CSC 2534— Decision Making Under Uncertainty
Assignment 1 — Solutions

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1. **[35 points]**

   (a) **[15]**

   (i) **[2]** While the initial ordering doesn't indicate strict preference among the outcomes, the fact that *any* lottery is *strictly* preferred to some other lottery (as indicated in each of the constraints) is sufficient to conlcude that some strict preference exists among outcomes (hence that the best is strictly better than the worst).

   (ii) **[11]** There are five outcomes to consider, which we analyze in an order so that each bound can build on the others:

   - z_1: $u(z_1) = 1$ (since it is best, and by normalization).
   - z_5: $u(z_5) = 0$ (since it is worst, and by normalization).
   - z_2: $u(z_2) \in (0.8, 0.9)$. This holds by the second preference constraint, since $u(z_1) = 1$, $u(z_5) = 0$, $EU(\langle 0.9, z_1; 0.1, z_5\rangle) = 0.9$ and $EU(\langle 0.8, z_1; 0.2, z_5\rangle) = 0.8$.
   - z_4: $u(z_4) \in (0.2, 0.3)$. We use the third preference constraint to derive its bound: notice that each of the three lotteries in this constraint is a mixture of z_4 and a standard gamble $EU(p)$ that selects outcome z_1 with probability p and z_5 with probability $1 - p$, hence whose expected utility is p. Specficially, we have:

   $$x = EU(\langle 0.42, z_1; 0.2, z_4; 0.38, z_5\rangle) \quad = EU(\langle 0.2, z_4; 0.8, EU(\frac{42}{80})\rangle) = 0.2u(z_4) + 0.42$$
   $$y = EU(\langle 0.3, z_1; 0.6, z_4; 0.1, z_5\rangle) \quad = EU(\langle 0.6, z_4; 0.4, EU(\frac{3}{4})\rangle) = 0.6u(z_4) + 0.3$$
   $$z = EU(\langle 0.38, z_1; 0.2, z_4; 0.42, z_5\rangle) \quad = EU(\langle 0.2, z_4; 0.8, EU(\frac{38}{80})\rangle) = 0.2u(z_4) + 0.38$$

   The third preference constraint tells us $x > y$: solving for $u(z_4)$ gives $0.3 > u(z_4)$. It also tells us $y > z$: solving for $u(z_4)$ gives $0.2 < u(z_4)$.
   - z_3: $u(z_3) \in (0.5, 0.7)$. We use the fourth preference constraint to derive its bound. The first lottery is a standard gamble, $\langle 0.7, z_1; 0.3, z_5\rangle$ with expected utility 0.7, hence $0.7 > u(z_3)$. The third lottery, $\langle 0.5, z_2; 0.5, z_4\rangle$, has expected utility $0.5u(z_2) + 0.5u(z_4)$. If we plug in the lower bounds for z_2 and z_4 derived above, we have $u(z_3) > 0.5(0.8) + 0.5(0.2) = 0.5$.

   (iii) **[2]** The utilities for these five outcomes cannot be set independently anywhere within their upper and lower bounds. Specifically, the lower bound on $u(z_3)$ depends on the values of both $u(z_2)$ and $u(z_4)$ via the inequality: $u(z_3) > 0.5u(z_2) + 0.5u(z_4)$.

   (b) **[10]** In each case, I'll call the first lottery/outcome A and the second B.

   (i) **[2]** Yes, it is easy to see B is preferred to A, by monotonicity: B is equivalent to A with one outcome z_4 replaced by a strictly preferred outcome z_2.

(ii) **[2]** Yes, A is preferred to B. A has its lowest utility relative to z_3 if we set outcomes z_2 and z_4 to have their lowest possible utility (recall z_1 has utility 1), so $EU(A) \geq 0.3 + 0.08 + 0.02 + 0.5u(z_3)$, so A is preferred to B if $0.4 > 0.5u(z_3)$. For any possible value $u(z_3) \in (0.5, 0.7)$, this inequality holds. (Note: the constraint that $u(z_3) > 0.5u(z_2) + 0.5u(z_4)$ holds in addition to the bounds derived above hold trivially when we set z_2, z_4 to their lower bounds.)

(iii) **[2]** Yes, it is easy to see B is preferred to A, again by monotonicity: B is equivalent to A with the exception that the probability of a preferred outcome z_3 is higher than in A, and the probability of a less preferred outcome z_4 is lower than in A.

(iv) **[4]** We can't prove that either of A or B is preferred to the other. First: By setting the outcome z_4 within A to its upper bound, we obtain $EU(A) = 0.65$. (I'm being loose here because the bounds are actually open intervals, but all equalities hold within an arbitrarily small ε. It's OK if you are similarly "loose.") With $u(z_4) = 0.3$ (its upper bound), we can't set z_3 in B to its lower bound because of the constraint $u(z_3) > 0.5u(z_2) + 0.5u(z_4)$ (see above). But $u(z_3)$ can be as low as 0.55 with this inequality holding (set z_2 to its lower bound and z_4, as above to its upper bound). So $EU(B) = 0.55$ in this case. So it is possible that A is preferred to B.

Second: By setting z_4 to its lower bound and z_3 to its upper bound (the constraint on z_3 is trivially satisfied then) we have $EU(A) = 0.6$ and $EU(B) = 0.62$, so it is also possible that B is preferred to A.

(c) **[10]**

(i) **[3]** PMR of A relative to B is the worst case difference $EU(B) - EU(A)$ which is $0.7 - 0.65 = 0.05$ (setting z_3 to its upper bound in B (there is no uncertainty in A's utility).

PMR of B relative to A is the worst case difference $EU(A) - EU(B)$ which is $0.65 - 0.5 = 0.15$ (setting z_3 to its lower bound in B. A has minimax regret (with max regret 0.05).

(ii) **[5]** PMR of C relative to D is the worst case difference $EU(D) - EU(C)$ which is $0.82 - 0.68 = 0.14$ (setting z_3 to its upper bound in D, and z_2, z_4 to their lower bounds in C.

PMR of D relative to C is the worst case difference $EU(C) - EU(D)$. This is less straightforward because of the constraint $u(z_3) > 0.5u(z_2) + 0.5u(z_4)$, so we can't simply set z_2, z_4 to their upper bounds in C without impacting the minimum value z_3 can take in D. However, it's not hard to see that the maximum advantage of C over D is attained by setting z_2 to its upper bound, z_4 to its lower bound, and then setting z_3 as low as permitted by the constraint. A simple justification: every δ increase in the value of z_2 causes an increase of 0.5δ in z_3 by the constraint; because of the lottery probabilities, it induces an increase of 0.4δ in the utility of C and 0.35δ in the utility of D (so advantage is maximized by *maximizing* z_2). Conversely, every δ increase in the value of z_4 also causes an increase of 0.5δ in z_3 by the constraint; because of the lottery probabilities, it induces an increase of 0.3δ in the utility of C and 0.35δ in the utility of D (so advantage is maximized by *minimizing* z_4).

Setting $u(z_2) = 0.9, u(z_4) = 0.2$, and $u(z_3) = 0.55$ has required by the constraint gives a PMR of $EU(C) - EU(D) = 0.72 - 0.785 = -0.065$. Hence D is always better than C no matter what the utilities: so D has minimax regret (and max regret of 0).

(iii) **[2]** A single bound query asking the decision maker whether B (i.e., z_3) is preferred to the standard lottery correponding to A (i.e., best outcome with probability 0.65 and worst with 0.35) trivially determines which of A or B is better.

2. **[25 points]** Let $\succeq$ be a preference function over lotteries satisfying the axioms. For any outcome $s \in S$, the "preference" for s refers to the preference for the degenerate lottery that gives s with probability 1.0.

   We first note that by orderability, transitivity, and the finiteness of S, we must have a best and worst outcome; that is, there is some $s_\top \in S$ s.t. $s_\top \succeq s$ for all $s \in S$, and some $s_\bot \in S$ s.t. $s_\bot \preceq s$ for all $s \in S$. Let $S = \{s_1, \ldots, s_n\}$.

   By continuity, for any $s_i \in S$, there exists a probability u_i s.t. $s_i \sim \langle u_i, s_\top; 1-u_i, s_\bot\rangle$. Furthermore, this u_i is unique, since—due to monontonicity and nontriviality—increasing or decreasing u_i results in a more or less preferred lottery. So let the u be the utility function $u : S \to [0,1]$ where $u(s_i) = u_i$. (Note that $u(s_\top) = 1$ and $u(s_\bot) = 0$.) We now show that u satisfies the requirements of the theorem.

   Suppose $l_1 \succ l_2$ for two lotteries:

   $$l_1 = \langle p_1^1, s_1; p_1^2, s_2; \ldots; p_1^n, s_n\rangle \text{ and } l_2 = \langle p_2^1, s_1; p_2^2, s_2; \ldots; p_2^n, s_n\rangle$$

   We then have

   $$\begin{aligned} l_1 &\sim \langle p_1^1, \langle u_1, s_\top; 1-u_1, s_\bot\rangle; p_1^2, s_2; \ldots; p_1^n, s_n\rangle \\ &\sim \langle p_1^1, \langle u_1, s_\top; 1-u_1, s_\bot\rangle; p_1^2, \langle u_2, s_\top; 1-u_2, s_\bot\rangle; \ldots; p_1^n, s_n\rangle \\ \ldots \\ &\sim \langle p_1^1, \langle u_1, s_\top; 1-u_1, s_\bot\rangle; p_1^2, \langle u_2, s_\top; 1-u_2, s_\bot\rangle; \ldots; p_1^n, \langle u_n, s_\top; 1-u_n, s_\bot\rangle\rangle \end{aligned}$$

   In other words, we replace each outcome s_i in sequence by its corresponding "standard gamble." The sequence of indifference statements is valid due to substitutability and transitivity. By decomposability (reduction of compound lotteries), we then have

   $$l_1 \sim \left\langle (\sum p_1^i u_i), s_\top; 1 - (\sum p_1^i u_i), s_\bot \right\rangle$$

   By identical reasoning

   $$l_2 \sim \left\langle (\sum p_2^i u_i), s_\top; 1 - (\sum p_2^i u_i), s_\bot \right\rangle$$

   Thus $l_1 \succ l_2$ iff

   $$\left\langle (\sum p_1^i u_i), s_\top; 1 - (\sum p_1^i u_i), s_\bot \right\rangle \succ \left\langle (\sum p_2^i u_i), s_\top; 1 - (\sum p_2^i u_i), s_\bot \right\rangle$$

   But by monotonicity, this holds iff $\sum p_1^i u_i > \sum p_2^i u_i$, which is equivalent to stating that $EU(l_1) > EU(l_2)$.

3. **[25 points]** The proof is straightforward. We'll call the first condition AX1 and the second AX2. Let c be a choice function satisfying AX1 and AX2. We'll define the following preference relation $\succeq$ based on c: for any $x, y \in X$, let $x \succeq y$ iff $x \in c(\{x,y\})$. First we show that $\succeq$ is a preference relation, i.e., connected and transitive.

   By definition of a choice function, either $x \in c(\{x,y\})$ or $y \in c(\{x,y\})$; so we have either $x \succeq y$ or $y \succeq x$, hence $\succeq$ is connected.

To show transitivity, suppose $x \succeq y$ (which means $x \in c(\{x, y\})$) and $y \succeq z$ (which means $y \in c(\{y, z\})$). We just need to show that $x \in c(\{x, z\})$ to prove transitivity. We will show that, in fact, $x \in c(\{x, y, z\})$, which implies $x \in c(\{x, z\})$ by AX1. By way of contradiction, suppose $x \notin c(\{x, y, z\})$. First we show y must be in $c(\{x, y, z\})$. If $y \notin c(\{x, y, z\})$, then $c(\{x, y, z\}) = z$, which by AX1 implies $z \in c(\{y, z\})$. But since $y \in c(\{y, z\})$, by AX2 we must have $y \in c(\{x, y, z\})$. So we know $y \in c(\{x, y, z\})$. But this means $y \in c(\{x, y\})$ by AX1. And together with the fact that $x \in c(\{x, y\})$, AX2 implies that $x \in c(\{x, y, z\})$. Hence $\succeq$ is transitive.

Second we must show that the choice function $c_{\succeq}$ induced by $\succeq$ (i.e., the choice function induced by selecting the best elements in any set according to $\succeq$) is identical to c. Let A be a non-empty subset of X. Suppose $x \in c(A)$. By AX1, we know $x \in c(\{x, y\})$ for any $y \in A$. By definition, $x \succeq y$ for any $y \in A$, so $x \in c_{\succeq}(A)$. Now suppose $x \in c_{\succeq}(A)$. This implies $x \succeq y$ for any $y \in A$, which by definition means $x \in c(\{x, y\})$ for any $y \in A$. Let z be some element of $c(A)$. If $z = x$, then clearly $x \in c(A)$, so suppose $z \neq x$. By AX1, $z \in c(\{x, z\})$, so $c(\{x, z\}) = \{x, z\}$. This fact, together with the fact that $z \in c(A)$ implies by AX2 that $x \in c(A)$. This means $x \in c(A)$ iff $x \in c_{\succeq}(A)$.

4. **[15 points]**

(i) Note that the expected monetary value of the investment for both Ali and Barb is \$5200.

(ii) Ali's expected utility for his current investment is given by

$$0.6u_a(10000) + 0.4u_a(-2000) = 0.6\ln(\frac{10000}{500} + 6) + 0.4\ln(\frac{-2000}{500} + 6) = 2.232117.$$

His certainty C_a equivalent must satisfy $u_a(C_a) = 2.232117$, or equivalently

$$C_a = (e^{2.232117} - 6)/500 = 1659.79.$$

So Ali requires at least \$1659.79 to sell his investment. Since this is less than the EMV, Ali is clearly risk averse (like much of the investment community in Toronto).

(iii) Barb's expected utility for her current investment is given by

$$0.6u_b(10000) + 0.4u_b(-2000) = 0.6\exp(\frac{x}{10000} - 2) + 0.4\exp(\frac{-2000}{3000} - 2) = 2.303994.$$

Her certainty C_b equivalent must satisfy $u_a(C_b) = 2.303994$, or equivalently

$$C_b = \ln(2.303994) + 2) \cdot 3000 = 8503.93.$$

So Barb requires at least \$8503.93 to sell her investment. Since this is more than the EMV, Barb is clearly risk seeking.

5. **[30 points]**

   (a) **[18 – 2 pts each]** For each you can either give an intuitive argument or simply appeal to the definition of d-separation. In the latter case, the hope is you will understand the underlying intuition of the formal, "graph-theoretic" criterion.

      (i) True (node I cuts the path from G to H, since it is a head-to-head node with no descendents as evidence)

      (ii) False (the path $H \rightarrow I \rightarrow J$ is not blocked: there is no evidence and there are no head-to-head nodes to block it)

- (iii) True (node M cuts the path from H to L, since it is a head-to-head node with no descendents as evidence)
- (iv) False (the path is not cut as in the previous question, since N–a descendent of M–is evidence)
- (v) True (node I blocks all paths from E to H, since it is a head-to-head node with no descendents as evidence)
- (vi) False (node I is now in evidence, and no longer blocks any of the paths between E and H)
- (vii) False (node C blocks the path $E \leftarrow C \leftarrow G \rightarrow I \leftarrow H$, as well as the similar paths $E \leftarrow D \leftarrow C \ldots$ and $E \rightarrow F \leftarrow D \leftarrow C \ldots$. But the path $E \leftarrow D \leftarrow B \leftarrow A \rightarrow C \leftarrow G \rightarrow I \leftarrow H$ becomes unblocked because C (and I) is in evidence)
- (viii) False (the paths $G \rightarrow C \rightarrow E \rightarrow F$, $G \rightarrow C \rightarrow D \rightarrow F$, and $G \rightarrow C \rightarrow D \rightarrow E \rightarrow F$, are all unblocked)
- (ix) False (C blocks the paths above, but now the path $G \rightarrow C \leftarrow A \ldots$ is unblocked, since C (head-to-head) is in evidence)
- (x) True (node D blocks all of the paths from A to F rendered active by the presence of C)

(b) **[12]** I'll just give the answers and some comments, not derivations.

- (i) **[2]** 0.418
- (ii) **[2]** 0.8 (directly from the CPT)
- (iii) **[3]** 0.4
- (iv) **[3]** 0.8977
- (v) **[2]** 0.8977. This follows from the previous question, since A is independent of E given C and D