dimension $2 \times (n-1)$, or break off two 1×2 pieces, and then proceeds with a chocolate bar of dimension $2 \times (n-2)$. So a suitable recurrence for this problem is:

$$
T(n) = \begin{cases} T(n-1) + T(n-2) & n \geq 3 \\ 2 & n = 2 \\ 1 & n = 1. \end{cases}
$$

   Note that this is almost the Fibonacci sequence from Question 3. In fact, with the exception of $n = 0$, on which T is not defined, $T(n) = fib(n+1)$, and so on its domain ($n \geq 1$):

$$
T(n) = \frac{(1+\sqrt{5})^{n+1} - (1-\sqrt{5})^{n+1}}{2^{n+1} \cdot \sqrt{5}}.
$$

   Proof: by induction. $P(n)$ is the above closed-form formula for $T(n)$. We want to show that for all $n \geq 1, P(n)$.

   Base case 1: $n = 1$.

$$
\begin{aligned}
\tfrac{(1+\sqrt{5})^2 - (1-\sqrt{5})^2}{4 \cdot \sqrt{5}} &= \tfrac{1+5+2\cdot\sqrt{5}-1-5+2\cdot\sqrt{5}}{4\cdot\sqrt{5}} \\
&= \tfrac{4\cdot\sqrt{5}}{4\cdot\sqrt{5}} \\
&= 1 \\
&= T(1).
\end{aligned}
$$

   Inductive case: Given $P(j)$ for all $j < 3 \leq n$, WTS $P(n)$. We begin by noting that both $x = (1+\sqrt{5})/2$ and $x = (1-\sqrt{5})/2$ satsify the equation $x^2 = x + 1$ (*):

$$
\begin{aligned}
(1+\sqrt{5})^2/4 &= (1 + 5 + 2\sqrt{5})/4 \\
&= (6 + 2\sqrt{5})/4 \\
&= (3 + \sqrt{5})/2 \\
&= (1 + \sqrt{5})/2 + 2/2 \\
&= (1 + \sqrt{5})/2 + 1
\end{aligned}
$$

$$\begin{aligned}(1-\sqrt{5})^2/4 &= (1+5-2\sqrt{5})/4\\ &= (6-2\sqrt{5})/4\\ &= (3-\sqrt{5})/2\\ &= (1-\sqrt{5})/2+2/2\\ &= (1-\sqrt{5})/2+1.\end{aligned}$$

Let $\alpha = (1+\sqrt{5})/2$ and $\beta = (1-\sqrt{5})/2$. Then:

$$\begin{aligned}\frac{(1+\sqrt{5})^n-(1-\sqrt{5})^n}{2^n\cdot\sqrt{5}} &= \frac{2^n\alpha^n-2^n\beta^n}{2^n\cdot\sqrt{5}}\\ &= \frac{\alpha^n-\beta^n}{\sqrt{5}}\end{aligned}$$

and so:

$$\begin{aligned}T(n) &= T(n-1)+T(n-2)\\ &= \frac{\alpha^n-\beta^n}{\sqrt{5}}+\frac{\alpha^{n-1}-\beta^{n-1}}{\sqrt{5}}\\ &= \frac{\alpha^n-\beta^n+\alpha^{n-1}-\beta^{n-1}}{\sqrt{5}}\\ &= \frac{\alpha^{n-1}(\alpha+1)-\beta^{n-1}(\beta+1)}{\sqrt{5}}\\ &= \frac{\alpha^{n-1}(\alpha^2)-\beta^{n-1}(\beta^2)}{\sqrt{5}}[\text{by}(*)]\\ &= \frac{\alpha^{n+1}-\beta^{n+1}}{\sqrt{5}}\\ &= \frac{(1+\sqrt{5})^{n+1}-(1-\sqrt{5})^{n+1}}{2^{n+1}\sqrt{5}}.\end{aligned}$$

5. (10 marks) Let M be the smallest set of real-valued matrices such that

- $\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}, \begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix} \in M$,
- if $m_1, m_2 \in M$, then $m_1 \cdot m_2 \in M$, and
- if $m \in M$, $r \in \mathbb{R}$ and $r \neq 0$, then $m_r \in M$, where m_r is obtained from m by multiplying every entry in the first row of m by r.

Prove inductively that every matrix in M is invertible (HINT: prove that the determinant is not 0).

SOLN:

A matrix is invertible iff its determinant is not 0, so it suffices to show that every element of M has a non-zero determinant. Recall that:

$$det(\begin{bmatrix} a & b \\ c & d \end{bmatrix}) = ad - bc.$$

We will prove by structural induction that for all $m \in M$, $P(m)$, namely "(1) m has a non-zero determinant, and (2) m is a 2×2 matrix."

BASE CASE 1: $det(\begin{bmatrix} 0 & 1 \\ 1 & 0 \end{bmatrix}) = 0 \cdot 0 - 1 \cdot 1 = -1$. (2) clearly holds.

BASE CASE 2: $det(\begin{bmatrix} 1 & 0 \\ 1 & 1 \end{bmatrix}) = 1 \cdot 1 - 0 \cdot 1 = 1$. (2) clearly holds.

The inductive hypothesis is that $P(m)$ holds for all structurally simpler $m \in M$.

RECURSIVE CASE 1: By the inductive hypothesis (2), m_1 and m_2 are 2×2 matrices. So is $m_1 \cdot m_2$, as it has the same number of rows as m_1, and the same number of columns as m_2. This proves (2).

Without loss of generality, assume that:

$$m_1 = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \quad m_2 = \begin{bmatrix} e & f \\ g & h \end{bmatrix}.$$

In this case:

$$m_1 \cdot m_2 = \begin{bmatrix} ae + bg & af + bh \\ ce + dg & cf + dh \end{bmatrix}$$

so:

$$\begin{aligned} det(m_1 \cdot m_2) &= (ae + bg)(cf + dh) - (af + bh)(ce + dg) \\ &= aecf + aedh + bgcf + bgdh - afce - afdg - bhce - bhdg \\ &= aedh - bhce - afdg + bgcf + aecf - aecf + bgdh - bgdh \\ &= eh(ad - bc) - fg(ad - bc) \\ &= (eh - fg)(ad - bc). \end{aligned}$$

By the inductive hypothesis (1), neither of these factors is 0, and therefore, their product is not zero.

RECURSIVE CASE 2: By construction, m_r has the same dimensions as m, which by the inductive hypothesis (2), is 2×2. Without loss of generality:

$$m_r = \begin{bmatrix} ra & rb \\ c & d \end{bmatrix}$$

where:

$$m = \begin{bmatrix} a & b \\ c & d \end{bmatrix}.$$

So $det(m_r) = rad - rbc = r(ad - bc)$. By the inductive hypothesis (1), the second factor is not 0, and by construction, $r \neq 0$, so the product is not 0 either.

6. (10 marks)

Prove that the following program is correct with respect to the following Precondition/Postcondition pair.
(Definition of **div**: if a, b are integers with $b > 0$, then a div b and a mod b are the unique integers such that $a = (a \text{ div } b)b + a \text{ mod } b$.)
(HINT: You may use without proof the fact that $y = \lfloor \sqrt{m} \rfloor$ if and only if $y^2 \leq m$ and $(y+1)^2 > m$.)

Precondition: m is an integer, $m \geq 0$.

Postcondition: The program returns $\lfloor \sqrt{m} \rfloor$.

```
SQRT(m) {
if m = 0 then
    return 0
else
    x := SQRT(m div 4)
    if (2 * x + 1) * (2 * x + 1) ≤ m then
```

```
        return 2 * x + 1
    else
        return 2 * x
    end if
end if }
```

SOLN:

Let $P(m)$ be: "The program call SQRT(m) returns $\lfloor\sqrt{m}\rfloor$." We will use complete induction to prove that $P(m)$ holds for every integer $m \geq 0$. Let $i \geq 0$ be an integer such that $P(j)$ holds for every integer j, $0 \leq j < i$. We will show $P(i)$.

CASE 1: $i = 0$. In this case we observe that the call SQRT(i) immediately halts, returning the correct answer 0.

CASE 2: $i \geq 1$. Since $i \geq 0$, i div $4 \geq 0$. Since $i > 0$, i div $4 \leq i/4 < i$. So by the inductive hypothesis, $P(i \text{ div } 4)$ holds, so the call SQRT$(i \text{ div } 4)$ halts and returns $\lfloor\sqrt{i \text{ div } 4}\rfloor$. Since $i \neq 0$, the call SQRT(i) invokes the call SQRT$(i \text{ div } 4)$, which returns and assigns to x the value $\lfloor\sqrt{i \text{ div } 4}\rfloor$.

So we have $1 \leq i = 4 \cdot (i \text{ div } 4) + (i \text{ mod } 4)$, and $0 \leq (i \text{ mod } 4) < 4$, and
$x^2 \leq (i \text{ div } 4)$, and $(x+1)^2 > (i \text{ div } 4)$, so $(x+1)^2 \geq (i \text{ div } 4) + 1$.

$(2x)^2 = 4x^2 \leq 4(i \text{ div } 4) \leq i$. In the case where $(2x+1)^2 > i$, the call SQRT(i) returns $2x$, which is indeed $\lfloor\sqrt{i}\rfloor$.

So assume $(2x+1)^2 \leq i$. In this case the call SQRT(i) returns $2x+1$, so it remains to show that $(2x+2)^2 > i$. We have $(2x+2)^2 = 4(x+1)^2 \geq 4((i \text{ div } 4) + 1) = 4(i \text{ div } 4) + 4 > i$.

So $P(i)$ holds.

7. (15 marks) Prove that the following program is correct with respect to the following Precondition/Postcondition pair. (Intuitively, the program is testing whether an array is sorted.)

Precondition: m is an integer, $m \geq 1$. A is an integer array.

Postcondition: If for all i, j such that $1 \leq i < j \leq m$ it is the case that $A[i] \leq A[j]$, then TRUE is returned; otherwise FALSE is returned.

```
k := 1
while k < m and A[k] ≤ A[k + 1] do
    k := k + 1
end while
if k = m then
    return TRUE
else
    return FALSE
end if
```

SOLN:

Loop Invariant Lemma: Given the preconditions and that there are at least l iterations, $k_l \leq m$, and for all $1 \leq i \leq j \leq k_l$, $A[i] \leq A[j]$.

Partial Correctness: Given that the loop terminates, there must be some final iteration n, and thus either:

Case 1: $k_n \geq m$. Yet by the LIL, $k_n \leq m$, so $k_n = m$. So the program returns TRUE. By the LIL, for all $1 \leq i \leq j \leq k_n = m$, $A[i] \leq A[j]$, as required by the postcondition. Or . . .

Case 2: $k_n < m$ and $A[k_n] > A[k_n + 1]$. So the program returns FALSE, and since $k_n + 1 \leq m$, k_n and $k_n + 1$ are a suitable i and j, respectively, to refute the claim that for all $1 \leq i \leq j \leq m$, $A[i] \leq A[j]$. So FALSE is predicted by the postcondition.

Proof of the LIL: by simple induction. $P(l)$ is the above. We want to show that $P(l)$ is true for all $l \in \mathbb{N}$.

$P(0)$: k_0 = 1, so $i = j = 1$. Also, by the precondition, $k_0 = 1 \leq m$.

Given $P(l)$, WTS $P(l + 1)$: Given that there at least $l + 1$ iterations, then $k_l < m$, so $k_{l+1} = k_l + 1 \leq m$. Also $A[k_l] \leq A[k_l + 1]$. By induction, for all $1 \leq i \leq j \leq k_l$, $A[i] \leq A[j]$, and so for all $1 \leq i \leq j \leq k_{l+1}$, $A[i] \leq A[j]$, since $k_{l+1} = k_l + 1$.

Termination: Given the precondition, and that there are at least l iterations, let $t_l = m - k_l$.

Claim 1: $t_l \geq 0$, where it is defined.

Proof: By the LIL, $k_l \leq m$, and therefore $0 \leq m - k_l = t_l$.

Claim 2: $t_{l+1} < t_l$, where they both exist.

Proof: They both exist where there have been at least $l + 1$ iterations, and so $k_{l+1} = k_l + 1$. Then $t_{k+1} = m - k_{l+1} = m - k_l - 1 < m - k_1 = t_l$.