Oblivious Queries over Data with Irregular Structure

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Abstract

In traditional query languages, formulating queries requires an exact knowledge of the schema. In particular, the user should explicitly take into account the possibility of incomplete information. Oblivious querying is an alternative approach that alleviates the need for an exact knowledge of the schema. Specifically, oblivious querying was devised for handling the following cases. First, when data have an irregular structure (e.g., data collected from heterogeneous sources). Second, when the schema of the data is complicated or is frequently changed. Third, when there is no schema or the schema is unknown.

Our approach to oblivious querying of semistructured data is twofold and is based on the paradigm of flexible queries and the paradigm of maximal answers. The paradigm of flexible queries facilitates, in a novel way, easy and concise querying that is independent of any particular representation of the relationships among entities. Two different semantics, *flexible* and *semiflexible*, that admit different levels of flexibility are introduced and investigated. The complexity of evaluating queries under the two semantics is analyzed. Query evaluation is polynomial in the size of the query, the database and the result in the following two cases. First, a semiflexible DAG query and a tree database. Second, a flexible tree query and an arbitrary database.

Query containment and equivalence are also investigated, as a first step towards developing optimization techniques. For both the semiflexible and flexible semantics, query containment and equivalence can be tested in polynomial time.

Under the semiflexible and flexible semantics, two databases could be equivalent even when they are not isomorphic. Database equivalence is formally defined and characterized. Moreover, the complexity of deciding equivalences among databases is analyzed. The implications of database equivalence on query evaluation are explained. In particular, transforming a database graph into an equivalent tree is investigated for both the semiflexible and the flexible semantics. Transforming a database to an equivalent tree is important for representing the data as an XML document with no references. *Tree transformable* databases are characterized and a transformation algorithm is provided.

The second paradigm, in our approach, is of queries with maximal answers. This paradigm is based on the notion of answers that are partial bindings of query variables to database objects. It facilitates a formulation of queries that is independent of whether the data is incomplete or not. The answer to a query consists of maximal rather than complete matchings, i.e., some query variables may be assigned null values. Answers that can be extended, by mapping more variables, are discarded since they are not maximal. In the relational model, a similar effect is achieved by computing the *full disjunction* (rather than the natural join or equijoin) of the given relations. For two semantics that allow maximal answers, the OR-*semantics* and the *weak semantics*, it is shown that query evaluation has a polynomial-time complexity in the size of the query, the database and the result.

This work shows how to combine the two paradigms and their proposed semantics. For combined semantics, query evaluation is polynomial in the size of the query, the database and the result in the following two cases. First, a tree query under the combination of the semiflexible semantics with either the weak or the OR-semantics. Secondly, a query and a database that are arbitrary graphs, under the combination of the flexible semantics with either the weak or the OR-semantics.

In the last part of the work, it is shown that the evaluation of full disjunctions is reducible to query evaluation under the weak semantics and hence can be done in polynomial time in the size of the input and the output. Complexity results are also given for the problem of evaluating a projection of the full disjunction. In the special case of γ-acyclic relation schemes, the projection problem has a polynomial-time algorithm in the size of the input and the output. In the general case, such an algorithm does not exist, assuming that $P \neq NP$. Finally, it is shown how to use full disjunction for transforming a semistructured database into a universal relation. This transformation admits querying a single relation with a known schema as an alternative to querying a graph whose schema may be unknown.

Contents

Chapter 1

Introduction

Querying Semistructured Data and XML

Semistructured data occur in situation where information has an irregular structure and is incomplete. Consequently, the semistructured data model is aimed at representing data that do not conform to a strict schema. The lack of a schema may be the result of frequent changes in the structure of the data. It can also be typical of an environment where many different users contribute data to the database, in a variety of forms. The World-Wide Web is such an environment [1, 2, 13].

The semistructured data model is formally described in Chapter 2. In the semistructured data model, databases are rooted labeled directed graphs. Nodes in the graph represent objects. Relationships between pairs of objects are represented by labeled edges. Values are attached to *atomic nodes*, i.e., nodes that have no outgoing edge. In many cases, the data is *self describing.* That is, there is no schema that explicitly defines the structure of the data. Instead, the structure of the data is implicitly described by the graph.

Recently, the eXtensible Markup Language (XML) [64] has become a popular notation for semistructured data. Essentially, XML has been developed for data exchange between applications. In data exchange, the sender and receiver do not necessarily use the same schema to represent data. Hence, XML documents do not have to conform to a schema. Moreover, schemas for XML (either DTD or XML Schema [63]) do not rigorously define the structure of conforming documents. Thus, in many cases, even XML data that conform to a schema are incomplete or do not comply with a strict pattern.

Traditional query languages and traditional querying methods are not well suited for semistructured data or XML. For one, the semistructured data model is based on a labeled directed graph. More importantly, traditional query languages are not adapted to data having no schema at all, nor to data whose schema may change considerably over time or from one data instance to another. An additional consideration is the size of the schema, which could be quite large compared to the size of schemas for structured data.

When the schema is large and complicated, querying the data could be rather difficult. An initial phase of querying the schema might be needed before the query can be formulated. Even with this additional step, the query could be quite large and hard to phrase, due to the need to cover many structurally similar, but not structurally identical data instances. Furthermore, an increase in the size of the query may cause an increase of the time complexity.

The size of the schema is not the only source of difficulties. A case in point is an XML repository of documents with DTDs designed for machine interchange (as well as DTDs that were actually generated by machines). In the eyes of a layperson, such DTDs might not be easy to grasp, regardless of the size.

Traditional query languages are based on the concept of *rigid matchings*. In a rigid matching there should be a perfect match between the conditions specified in the query and the data. Rigid matchings are very sensitive to variations in the schema and to incompleteness of the data. A query that is posed having a specific schema in mind, is likely to yield only some of the answers or no answer at all, even when only minor changes occur in the structure of the data while the data itself does not change at all.

In recent years, a considerable amount of work has been done on querying semistructured data, in general, and XML data, in particular. Many query languages for semistructured data [4, 7, 8, 15, 46, 50] and for XML [2, 9, 17, 22, 30] have been proposed. In order to deal with the special characteristics of semistructured data, some of these languages use regular expressions. Issues related to the evaluation and optimization of query languages with regular expressions are discussed in

[6, 51]. Additional work has been done on constraining semistructured data [14, 16], describing the structure of the data for query formulation and optimization [36], and extracting information on changes in the data [18].

The query languages cited above are based on rigid matchings, and some of these languages use regular expressions to express possible variations in the structure of the data. But the approach of relying on regular expressions is problematic in at least two aspects. First, the ounce of responsibility is on the user, who should phrase the regular expressions in a way that will cover the possible variations in the structure of the data and yet will *not* cover too much. Secondly, the tasks of query evaluation and optimization become more complex in the presence of regular expressions.

An alternative approach is that of extracting the *ontology* from the schemas (e.g., DTDs). The ontology consists of the list of names given to the elements and to the attributes of the schemas, and is much easier to comprehend than the full structural details of the schemas. In many cases, queries can be phrased simply and succinctly using just the ontology.

Semiflexible and Flexible Queries

In Chapter 3, we present two semantics that are suitable for ontology-based querying of semistructured data. Similarly to existing query languages, our queries are also represented as rooted labeled directed graphs. Thus, preserving the similarity between queries and databases. However, instead of a semantics that is based on rigid matchings, we introduce two new semantics that are based on *semiflexible matchings* and *flexible matchings*. We believe that these semantics capture the intended meaning of many common queries. A user only needs to be familiar with the ontology of the database in order to phrase a query, and hence query formulation is more intuitive and simpler compared to query languages that use regular expressions.

In a rigid-matching semantics, the *implicit* relationships between objects are expressed by labeled edges. More complicated relationships have to be constructed explicitly by formulating queries in terms of labeled directed graphs. The semiflexible semantics, in comparison, also uses labeled directed graphs to describe relationships between objects, but it assumes that database objects are implicitly related if they are connected by a directed path (and not just by an edge). Thus, a path π of the query can match a path ϕ of the database if ϕ includes all the labels of π; however, the inclusion need not be contiguous or in the same order as in π. The flexible semantics further extends this idea by applying transitivity to the implicit relationships that hold under the semiflexible semantics.

In Chapter 3, we investigate query evaluation under the semiflexible and the flexible semantics. Other topics that are included in this chapter are the followings. First, we characterize query equivalence and query containment. Secondly, a novel concept of database equivalence is presented. We provide a characterization of equivalent databases, for both the semiflexible and the flexible semantics. Thirdly, it is shown that some databases could be transformed into equivalent trees. We characterize these databases and provide an algorithm that performs the transformation. The work that is presented in Chapter 3 has been published in [43]. A query language that can query simultaneously relations and XML documents and uses the flexible semantics has been presented in [23].

Queries with Maximal Answers

Queries under the rigid, the semiflexible and the flexible semantics contain variables and constraints on these variables. An answer to a query is a binding of all the query variables to database objects, according to the query constraints. However, when querying incomplete data, the binding of query variables should be relaxed to allow *partial answers*, i.e., answers in which some of the variables do not have a binding to a database object. Otherwise, the user might receive only some of the answers that she expected. Alternatively, the user should know where data might be incomplete and formulate the query accordingly. However, when the structure of the data is unknown or when the schema is complicated, it is practically impossible to know where data might be incomplete.

An important issue in queries with partial answers is subsumptions among answers. An answer is *subsumed* by another answer if the subsuming answer has all the bindings of the subsumed answer plus some additional bindings. There is no need to keep subsumed answers in the result. For example, if the tuple (a, b, c) is an answer, then there is no need to include also the tuple $(a, b, \perp)$ as an answer. If subsumption is not handled judiciously during query evaluation, then it could add an exponential factor to the size of the query result without adding new information. Thus, queries should return only *maximal answers*, i.e., answers that are not subsumed by any other answer.

Several semantics for queries with maximal answers over semistructured data were proposed in [41, 42]. The semantics differ in their degree of incompleteness. Query formulation remains the same under all these semantics, but the user can choose the degree of incompleteness that suits her best.

A significant issue is the complexity of evaluating queries with incomplete answers. Traditionally, complexity of query evaluation is measured in terms of either data complexity or query complexity [61]. But neither one can capture the differences in time complexity between evaluation of queries under complete semantics versus the incomplete semantics of [41, 42]. For all these semantics, data complexity is polynomial and query complexity is exponential.

The move from the traditional complete semantics to an incomplete semantics could potentially have two opposing effects. On one hand there are more answers, but on the other hand finding some answer could be easier, since it is not necessary to satisfy all the conditions of the query.

A more suitable measure of complexity is *input-output* complexity, in which the time to evaluate a query is measured as a function of the size of the input (i.e., both the query and the database) and the output. Yannakakis [65] used this complexity measure to show that there is a polynomial-time algorithm for evaluating the natural join of n relations if the relation schemes are α-acyclic. In the general case, deciding non-emptiness of the natural join of n relations is NP-complete [49].

Two of the semantics, namely, the OR-*semantics* and the *weak semantics* that were presented in [41, 42] are investigated in our work. In [41, 42], it was shown that for both the OR-semantics and the weak semantics, tree as well as DAG queries (without selections and projections) can be evaluated in polynomial time in the size of the input and output. In Chapter 4, we extend this result to queries that may have cycles. The work that is presented in Chapter 4 has been published in [44].

Combining the Paradigms of Flexible Queries with the Paradigm of Maximal Answers

Thus far, we discussed two paradigms. The paradigm of flexible queries and the paradigm of maximal answers. The first paradigm has been developed for managing data that has an irregular structure or an unknown schema. The second paradigm handles incomplete information. As mentioned previously, semistructured data occur in situation where information has an irregular structure and is incomplete. Hence, when querying semistructured data, the two paradigms should be combined.

In Chapter 5, four new semantics are presented. Each new semantics is a combination of a semantics of the flexible-queries paradigm (i.e., the semiflexible and the flexible semantics) with a semantics of the maximal-answers paradigm (i.e., the weak and the OR-semantics). The new semantics facilitate the querying of data that is both irregular and incomplete. Query evaluation is investigated under the new semantics.

Full Disjunctions

Incomplete information may also occur in relational data. Usually, relational query language return *complete* answers, that is, answers that provide bindings to all the variables of the query. If the database has incomplete information, the user has to formulate the query differently, e.g., by using outerjoins or conditions that check whether some attributes are null. The need to be aware of the possibility of incomplete information is further complicated by the fact that the semantics of outerjoins is problematic, since outerjoins are not associative.

Galiando-Legaria [34] proposed *full disjunctions*, which are commutative and associative, as an alternative to outerjoins. Essentially, the full disjunction of a set of relations is obtained by taking all joins of any subset of relations, excluding joins that involve a Cartesian product, and removing subsumed tuples. Rajaraman and Ullman [55] showed that the full disjunction of n relations can be computed by a sequence of natural outerjoins if and only if the relation schemes form a connected and γ-acyclic hypergraph. Rajaraman and Ullman also enunciated full disjunctions as a mechanism for integrating information from the Web, since the user cannot have any a priori knowledge of where information might be incomplete.

The size of a full disjunction could be large and sometimes even exponential in the size of the input. Thus, it is important that the computation of full disjunctions will be performed in polynomial time, under input-output complexity. For relations with relation schemas that form a connected and γ-acyclic hypergraph, a polynomial-time algorithm, under input-output complexity, has been proposed by Rajaraman and Ullman [55]. For the general case, it has been unknown whether a polynomial-time algorithm exists.

In Chapter 6, we show that computing full disjunctions is reducible to evaluating queries under the weak semantics. Thus, it follows that full disjunctions can always be computed in polynomial time in the size of the input and the output. We also give complexity results for two related problems. One is evaluation of a projection of the full disjunction. The other is evaluation of all tuples of the full disjunction that are not null on a given set of attributes.

We also propose a novel way, for oblivious querying of semistructured data, that is based on full disjunctions. This querying method can be applied to tree databases that do not have repeated labels in their paths. In this querying method, the semistructured database is transformed into a universal relation. Consequently, the user can query a relation with a known schema instead of querying the semistructured data. A similar approach has been used in [24, 25] for queries that are just selection and projection over XML. Our approach is different from the approach of [24, 25] in the following aspect. It finds connections between objects based on paths from the root to the leaves of the database, as opposed to paths from the root to specific objects.

When integrating information, full disjunctions are rather limited, since they are based solely on equalities among attributes, as in natural joins and equijoins. In many cases, there is a need to integrate information based on conditions that are more general than merely equalities among attributes. We show how our approach leads to a method of integrating information by means of general types of conditions and not just equality conditions. The work that is presented in Chapter 6 has been published in [44].

Related Work

A somewhat similar approach to ours is the approach of searching in XML documents. Information retrieval deals with the problem of searching for relevant documents in a repository of documents. When searching in XML repositories it is frequently not enough to find relevant documents, since documents are large and may contain a massive amount of data. The user should receive the most relevant fragments of documents rather then whole documents. This problem has been the focus of several papers. These papers made an attempt to combine the paradigm of querying with the paradigm of searching [21, 26, 38]. However, in a search, the result does not always include all the answers to the query. In addition, the result of a search may include items that are not an answer to the query. Thus, searching is different from querying.

A lot of research has been done, over the years, on querying incomplete data. The main issues that were considered are representation of missing information and query evaluation. In logical databases, one could use either a proof-theoretic approach [56] or a model-theoretic approach [62]. If there is some partial knowledge about the missing information, then it could be represented in the form of conditions that restrict the possible values of unknown data items [40]. The complexity of query evaluation in these approaches is investigated in [3, 40, 62].

The outerjoin [20, 29, 45] was introduced in order to preserve tuples that are lost during an ordinary join. Since outerjoins are not associative, it is rather difficult to formulate queries that use them. Moreover, the lack of associativity restricts the ability to optimize queries with outerjoins. Efficient evaluation methods for outerjoin queries is the focus of several papers [12, 19, 33, 57].

Full disjunctions were proposed by Galiando-Legaria [34] as an alternative to outerjoins, since full disjunctions are commutative and associative. Full disjunctions bear some similarity to the weak instances that were investigated extensively in the framework of the universal-relation model [37, 39, 58, 59]. In both approaches, all the tuples of the base relations are preserved. In full disjunctions, tuples are joined together whenever possible. The joins should be connected in order to avoid combinations of tuples that are evidently not related. In weak instances, tuples are joined together as implied by the dependencies and queries can be evaluated by taking the union of all lossless joins [58, 59].

Different notions of acyclicity for hypergraphs were investigated in [31], including α-acyclicity and γ-acyclicity. Properties of acyclic database schemes were investigated in [11]. (Note that a set of relation schemes can be represented as a hypergraph by viewing the attributes as nodes and viewing each relation scheme as a hyperedge.) Acyclicity leads to another form of query evaluation over universal relations [32].

Rajaraman and Ullman [55] enunciated the importance of full disjunctions in the context of integrating information from the Web. They showed how full disjunction could be used to find maximal join-consistent connections in trees of the OEM model of [53, 54]. Their main result is that a full disjunction can be evaluated by a natural outerjoin sequence if and only if the relation schemes are connected and γ-acyclic. Their algorithm was implemented in the Information Manifold System [47, 48] that integrates information from the Web.

Chapter 2

The Data Model

In this chapter, we present the data model and the query formulation of our work.

2.1 Databases

The data model that we use in this work is the *semistructured data model*, which is based on the Object Exchange Model (OEM) of [53]. The data are represented by rooted labeled directed graphs. Each node represents an *object* and has an identifier (*oid*). Values are attached to *atomic nodes* (i.e., nodes without outgoing edges). For simplicity, we assume that all atomic nodes are of type PCDATA (i.e., *string*). Nodes with outgoing edges represent *complex objects*. A database has a *root* and every node in the database is reachable from the root.

Definition 2.1 (Database) *Let $\mathcal{L}$ be a set of labels. A* database *D over $\mathcal{L}$ is a 4-tuple $D = (O, E_D, r_D, \alpha)$ that satisfies the following conditions. The set O is a finite set of objects. The set E_D is a finite set of 3-tuples (s, t, l), called labeled edges. In an edge, s and t are objects of O, called the* source *and* target *of the edge, respectively; and l is an element of $\mathcal{L}$, called the label on the edge. The object r_D is the root, and every object of O is reachable from the root via a directed path in D. Finally, α is a function that maps each atomic node to a value. The set of labels $\mathcal{L}$ on the edges of E_D is called the* ontology *of D.*

Example 2.2 Figure 2.1 shows a movie database. Nodes are depicted as circles and the oid of each node appears inside the circle. The values that are attached to

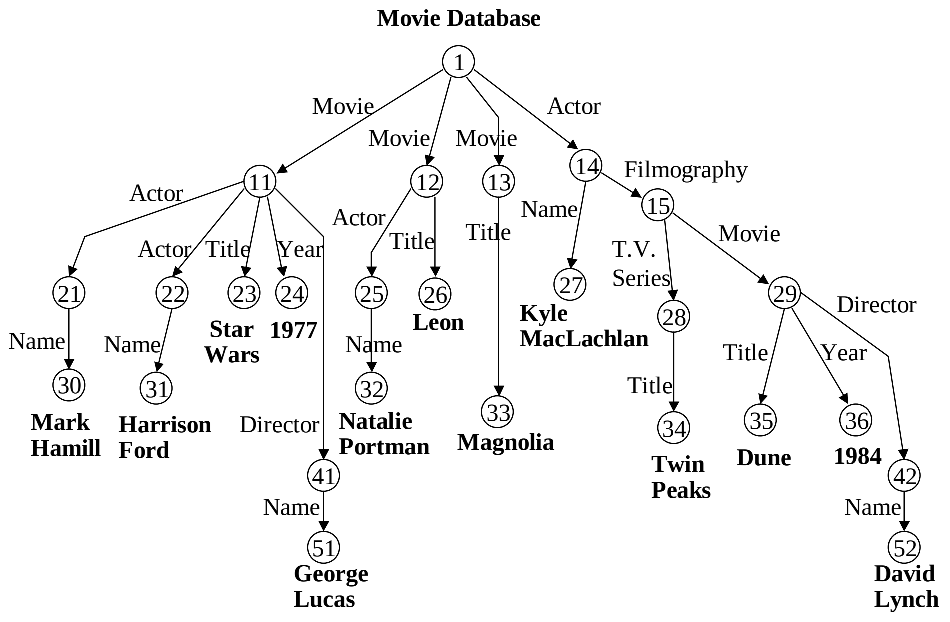


Figure 2.1: A movie database with information on movies, T.V. Series, actors and directors.

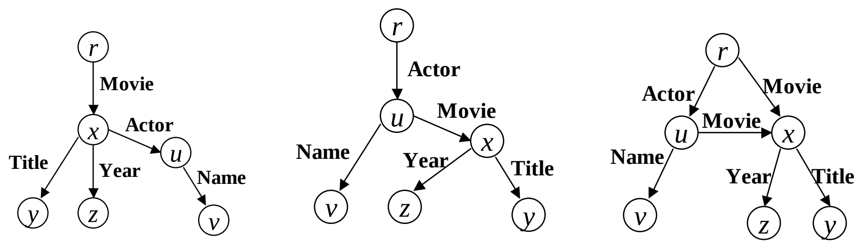
atomic nodes are shown below those nodes, in boldface. Labels appear next to the edges. The ontology of the movie database is the set: {*Movie*, *T.V. Series*, *Actor*, *Director*, *Name*, *Title*, *Year*}.

2.2 Queries

Like databases, queries are also represented as labeled directed graphs and are similar to the queries defined in [41]. Nodes of queries represent variables rather than objects, and no value is associated with any node.

Definition 2.3 (Query) *A* query *is 3-tuple $Q = (V, E_Q, r_Q)$, where V is a finite set of* variables*, E_Q is a set of labeled edges and r_Q is the root of the query.*

Queries are posed to databases. The result of posing a query to a database consists of assignments of database objects to query variables. Queries impose constraints according to their semantics. Thus, a query result comprises of assignments that satisfy the imposed constraints.



(a) Query 1 (b) Query 2 (c) Query 3

Figure 2.2: Three different queries for finding, in the movie database, pairs of an actor and a movie, such that the actor acted in the movie. For actors, the queries return their names. For movies, the queries return the title and the year of production.

Definition 2.4 (Matching) *Suppose that $Q = (V, E_Q, r_Q)$ is a query and $D = (O, E_D, r_D, \alpha)$ is a database. A* satisfying assignment *(or a* matching*) of Q with respect to (w.r.t.) D is a mapping $\mu\colon V \to O$ that satisfies the* constraints *imposed by Q.*

In the following chapters, we will presents several types of query semantics. In each semantics, queries impose different constraints. Now, we introduce two types of constraints for a query Q. In the next two definitions, D denotes a database and μ denotes an assignment of Q w.r.t. D.

Definition 2.5 (Root Constraint) *A* root constraint *(rc) is satisfied by matchings that map the root of Q to the root of D.*

Definition 2.6 (Edge Constraint) *An* edge constraint *(ec) is written as ulv, where ulv is an edge of Q. The ec ulv is satisfied by the mapping μ if D has an edge labeled with l from $\mu(u)$ to $\mu(v)$.*

The usual semantics of queries is defined in terms of rigid matchings. In a rigid matching, two query variables that are connected by an edge with a label l are mapped to database objects that are also connected by an edge with the label l.

Query	The root (r)	Movie (x)	Title (y)	Year (z)	Actor (u)	Name (v)
Query 1	1	11	Star Wars (23)	1977 (24)	21	Mark Hamill (30)
	1	11	Star Wars (23)	1977 (24)	21	Harrison Ford (31)
Query 2	1	29	Dune (35)	1984 (36)	14	Kyle MackLachlan

Table 2.1: The rigid matchings of Query 1 and Query 2 (Figure 2.2) w.r.t. the movie database of Figure 2.1. For Query 3, there are no rigid matchings w.r.t. the movie database. The columns of the table are the query variables and the rows are matchings. Under each column there are the oid's that are assigned to the variable that the column represents. For atomic objects, their values are presented, in addition to the oid.

Definition 2.7 (Rigid Matching) *Consider a query Q and a database D. A* rigid matching *of Q w.r.t. D is an assignment that satisfies the rc and all the ec's of Q. The set of all rigid matchings of Q w.r.t. D is denoted as $\mathcal{M}_r(Q, D)$.*

Note that if μ is a rigid matching of a query Q w.r.t. a database D, then there is a subgraph G of D such that the nodes of G are the objects assigned by μ to the variables of Q and G is isomorphic to Q.

Example 2.8 In Figure 2.2, three different queries are depicted. All the tree queries aimed at finding pairs of an actor and a movie such that the actor acted in the movie. For actors, the queries return their names. For movies, the queries return the title and the year of production. Without any knowledge about the structure of the database, it is impossible to tell whether among the three queries there is one that returns all the required actor-movie pairs. It is also impossible to say if one of these queries is better than the others, i.e., finds more actor-movie pairs than the other two queries. Table 2.1 presents the matchings that are produced when the queries of Figure 2.2 are posed to the movie database. Note that none of the queries return all the required pairs.

Chapter 3

Flexible and Semiflexible Queries

In this chapter, we present two new query semantics. The new semantics facilitate querying of databases where the hierarchy between objects is unknown or irregular. The outline of this chapter is as follows. Section 3.1 defines the flexible and the semiflexible semantics. Section 3.2 provides complexity results for query evaluation under the semiflexible semantics. Section 3.3 gives complexity results for query evaluation under the flexible semantics. The running times of query-evaluation algorithms are cast as functions of the combined size of the query, the database and the result. This approach of analyzing the complexity w.r.t. the size of the query, the database and the result (rather than the more common data complexity) can discern between cases that are computationally hard, even when the result is small, and cases where the running time is governed merely by the size of the result. Section 3.4 gives characterizations and complexity results for containment and equivalence of queries. Under the semiflexible and flexible semantics, two databases could be equivalent (even in nontrivial cases) in the sense that they give the same result for all queries. Section 3.5 characterizes equivalence of two databases. It also characterizes when a given database is equivalent to a tree and gives an algorithm for transforming the database to a tree. This result is important in the context of query evaluation, since queries over tree databases can be evaluated more efficiently, as shown in Section 3.2. Finally, in Section 3.6, we summarize the contributions of this chapter.

3.1 Flexible and Semiflexible Matchings

Before we define the new semantics, we need to introduce two new types of constraints. In the next two definitions, let Q be a query and D be a database.

Definition 3.1 (Weak Edge Constraint) *A* weak edge constraint *(wec) is written as lv, where variable v of Q has an incoming edge labeled with l. The wec lv is satisfied by a mapping μ if $\mu(v)$ has an incoming edge labeled with l.*

Definition 3.2 (Quasi Edge Constraint) *A* quasi edge constraint *(qec) is written as ulv, where u and v are variables of Q and l is the label of an edge from u to v. The qec ulv is satisfied by a mapping μ if D either has a path from $\mu(u)$ to $\mu(v)$ or vice-versa.*

Other types of constraints also include *filtering constraints* which are essentially selections. Considering filtering constraints is beyond the scope of this work. Some of the techniques developed in [41] could be used to extend our results also to queries with filtering constraints.

We introduce some notations. An edge from node u to node v that is labeled with l is denoted as ulv. A path from node u to node v that ends with a label l is denoted as $u{*}lv$. This notation is generalized in the natural way. For example, $ulv{*}kw$ denotes a path that starts at u, continues to v through an edge labeled with l, and then continues to w through a path that ends with the label k.

Next, we introduce two new notions of matchings. The assignment μ is a *semiflexible matching* if it satisfies the rc of Q as well as the following two conditions.

1. For each finite path $v_0 l_1 v_1 l_2 v_2 \cdots l_n v_n$ of Q, where v_0 is the root, the images $\mu(v_i)$ $(0 \leq i \leq n)$ lie on some path ϕ of D, such that for all i $(1 \leq i \leq n)$, the edge of ϕ that enters $\mu(v_i)$ is labeled with l_i. Note that $\mu(v_0)$ is the root of D and, hence, is the first node on ϕ, but the rest of the $\mu(v_i)$ are not necessarily arranged on ϕ in the order $\mu(v_1), \ldots, \mu(v_n)$. Moreover, other nodes could be on ϕ between the $\mu(v_i)$.

2. For every strongly connected component C in Q, the image of C is a strongly connected component in D.

A strongly connected component in a graph is a set of nodes, such that between every two nodes there is a path in both directions. The preservation of the strongly connected components is needed in order to deal with paths that have cycles.

Condition 1 above can be defined equivalently as follows.

Definition 3.3 (SF-Condition) *A path $v_0 l_1 v_1 l_2 v_2 \cdots l_n v_n$ of Q satisfies the* SF-Condition *w.r.t. μ if there is a permutation σ of $0, \ldots, n$, such that $\sigma(0) = 0$ and D has a path of the following form:*

$$\mu(v_{\sigma(0)}) {*} l_{\sigma(1)} \mu(v_{\sigma(1)}) {*} l_{\sigma(2)} \mu(v_{\sigma(2)}) \cdots {*} l_{\sigma(n)} \mu(v_{\sigma(n)}).$$

Now we can give an equivalent definition of a semiflexible matching.

Definition 3.4 (Semiflexible Matching) *An assignment μ of a query Q w.r.t. a database D is a semiflexible matching if the next three conditions hold.*

1. *The assignment μ satisfies the rc of Q.*
2. *Every finite path $v_0 l_1 v_1 l_2 v_2 \cdots l_n v_n$ of Q, where v_0 is the root, satisfies the SF-Condition w.r.t. μ.*
3. *For every set $C = \{v_{i_1}, \ldots, v_{i_m}\}$, if C is a strongly connected component in Q, then the set $\{\mu(v_{i_1}), \ldots, \mu(v_{i_m})\}$ is a strongly connected component in D.*

We denote the set of all semiflexible matchings of Q w.r.t. D as $\mathcal{M}_{sf}(Q, D)$.

The second new notion of matchings that we introduce in this chapter is the notion of *flexible matching* that is defined next.

Definition 3.5 (Flexible Matching) *A* flexible matching *of a query Q w.r.t. a database D is an assignment μ for which the following three conditions hold.*

1. *The rc is satisfied by μ.*
2. *All the wec's of Q are satisfied by μ.*
3. *All the qec's of Q are satisfied by μ.*

We denote the set of all flexible matchings of Q w.r.t. D as $\mathcal{M}_f(Q, D)$.

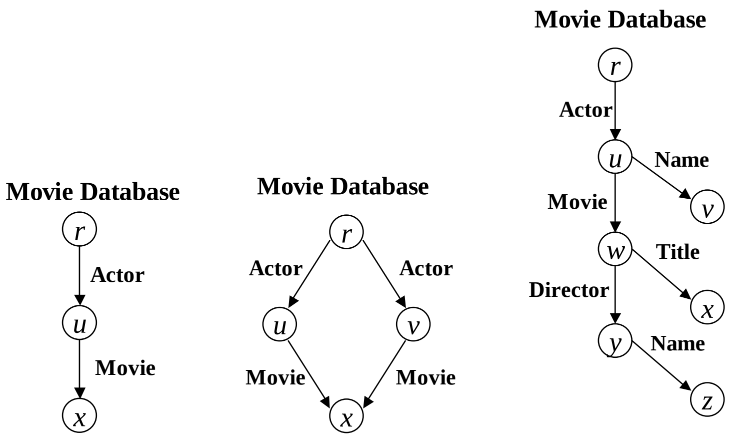
Intuitively, in both the semiflexible and flexible semantics, we assume that nodes of D that are connected by a path are semantically related. The difference is that in the flexible semantics, we also assume that this relationship is transitive. This difference entails another important difference between the semiflexible and the flexible semantics. In the flexible semantics, cycles in the query are not necessarily mapped to cycles in the database. Semiflexible matchings, however, require strongly connected components in the query to be mapped to strongly connected components in the database.

Proposition 3.6 (Increasing Flexibility) *Given a database D and a query Q, the following holds: $\mathcal{M}_r(Q,D) \subseteq \mathcal{M}_{sf}(Q,D) \subseteq \mathcal{M}_f(Q,D)$.*

Proof. Let μ be a rigid matching of Q w.r.t. D. It is easy to see that μ is also a semiflexible matching of Q w.r.t D. First, μ satisfies the rc of Q. Second, consider a finite path ϕ in Q that starts at the root. By taking the identity permutation, it is easy to see that ϕ satisfies the SF-Condition w.r.t. μ because μ satisfies all the ec's of Q. Third, if $C = \{v_{i_1}, \ldots, v_{i_m}\}$ is a strongly connected component in Q, then $\{\mu(v_{i_1}), \ldots, \mu(v_{i_m})\}$ is a strongly connected component in D, because μ satisfies all the ec's of Q.

Let μ be a semiflexible matching of Q w.r.t. D. We show that μ is also a flexible matching of Q w.r.t. D. First, μ satisfies the rc of Q. Secondly, we show that μ satisfies all the wec's in Q. Given a wec lv, since v is reachable from the root of Q, there is a path $v_0 l_1 v_1, \cdots l_n v_n$ in Q such that v_0 is the root of Q, l_n is l and v_n is v. This path satisfies the SF-Condition w.r.t. μ. Thus, there is a permutation σ such that D contains the path $\mu(v_{\sigma(0)}) {*} l_{\sigma(1)} \mu(v_{\sigma(1)}) {*} l_{\sigma(2)} \mu(v_{\sigma(2)}) \cdots {*} l_{\sigma(n)} \mu(v_{\sigma(n)})$. In this path, $l_{\sigma(i)} \mu(v_{\sigma(i)})$, where $1 \leq i \leq n$ and $\sigma(i) = n$, shows that μ satisfies the wec lv.

Thirdly, we show that μ satisfies all the qec's in Q. Given a qec ulv in Q, since u is reachable from the root of Q, there is a path $v_0 l_1 v_1, \cdots l_{n-1} v_{n-1} l_n v_n$ in Q such that v_0 is the root of Q, v_{n-1} is u, l_n is l and v_n is v. This path satisfies the SF-Condition, so there is a path in D that contains both $\mu(v)$ and $\mu(u)$. Thus, either there is a path in D from $\mu(v)$ to $\mu(u)$ or vice-versa. $\square$



(a) Query 1 (b) Query 2 (c) Query 3

Figure 3.1: Three queries over the movie database.

Some languages for semistructured data use regular expressions as a tool for formulating queries when the structure is not completely known to the user. In principle, the expressive power of regular expressions subsumes the semiflexible and flexible semantics. However, in addition to regular expressions, one may also need unions and joins in order to fully express these semantics. In any case, the semiflexible and flexible semantics facilitate a much more succinct and easier style of writing queries. Moreover, we will show that evaluation and optimization of semiflexible and flexible queries are easier than evaluation and optimization of queries with regular expressions.

3.1.1 Examples

A movie database that holds information on movies, T.V. series, actors and directors is depicted in Figure 2.1. Suppose that Alice wants to query the data. She is familiar with the ontology of the database, but does not know how the information is organized. We examine a few cases.

Example 3.7 Alice tries to look in the database for information on movies. She

Matchings	r	u	v (Actor Name)	w	x (Movie Title)	y	z (Director Name)
	1	21	Mark Hamill (30)	11	Star Wars (23)	41	George Lucas (51)
	1	22	Harrison Ford (31)	11	Star Wars (23)	41	George Lucas (51)
	1	14	Kyle MacLachlan (27)	29	Dune (35)	42	David Lynch (52)

Table 3.1: The flexible matchings for Query 3 of Figure 3.1(c) and the database of Figure 2.1.

assumes that the information on each actor includes the movies in which the actor participated. Thus, she formulates Query 1 of Figure 3.1(a). There are no rigid matchings of Query 1 w.r.t. the movie database. However, under the semiflexible semantics, the answer of Query 1 will include a mapping of r, u and x to 1, 14 and 29, respectively, even though 14 and 29 are not adjacent. The answer will also include a mapping of r, u and x to the next object triplets: *(1)* to 1, 21 and 11; *(2)* to 1, 22 and 11; and *(3)* to 1, 25 and 12. Note that in the last three mappings the movie is above the actor while in the query the actor is above the movie.

Example 3.8 Alice wants to find two actors who played in the same movie. She formulates Query 2 of Figure 3.1(b). Obviously, there are no rigid matchings in this case since the database is a tree and the query is a dag. However, under either the semiflexible or the flexible semantics, Alice will find Mark Hamill and Harrison Ford as two actors who played in the same movie (Star Wars). This is due to the fact that assigning r, u, v, x to $1, 21, 22, 11$, respectively, is a semiflexible (and, hence, also a flexible) matching of the query w.r.t. the movie database.

Example 3.9 When Query 3 of Figure 3.1(c) is posed to the movie database as a rigid query, there are no matchings of the query w.r.t. the database. Under the semiflexible semantics, the query has only a single matching (r, u, v, w, x, y and z are mapped to 1, 14, 27, 29, 35, 42 and 52, respectively), although the database has two movies with a director. However, under the flexible semantics, we get the assignments that are shown in Table 3.1. For atomic nodes, the value is presented in addition to the oid.

3.1.2 More on Semiflexible Matchings

Consider a query Q, a database D and an assignment μ of Q w.r.t. D. Obviously, checking whether μ is a flexible matching can be done in polynomial time in the size of the query and the database. This follows from the next three facts. First, the number of qec's and the number of wec's, in a query, is equal to the number of query edges. Second, checking satisfaction of the root constraint and satisfaction of the wec's is immediate, i.e., in $O(1)$. Third, checking satisfaction of a qec requires to check that two database nodes are connected by a path and this can be done in linear time in the size of the database.

It is not obvious how to check in polynomial time whether an assignment to the variables of a query Q is a semiflexible matching. The reason is that all the paths of Q have to be considered. If Q is a dag (directed acyclic graph), the number of paths could be exponential, and if Q has a cycle, then there are infinitely many paths. In this section, we show that for any given query Q, checking whether an assignment is a semiflexible matching could be done in polynomial time in the size of Q and D. For the case of a cyclic query, we first show that it is sufficient to consider only simple paths and simple cycles in order to check whether a given assignment is a semiflexible matching. Based on this result, we provide sufficient and necessary conditions that can be checked in polynomial time in the size of the query and the database, such that if the conditions hold then the assignment is a semiflexible matching. Note that as a matter of terminology, a cyclic query (database) has at least one cycle. Thus, the class of cyclic queries (databases) does not include any dag query (database). A tree query (database), however, is a special case of a dag query (database).

Consider a query Q, a database D and an assignment μ of Q w.r.t. D. The next definition provides a characterization for satisfiability of the SF-Condition w.r.t. μ by simple paths of Q. This characterization holds even when the database is cyclic.

Definition 3.10 (Connectivity-Preserving Mapping) *Let Q be a query and D be a database. We say that the assignment $\mu\colon V \to O$ is a* connectivity-preserving mapping *if it satisfies the following conditions.*

- *For all labels l, all nodes v and the root r of Q, if Q has a simple path of the form $r{*}lv$, then D has a path of the form $\mu(r){*}l\mu(v)$; and*
- *For all labels l and k, and all nodes u, v and w of Q, if Q has a simple path of the form $ulv{*}kw$, then D either has a path of the form $\mu(v){*}k\mu(w)$ or a path of the form $\mu(w){*}l\mu(v)$.*

the next lemma shows that the conditions in Definition 3.10 are necessary and sufficient for satisfiability of the SF-Condition w.r.t. μ by simple paths of Q.

Lemma 3.11 (The SF-Condition for Simple Paths) *Consider a query Q, a database D (that may have cycles) and an assignment $\mu\colon V \to O$ of Q w.r.t. D. Every simple path in Q satisfies the SF-Condition w.r.t. μ if and only if μ is a connectivity-preserving mapping.*

Proof. It is easy to see that if every simple path in Q satisfies the SF-Condition w.r.t. μ, then μ is a connectivity-preserving mapping.

Before proving the other direction, we present and prove a claim regarding the existence and non-existence of specific paths in a strongly connected component.

Claim 3.12 *Consider n wec's $l_1v_1, \ldots, l_nv_n$, a database D and a mapping μ of $v_1, \ldots, v_n$ to objects of D such that $\mu(v_1), \ldots, \mu(v_n)$ are all in a strongly connected component of D. Suppose that for each two wec's l_iv_i and l_jv_j, D either has a path of the form $\mu(v_i){*}l_j\mu(v_j)$ or a path of the form $\mu(v_j){*}l_i\mu(v_i)$. If there are v_a and v_b, among $v_1, \ldots, v_n$, such that D has no path of the form $\mu(v_b){*}l_av_a$ then the following three assertions will hold.*

1. *For each $1 \le i \le n$, D has no path of the form $\mu(v_i){*}l_a\mu(v_a)$.*
2. *For each $1 \le i \le n$, D has a path of the form $\mu(v_a){*}l_i\mu(v_i)$.*
3. *For each $1 \le i \le n$ and $1 \le j \le n$, if $v_i \ne v_a$ then D has a path of the form $\mu(v_j){*}l_i\mu(v_i)$.*

We show correctness of the first assertion. Suppose that there was $1 \le i \le n$ such that D has a path of the form $\mu(v_i){*}l_a\mu(v_a)$. This path can be combined with a path from $\mu(v_b)$ to $\mu(v_i)$ (such a path must exists in a strongly connected component) and this yield a contradiction to the assumption that D has no path of the form $\mu(v_b)*l_a v_a$.

The correctness of the second assertion follows from the first assertion and the assumption that for each two wec's $l_i v_i$ and $l_j v_j$, D either has a path of the form $\mu(v_i)*l_j\mu(v_j)$ or a path of the form $\mu(v_j)*l_i\mu(v_i)$.

We show the correctness of the third assertion. For each $1 \leq j \leq n$, D has a path from $\mu(v_j)$ to $\mu(v_a)$, since $\mu(v_j)$ and $\mu(v_a)$ are two objects on a strongly connected component. According to the second assertion, D has a path of the form $\mu(v_a)*l_i\mu(v_i)$. Combining these two paths yield a path of the form $\mu(v_j)*l_i\mu(v_i)$.

In the notations of Claim 3.12, we call $\mu(v_a)$ the *entry point* of $\mu(v_1),\ldots,\mu(v_n)$. It is easy to see that under the conditions of the claim, $\mu(v_1),\ldots,\mu(v_n)$ cannot have more than one entry point. Specifically, if one of the objects, $\mu(v_r)$, is the database root, then for each v_j other than the root, D has a path of the form $\mu(v_r)*l_j\mu(v_j)$. In this case, $\mu(v_r)$, i.e., the database root, is the entry point of $\mu(v_1),\ldots,\mu(v_n)$.

Now, we can prove the other direction of the lemma. Suppose that μ is a connectivity-preserving mapping and consider a simple path $\pi = v_0 l_1 v_1 l_2 v_2 \cdots l_n v_n$ of Q. We need to show that for some permutation σ, the database D has a path of the form

$$\mu(v_{\sigma(0)})*l_{\sigma(1)}\mu(v_{\sigma(1)})*l_{\sigma(2)}\mu(v_{\sigma(2)})\cdots *l_{\sigma(n)}\mu(v_{\sigma(n)}).$$

As a preliminary step, we partition the nodes $\mu(v_0),\mu(v_1),\ldots,\mu(v_n)$ according to the maximal strongly connected components of D. Let $C_1,\ldots,C_m$ be the maximal strongly connected components sorted topologically (see [27] for topological-sort algorithms).

We choose a permutation σ of $0,\ldots,n$ that has the following properties. First, $\sigma(0)=0$. Second, the sequence of objects $\mu(v_{\sigma(0)}),\mu(v_{\sigma(1)}),\ldots,\mu(v_{\sigma(n)})$ conforms to the topological ordering of $C_1,\ldots,C_m$. That is, if $\mu(v_{\sigma(i)}) \in C_{t_1}$, $\mu(v_{\sigma(j)}) \in C_{t_2}$ and $i < j$ then $t_1 \leq t_2$. Third, if $\mu(v_{\sigma(i_1)}),\ldots,\mu(v_{\sigma(i_k)})$ are objects in a strongly connected component of D and $\mu(v_{\sigma(i_j)})$ is an entry point of $\mu(v_{\sigma(i_1)}),\ldots,\mu(v_{\sigma(i_k)})$ then $i_j \leq i_h$, for each $1 \leq h \leq k$. That is, among the objects $\mu(v_{\sigma(i_1)}),\ldots,\mu(v_{\sigma(i_k)})$,

$\mu(v_{\sigma(i_j)})$ appears first in the path that is defined by μ and σ. The order of the other objects is chosen arbitrarily. If there is no entry point of $\mu(v_{\sigma(i_1)}), \ldots, \mu(v_{\sigma(i_k)})$ then the order of all these objects is chosen arbitrarily, i.e., there can be any order for $i_1, \ldots, i_k$. Note that we can choose such σ because $C_1, \ldots, C_m$ are maximal strongly connected components and they are sorted topologically.

Note that $v_{\sigma(0)}$ and $\mu(v_{\sigma(0)})$ are the roots of Q and D, respectively. Moreover, if both $\mu(v_{\sigma(i-1)})$ and $\mu(v_{\sigma(i)})$ $(1 \leq i \leq n)$ lie on the same path ϕ of D, then one of the next two conditions must be true.

- The path ϕ is from $\mu(v_{\sigma(i-1)})$ to $\mu(v_{\sigma(i)})$.
- The objects $\mu(v_{\sigma(i-1)})$ and $\mu(v_{\sigma(i)})$ both belong to the same strongly connected component of D.

We claim that D has a path of the form

$$\mu(v_{\sigma(0)}) {*} l_{\sigma(1)} \mu(v_{\sigma(1)}) {*} l_{\sigma(2)} \mu(v_{\sigma(2)}) \cdots {*} l_{\sigma(n)} \mu(v_{\sigma(n)}).$$

To prove the claim, we show that for all $1 \leq i \leq n$, D has a path of the form $\mu(v_{\sigma(i-1)}) {*} l_{\sigma(i)} \mu(v_{\sigma(i)})$. For $i = 1$, node $v_{\sigma(0)}$ is the root of Q, and so, Q has a path of the form $v_{\sigma(0)} {*} l_{\sigma(1)} v_{\sigma(1)}$. Thus, by the first part of the condition in Definition 3.10, D has a path of the form $\mu(v_{\sigma(0)}) {*} l_{\sigma(1)} \mu(v_{\sigma(1)})$.

For $i > 1$, there are three cases to be considered. First, we consider the case where $\mu(v_{\sigma(i-1)})$ and $\mu(v_{\sigma(i)})$ belong to the same strongly connected component. According to the construction of σ, $\mu(v_{\sigma(i)})$ is not the entry point of the nodes in the strongly connected component. This is because the entry point appears in the path before all the other nodes of the strongly connected component. Hence, according to the third assertion of Claim 3.12, D has a path of the form $\mu(v_{\sigma(i-1)}) {*} l_{\sigma(i)} \mu(v_{\sigma(i)})$.

In the next two cases, $\mu(v_{\sigma(i-1)})$ and $\mu(v_{\sigma(i)})$ do not belong to the same strongly connected component. In one case, $v_{\sigma(i-1)}$ appears before $v_{\sigma(i)}$ on the path π of Q. In the other case, $v_{\sigma(i-1)}$ appears after $v_{\sigma(i)}$ on π.

If $v_{\sigma(i-1)}$ appears first, then Q has a path of the form $v_{\sigma(i-1)} {*} l_{\sigma(i)} v_{\sigma(i)}$. Thus, according to the second part of the condition in Definition 3.10, D must have a path of the form $\mu(v_{\sigma(i-1)}) {*} l_{\sigma(i)} \mu(v_{\sigma(i)})$. Alternatively, if $v_{\sigma(i)}$ appears first, then there is a j $(0 \leq j \leq n-2)$, such that Q has a path of the form $v_j l_{\sigma(i)} v_{\sigma(i)} {*} l_{\sigma(i-1)} v_{\sigma(i-1)}$. Thus, according to the second part of the condition, D must have a path of the form $\mu(v_{\sigma(i-1)}){*} l_{\sigma(i)} \mu(v_{\sigma(i)})$. □

Consider a query Q, a database D and an assignment of Q w.r.t. D. Lemma 3.11 provides a test for checking if a simple path π in Q satisfies the SF-Condition. In this test, we just need to check that the conditions of Definition 3.10 are satisfied w.r.t. π. It is easy to see that the test can be done in polynomial time in the size of π and D.

Corollary 3.13 (DAG Queries) *Consider a dag query Q and a database D that may have cycles. The assignment $\mu\colon V \to O$ is a semiflexible matching of Q w.r.t. D if and only if (1) μ is a connectivity-preserving mapping, and (2) μ maps the query root to the database root.*

Proof. If μ is a semiflexible matching of Q w.r.t. D then, by Lemma 3.11, μ is a connectivity-preserving mapping; and from the first condition of Definition 3.4 follows that μ maps the query root to the database root.

For the other direction, since Q is a dag, we only need to show that μ satisfies the first two conditions of Definition 3.4. Condition 1 of Definition 3.4 requires satisfaction of the rc. The condition holds because μ maps the query root to the database root. Condition 2 requires satisfaction the SF-condition w.r.t. μ, for every finite path in Q and Lemma 3.11 proves that this condition holds. □

From Corollary 3.13 it follows that when Q is a dag, the complexity of checking whether μ is a semiflexible matching is in polynomial time in the size of Q and D.

Next, we will consider cyclic queries. Since cyclic queries have infinitely many paths, a naive test based on the definition of a semiflexible matching does not terminate. The following lemma alleviates this difficulty.

The path $v_0 l_1 v_1 l_2 v_2 \cdots l_n v_n$ (where v_0 is not necessarily the root) is a *simple path* if all the v_i are distinct. It is a *cycle* if $v_0 = v_n$, and it is a *simple cycle* if $v_0 = v_n$ and $v_1, v_2, \ldots, v_n$ are distinct.

Lemma 3.14 (Condition for Cyclic Queries) *Let Q be a cyclic query and D be a cyclic database. The assignment μ is a semiflexible matching of Q w.r.t. D if and only if the following conditions are satisfied.*

1. *The assignment μ satisfies the rc of Q.*

2. *All the simple paths of Q that start at the root satisfy the SF-Condition w.r.t. μ.*

3. *For all the simple cycles $u_0 l_1 u_1 \cdots l_k u_k$ of Q (where u_0 is not necessarily the root), D has a cycle (which is not necessarily a simple cycle) of the form $\mu(u_0){*}l_1\mu(u_1)\cdots{*}l_k\mu(u_k)$.*

Proof. If μ is a semiflexible matching, then, by Definition 3.4, Condition 1 and Condition 2 hold. Next, we will show the necessity of Condition 3.

Let μ be a semiflexible matching of Q w.r.t. D, and let $C = u_0 l_1 u_1 \cdots l_k u_k$, where $u_0 = u_k$, be a simple cycle of Q.

According to the definition of a semiflexible matching (Definition 3.4), the set $\{\mu(u_0), \ldots, \mu(u_k)\}$ is contained in a strongly connected component of D, because $\{u_0, \ldots, u_k\}$ is contained in a strongly connected component of Q. Therefore, there is a path from $\mu(u_i)$ to $\mu(u_{i+1})$ $(0 \leq i \leq k-1)$.

The second observation is that D has a path of the form $\mu(u_i){*}l_i\mu(u_i)$ $(1 \leq i \leq k)$. In proof, for each node u_i $(1 \leq i \leq k)$ in C, there is a path that starts at the root of Q, continues to a node in C and then goes to u_i through the edge $u_{i-1}l_i u_i$. This path can be extended by going around the cycle C from u_i back to itself. Thus, Q has a path of the form $r_Q{*}l_i u_i{*}l_i u_i$, where r_Q is the root of Q. This path satisfies the SF-Condition (Definition 3.3), since μ is a semiflexible matching. Therefore, D must have a path of the form $\mu(u_i){*}l_i\mu(u_i)$ $(1 \leq i \leq k)$.

By combining the path from $\mu(u_{i-1})$ to $\mu(u_i)$ with the path $\mu(u_i){*}l_i\mu(u_i)$ $(1 \leq i \leq k)$, it follows that D has a path of the form $\mu(u_{i-1}){*}l_i\mu(u_i)$ $(1 \leq i \leq k)$. Therefore, D also has a path of the form $C_D = \mu(u_0){*}l_1\mu(u_1)\cdots{*}l_k\mu(u_k)$. Since $u_0 = u_k$, it follows that $\mu(u_0) = \mu(u_k)$ and, thus, C_D is a cycle in D. Thus, we have shown that Condition 3 is satisfied.

For the other direction, suppose that μ is a mapping of the variables of Q to objects of D, such that conditions 1–3 are satisfied. By Condition 3, μ maps strongly connected components of Q to strongly connected components of D. Next, we will show that finite paths of Q satisfy the SF-Condition w.r.t. μ.

Let $\pi = v_0 l_1 v_1 \cdots l_n v_n$ be a path of Q, where v_0 is the root of Q. We will show by induction on n that π satisfies the SF-Condition w.r.t. μ.

For $n = 1$, π is either a simple path (if $v_0 \neq v_1$) or a simple cycle (if $v_0 = v_1$). In the first case, π satisfies the SF-Condition by Condition 2, and in the second case—by Condition 3.

Next, suppose that the claim holds for paths of length less that n, where $n > 1$. We have to show that the claim holds for the path π.

Once again, if π is a simple path or a simple cycle, then either Condition 2 or Condition 3 implies that π satisfies the SF-Condition.

If π is neither a simple path nor a simple cycle, then it must have variables v_i and v_k $(i < k)$, such that the path $\pi_c = v_i l_{i+1} v_{i+1} \cdots l_k v_k$ is a simple cycle, i.e., $v_i = v_k$. By Condition 3, D has a cycle of the form $\phi_c = \mu(v_i) {*} l_{i+1} \mu(v_{i+1}) \cdots {*} l_k \mu(v_k)$.

By the induction hypothesis, the path

$$\pi_p = v_0 l_1 v_1 \cdots l_i v_i l_{k+1} v_{k+1} \cdots l_n v_n$$

satisfies the SF-Condition. Thus, there is a permutation σ_p of $0, 1, \ldots, i, k+1, \ldots, n$, such that $\sigma_p(0) = 0$ and D has a path of the following form.

$$\begin{aligned}
\phi_p \;=\; & \mu(v_{\sigma_p(0)}) {*} l_{\sigma_p(1)} \mu(v_{\sigma_p(1)}) \cdots \\
& {*} l_{\sigma_p(i)} \mu(v_{\sigma_p(i)}) {*} l_{\sigma_p(k+1)} \mu(v_{\sigma_p(k+1)}) \cdots \\
& {*} l_{\sigma_p(n)} \mu(v_{\sigma_p(n)})
\end{aligned}$$

We will now show that the cycle ϕ_c and the path ϕ_p can be combined into a single path ϕ that shows that π satisfies the SF-Condition. Intuitively, this is done by inserting the cycle ϕ_c into the path ϕ_p starting at an occurrence of v_i in ϕ_p. To do that, we need to define a permutation σ of $0, 1, \ldots, n$ that combines the effect of the permutation σ_p with the effect of the identity permutation of $i+1, \ldots, k$, which relates the simple cycle π_c to the cycle ϕ_c.

The permutation σ_p is defined over $0, 1, \ldots, i, k+1, \ldots, n$, i.e., it has a gap between $i+1$ and k. However, this gap is not necessarily the place to put the cycle ϕ_c. Therefore, we first have to shift some positions in order to create the gap in the right place.

Formally, there is a j, such that $\sigma_p(j) = i$. Note that j is the position of $\mu(v_i)$ in the path ϕ_p. There are two cases for creating the gap, starting at position $j+1$, and for each case σ is defined differently.

If $j + 1 \leq i$, then the gap has to be created by shifting right; that is, for $j+1 \leq m \leq i$, we define $\sigma(m+k-i) = \sigma_p(m)$.

If $k+1 \leq j$, then the gap has to be created by shifting left; that is, for $k+1 \leq m \leq j$, we define $\sigma(m-(k-i)) = \sigma_p(m)$.

Now, we fill the gap with the cycle ϕ_c by defining $\sigma(j+m) = i+m$ for $1 \leq m \leq k-i$. Finally, we complete the definition of σ by defining $\sigma(m) = \sigma_p(m)$, for all other values of m.

It thus follows that D has a path ϕ of the form

$$\mu(v_{\sigma(0)}){*}l_{\sigma(1)}\mu(v_{\sigma(1)}){*}l_{\sigma(2)}\mu(v_{\sigma(2)})\cdots{*}l_{\sigma(n)}\mu(v_{\sigma(n)})$$

and this path shows that π satisfies the SF-Condition. □

Lemma 3.14 shows that when Q is a cyclic query, verifying that a matching is a semiflexible matching is decidable. However, the verification requires to check that the image of every simple cycle in Q is on a cycle in D. Since the number of simple cycles in a query can be exponential in the size of the query, verification according to Lemma 3.14 may require exponential time. The next definition provides a condition that can be verified in polynomial time in the size of the query and the database. This condition will assist in checking if simple cycles of the query are mapped to cycles in the database.

Definition 3.15 (Cycle-Preserving Mapping) *Let Q be a cyclic query, D be a cyclic database and $\mu : Q \to D$ be an assignment of Q w.r.t. D. We say that μ is a* cycle-preserving mapping *if for every pair of variable v and u, if Q has a path of the form $u{*}lv$ and a path of the form $v{*}l'u$, then D has a path of the form $\mu(u){*}l\mu(v)$ and a path of the form $\mu(v){*}l'\mu(u)$.*

The next lemma shows that the condition in Definition 3.15 is sufficient and necessary for mapping simple cycles of the query into cycles of the database. Note that the two variables u and v in Definition 3.15 could be equal.

Lemma 3.16 (Cycle Preserving) *Let Q be a cyclic query, D be a cyclic database and $\mu : Q \to D$ be an assignment of Q w.r.t. D. The following conditions are equivalent.*

1. *For every simple cycle $u_0 l_1 u_1 \cdots l_k u_k$ of Q (where u_0 is not necessarily the root), D has a cycle (which is not necessarily a simple cycle) of the form $\mu(u_0) * l_1 \mu(u_1) \cdots * l_k \mu(u_k)$.*

2. *The assignment μ is a cycle-preserving mapping.*

Proof. Assume that Condition 2 holds and let $u_0 l_1 u_1 \cdots l_k u_k$ be a simple cycle in Q. Then ,for each pair of nodes u_i and u_{i+1} ($0 \leq i \leq k-1$), Q has a path $u_i * l_{i+1} u_{i+1}$ and a path $u_{i+1} * l_i u_i$ (note that $u_0 = u_k$, thus $l_0 = l_k$). By Definition 3.15, in D there is a path of the form $\mu(u_i) * l_{i+1} \mu(u_{i+1})$. This shows that D has a cycle of the form $\mu(u_0) * l_1 \mu(u_1) \cdots * l_k \mu(u_k)$.

For the other direction, we start by proving the following claim.

Claim 3.17 *Given that Condition 1 holds, for every pair of variables u and v that are on a cycle (not necessarily a simple cycle), the images of u and v, i.e., $\mu(u)$ and $\mu(v)$, are on a cycle in D.*

The proof of the claim is by an induction on the number of nodes in the cycle that contains u and v. First, we consider the case where u and v are equal and the cycle is a loop, i.e., contains only one edge. In this case, the cycle is simple and has the form ulu. By Condition 1, D has a cycle $\mu(u) * l \mu(u)$ and the claim holds.

Secondly, we consider the case where the cycle contains only two nodes. In this case, the cycle has an edge from u to v and vice-versa. Thus, there is a simple cycle that contains u and v. According to Condition 1, the image of this cycle w.r.t. μ is on a cycle in D.

Next, we show that if the claim holds for each u and v that are on a cycle with less than k nodes then the claim also holds for each u and v that are on a cycle with k nodes. Let u and v be nodes on a cycle C with k nodes. If C is a simple cycle then, according to Condition 1, the nodes in the set $\{\mu(w) \mid w \text{ on } C\}$ are on a cycle in D and, in particular, $\mu(u)$ and $\mu(v)$ are on a cycle in D. If C is not a simple cycle then there is a node w, such that w appears twice in C. At least one of the following three cases must hold.

1. C is a path of the form $u{*}w{*}w{*}v{*}u$;
2. C is a path of the form $u{*}v{*}w{*}w{*}u$;
3. C is a path of the form $u{*}w{*}v{*}w{*}u$;

In all three cases, C is a composition of two smaller cycles. In the first case, Q has a cycle of the form $u{*}w{*}v{*}u$ with less than k nodes and thus, by the induction hypothesis, $\mu(u)$ and $\mu(v)$ are on a cycle in D. The second case is similar to the first case. In the third case, C contains the following two sub-cycles: A cycle C_1 of the form $u{*}w{*}u$ and a cycle C_2 of the form $v{*}w{*}v$. The size of each one of the two cycles is smaller than k. Thus, according to Condition 1, D has a cycle that contains $\mu(u)$ and $\mu(w)$ and D also has a cycle that contains $\mu(v)$ and $\mu(w)$. Since these two cycles have a shared node, $\mu(w)$, they can be combined to form a cycle that contains $\mu(u)$ and $\mu(v)$. This complete the proof of the claim.

Now, assume that Condition 1 holds and let u and v be variables of Q, such that there are paths of the form $u{*}lv$ and $v{*}l'u$, in Q. It must hold that Q has *(1)* a simple cycle $v{*}lv$; *(2)* a simple cycle $u{*}l'u$; and *(3)* a cycle $v{*}u{*}v$. The reason for having the first (second) cycle is that v (u) is on a cycle that goes into v (u) through an edge labeled with l (l'). From the existence of the first two cycles and from Condition 1, it follows that D has a path of the form $\mu(v){*}l\mu(v)$ and a path of the form $\mu(u){*}l'\mu(u)$. From the third cycle and the claim above, it follows that D has a cycle $\mu(v){*}\mu(u){*}\mu(v)$. Combining these cycles yields a cycle $\mu(v){*}l'\mu(u){*}l\mu(v)$. Therefore, D has a path of the form $\mu(u){*}l\mu(v)$ and a path of the form $\mu(v){*}l'\mu(u)$.

□

Using the above lemmas, we can introduce a condition that is both necessary and sufficient for an assignment μ of a cyclic query w.r.t. a cyclic database to be a semiflexible matching. The condition requires local tests in the sense that only pairs of nodes are checked each time and the check for each pair can be done in polynomial time in the size of the query and the database.

Corollary 3.18 (Cyclic Queries) *Suppose that Q is a cyclic query and D is a cyclic database. The assignment μ is a semiflexible matching of Q w.r.t. D if and only if the following conditions are satisfied.*

1. *The assignment μ satisfies the rc of Q.*
2. *The assignment μ is a connectivity-preserving mapping.*
3. *The assignment μ is a cycle-preserving mapping.*

Proof. This consequence follows from Lemma 3.11, Lemma 3.14 and Lemma 3.16. □

Based on the above condition we get the next theorem.

Theorem 3.19 (Polynomial-Time Verification) *Deciding whether a given assignment μ of a query w.r.t. a database is a semiflexible matching is in polynomial time in the size of the query and the database.*

In summary, Theorem 3.19 states that for all types of queries, testing whether μ is a semiflexible matching has a running time that is polynomial in the size of the query and the database.

3.2 Query Evaluation in Semiflexible Semantics

In *data complexity*, query evaluation is measured in terms of the size of the database. That is, we assume that the size of the query is fixed. Query evaluation has a polynomial-time data complexity under either the semiflexible or flexible semantics.

In *combined complexity*, both the query and the data are considered as input. The combined complexity of query evaluation under either the semiflexible and the flexible semantics is exponential, since the size of the result could be exponential in the size of the query and the data.

A better approach is to analyze the *input-output complexity* that is measured in terms of the size of the query, the database and the result. The motivation for using input-output complexity is that it allows us to distinguish between the following two cases. First, a case where the runtime of the algorithm is exponential simply because the size of the result is exponential. Secondly, a case where the runtime of the algorithm is exponential even though the size of the result is small (i.e., less than exponential in the size of the input). The second case could be a consequence of having a naive algorithm, i.e., an algorithm that does a lot of redundant work. A different cause could be that the problem is NP-hard.

Recall that in the relational case, merely checking whether a join of n relations is not empty is NP-complete [49]; hence, the input-output complexity is exponential. However, for the important case of acyclic joins, the input-output complexity is polynomial [65]. In this section, we will discuss the input-output complexity of query evaluation under the semiflexible semantics.

3.2.1 Path Queries

In this section we consider *path queries.* A path query has the form $v_0 l_1 v_1 l_2 v_2 \ldots l_n v_n$, where v_0 is the root and all the v_i are distinct. First, we will discuss the case of evaluating a path query over a database that is also a path. Note that even in this case, the result could be exponential in the size of the query and the database (provided that some labels are repeated along paths of the database). However, the input-output complexity is polynomial.

For a query node v, the *correspondence set* of v, denoted C_v, is the set of all database objects o, such that o satisfies some wec of v (i.e., there is a label l, such that both o and v have incoming edges labeled with l). The correspondence set of the root r_Q of Q consists of the root r_D of the database.

Proposition 3.20 (Path Query and Path Database) *Let Q be a path query of the form $v_0 l_1 v_1 l_2 v_2 \dots l_n v_n$, and D be a path database over a set of objects O. Consider the set $\mathcal{M}$ that the next equation defines.*

$$\mathcal{M} = \{\mu : V \to O \mid \mu(v_i) \in C_{v_i} \textit{ for } 0 \leq i \leq n, \textit{ and } \mu(v_i) \neq \mu(v_j) \textit{ for } i \neq j\}. \quad (3.1)$$

Then the set $\mathcal{M}_{sf}(Q, D)$ of the semiflexible matchings of Q w.r.t. D is equal to $\mathcal{M}$. Computing $\mathcal{M}_{sf}(Q, D)$ has linear-time input-output complexity.

Proof. First, we show that $\mathcal{M} = \mathcal{M}_{sf}(Q, D)$. Consider a mapping μ in $\mathcal{M}$. Obviously, μ satisfies the root constraint because the correspondence set of the root, C_{r_Q}, contains only the database root. In addition, μ maps the variables of Q into a database path. Thus, μ is a connectivity-preserving mapping (see Definition 3.10) and according to Corollary 3.13, μ is a semiflexible matching. For the other direction, suppose that μ is in $\mathcal{M}_{sf}(Q, D)$. The single path in Q satisfies the SF-Condition w.r.t. μ. Thus, for each query variable v_i, $\mu(v_i) \in C_{v_i}$. Furthermore, $\mu(v_i) \neq \mu(v_j)$ for every $i \neq j$. Consequently, μ satisfies the condition in Equation 3.1, and hence, μ is in $\mathcal{M}$.

To show that the computation can be done in linear time, let $|Q|$, $|D|$ and $|M|$ be the sizes of the query, the database and the result, respectively.

Computing $\mathcal{M}_{sf}(Q, D)$ is done as follows. First, we verify that for each label l in Q if l is attached to k edges of Q, then there are at least k different nodes in D with an incoming edge labeled with l. If this test fails, then $\mathcal{M}_{sf}(Q, D)$ is empty. The test requires a runtime of $O(|Q| + |D|)$, since counting the labels can be done in a single pass over Q and D.

Next, the correspondence sets C_{v_i} $(1 \leq i \leq n)$ are constructed. Using a suitable data structure, such as a hash table, this construction can be done in $O(|Q| + |D|)$.

Finally, the matchings are created. A matching is constructed by choosing a single object from each correspondence set, verifying that no object is chosen twice. This can be done using nested iterations over the sets C_{v_i} $(1 \leq i \leq n)$. Constructing $\mathcal{M}$ from the correspondence sets requires $O(|M|)$ runtime. Thus, the runtime of the algorithm is in $O(|Q| + |D| + |M|)$. □

Next, we investigate the case where the database is a tree. A simple evaluation algorithm for a path query Q w.r.t. a tree database D would be as follows. For each path π in D, from the root to a leaf, evaluate Q w.r.t. π. Consider the sets of semiflexible matchings that are the result of evaluating Q w.r.t. the different paths. The union of these sets is the result of evaluating Q w.r.t. the whole database. Evaluating Q w.r.t. the paths of D can be done as described in Proposition 3.20.

The runtime complexity of such evaluation will be a function of $|D|^2$. This is because the number of paths in D, from the root to a leaf, could be $O(|D|)$. In addition, the size of such paths could be $O(|D|)$.

The simple evaluation algorithm that was described in the last paragraphs could be improved so that the runtime will not be a function of $|D|^2$. The algorithm is presented in Appendix C.

Theorem 3.21 (Path Query and Tree Database) *Let Q be a path query and D be a tree database. There is an algorithm that computes the semiflexible matchings of Q w.r.t. D in $O(|Q||D| + |M|)$ runtime, where $|Q|$, $|D|$ and $|M|$ are the sizes of the query, the database and the result, respectively.*

The proof of Theorem 3.21 is presented in Appendix C.

Next, we show that query evaluation, under the semiflexible semantics, is not likely to have a polynomial-time input-output complexity when the database is a dag. We use a reduction of 3SAT. A formula φ is in 3CNF if φ is a conjunction of clauses $c_1, \ldots, c_m$, where each clause c_i is a disjunction of three literals $l_{i_1}, l_{i_2}, l_{i_3}$ and a literal is either a propositional letter or a negation of a propositional letter. Deciding whether a formula in 3CNF has a satisfying assignment is NP-complete [35].

Lemma 3.22 (Reduction of 3SAT) *Consider a 3CNF formula φ over a set of propositional letters U. One can construct in polynomial time a path query Q and a dag database D such that the following are equivalent:*

- *There is a semiflexible matching of Q w.r.t. D.*
- *There is an assignment for the propositional letters in U that satisfies φ.*

The proof of Lemma 3.22 is given in Appendix C.

Theorem 3.23 (NP-**Completeness)** *Given a path query Q and a dag database D, deciding whether $\mathcal{M}_{sf}(Q, D)$ is not empty is* NP-*complete.*

Proof. NP-hardness follows from Lemma 3.22. Membership in NP is because one can guess in polynomial time a mapping from the variables of Q to the objects of D and verify, by Corollary 3.13, that this mapping is a semiflexible matching. □

Proposition 3.24 (Path Query and Cyclic Database) *Given a path query Q and a cyclic database D, query evaluation has an $O(|D|^{|Q|-1}|Q|^2)$ runtime under input-output complexity.*

Proof. As a preliminary step, we compute a table that holds an entry for each pair of nodes o_1 and o_2 in D. The entry for o_1 and o_2 contains the set of labels l such that D has a path of the form $o_1{*}lo_2$. If there is no path in D from o_1 to o_2, then the entry contains an empty set. In addition, we topologically sort the nodes of D and compute the strongly connected components of D in $O(|D|)$ (see in [27] how to compute strongly connected components in linear time). All together, the preliminary step is done in $O(|D|^2)$.

Next, we iteratively test all the possible assignments. For each assignment it is checked whether it is a semiflexible matching. Since the root of the query is always mapped to the root of the database, there are $|D|^{|Q|-1}$ possible mappings to check. Verifying that an assignment is a semiflexible matching is done as described in Corollary 3.13. According to Corollary 3.13, it should be checked that the assignment is a connectivity-preserving mapping (see Definition 3.10); and this requires a test w.r.t. each pair of query variables, i.e., $O(|Q|^2)$ tests. □

3.2.2 Tree and DAG Queries over a Tree Database

XML documents are frequently trees. Thus, querying tree databases has a particular importance. In this section, we show that, over a tree database, evaluation of tree and dag queries can be done in polynomial time, in the size of the input and the output, provided that labels are not repeated in query paths.

When both the query and the database are acyclic, queries that have repeated label on a path are not practical, for the following reasons. Acyclic databases seldom have the same label repeated on a path, since it may cause semantic confusion. In the rare cases, when the database does have repeated label on a path, the semantic meaning of objects with the same incoming label is defined by the order of their appearance on the path. In semiflexible matchings, however, this order is not preserved.

We say that a query is *plain* if there are no repeated labels on query paths. Consider a plain tree query and a tree database. In the first part of this section, we will present an algorithm that computes the semiflexible matchings of the query w.r.t. the database, in polynomial time in the size of the input and output. In the second part of this section, we will show how to modify the algorithm in order to compute, in polynomial time, semiflexible matchings of a simple dag query w.r.t. the database. We start by presenting some definitions and notations.

Preliminaries

Suppose that a tree or a dag query is posed to a tree database. If there is a variable in the query that has two different incoming labels (on two different edges), then there are no semiflexible matchings of the query w.r.t. the database. This is because, in a tree database, an object (other than the root) has exactly one incoming edge and a single incoming label. Thus, we will consider queries in which the mapping of nodes to their incoming labels is a function, i.e., for each node, all the edges that enter the node have the same label. We denote by $label(v)$ the label on the edges that enter v. The same notation will also be used for the mapping of objects to labels in a tree database. We assume that for the root of the query and for the root of the database, the $label()$ function returns the unique label $ROOT$.

In a tree database D, the topological order of the nodes is a partial order that can be used to distinguish between pairs of nodes that are connected by a path and pairs of nodes that are not connected by a path. We say that a node o_1 is *above* o_2

and o_2 is *below* o_1, denoted $o_1 \succ o_2$, if there is a path in D from o_1 to o_2.[1] By $o_1 \succeq o_2$ we denote the case where $o_1 \succ o_2$ or $o_1 = o_2$. Note that $r_D \succeq o$ for the root of the database r_D and any object o. Moreover, if $o \neq r_D$ then $r_D \succ o$. Two nodes o_1 and o_2 are *on a path* in D if o_1 is either above o_2 or below o_2. For acyclic queries, "above" and "below" are defined similarly.

Consider a set of objects O in a tree database D. Suppose that O' are all the objects in D that are on a path with each object of O. The following lemma shows how to find O'.

Lemma 3.25 *Let D be a tree database.*

1. *Let $o_1, \dots, o_k$ be objects that are all on one path in D and let $o_\downarrow$ be the lowest node among $o_1, \dots, o_k$, i.e., the object that is the farthest from the root. Then an object o in D is on one path with all the nodes $o_1, \dots, o_k$ if and only if $(o \preceq o_\downarrow) \vee (o \succeq o_\downarrow)$, i.e., o is in the set $\{x \mid (x \preceq o_\downarrow) \vee (x \succeq o_\downarrow)\}$.*

2. *Let $o_1, \dots, o_k$, where $k \geq 2$, be objects such that no pair, among these objects, is connected by a path. Let $o_\uparrow$ be the lowest common ancestor of $o_1, \dots, o_k$. Then, an object o of D is connected by a path to each one of the objects $o_1, \dots, o_k$ if and only if $o \succeq o_\uparrow$, i.e., o is in the set $\{x \mid x \succeq o_\uparrow\}$.*

Proof. We start with Part 1 of the lemma. If o is below $o_\downarrow$ (i.e., $o \preceq o_\downarrow$) then for each $1 \leq i \leq k$ there is a path from o_i to $o_\downarrow$ and a path from $o_\downarrow$ to o. Thus, there is a path to o from each one of the k objects. If o is above $o_\downarrow$ (i.e., $o \succeq o_\downarrow$) then there is a path from o to $o_\downarrow$. If there was a node among $o_1, \dots, o_k$ that did not have a path to o and was not reachable by a path from o, then there would be two different paths in D from the root to $o_\downarrow$. This would be a contradiction to the fact that D is a tree. For the other direction, if $(o \not\preceq o_\downarrow) \wedge (o \not\succeq o_\downarrow)$ then there is no path from o to $o_\downarrow$ and there is no path from $o_\downarrow$ to o. That is, o is not on one path with $o_\downarrow$. (Recall that $o_\downarrow$ is one of the nodes $o_1, \dots, o_k$.)

We now prove Part 2 of the lemma. If $o \succeq o_\uparrow$ then there is a path from o to $o_\uparrow$. There is a path from $o_\uparrow$ to each one of the objects $o_1, \dots, o_k$, since $o_\uparrow$ is an ancestor

[1]In some places we use the terms "ancestor" and "descendent" instead of the terms "above" and "below". By definition, o_1 is an ancestor (descendent) of o_2 when o_1 is above (below) o_2.

of all these objects. So, o is on a path with each one of the k objects. For the other direction, assume that $o \not\succeq o_\uparrow$. There are three possible cases to consider. The first case is that o is below $o_\uparrow$ and there is a path from o to all the objects $o_1, \ldots, o_k$. This would be a contradiction to $o_\uparrow$ being the least common ancestor. The second case is that o is below $o_\uparrow$ and it is also below the object o_i, for some $1 \leq i \leq k$. In this case, there cannot be an o_j, $j \neq i$, such that o is also below o_j, because D is a tree and there is no path between o_i and o_j. So, o is below o_i and o_j is below o. This contradicts the condition that no pair of objects among $o_1, \ldots, o_k$ are connected by a path. The third case is when o is not on the same path with $o_\uparrow$. In this case, there is no path from any of the objects $o_1, \ldots, o_k$ to o. There cannot be a path from o to any of the nodes $o_1, \ldots, o_k$, because D is a tree and there is exactly one path from the root to each node. □

The partial ordering $\succ$ is used for defining *SF-properties*.

Definition 3.26 (SF-Property) *Suppose that Q is a query and D is a database. A* matching property *w.r.t. Q and D is a 4-tuple $P = (v, o, V_P, S_p)$ where v is a variable of Q, o is an object of D, V_P is a set of variables of Q and S_P is a set of objects of D. The matching property P is an* SF-property *if the following holds. For every semiflexible matching μ of Q w.r.t. D, if μ maps v to o, then μ maps each variable of V_P to an object in the set S_P.*

Later in this section, we will present an algorithm that computes semiflexible matchings of a tree query w.r.t. a tree database. In the algorithm, there are two phases. In the first phase, the algorithm computes SF-properties w.r.t. the given query and database. The SF-properties are expressed using the partial ordering $\succ$. This is demonstrated in the following two examples.

Example 3.27 Consider a query Q and a variable v in Q. suppose that v is mapped to a database node o by a semiflexible matching μ. Then, μ maps all the ancestors of v to nodes that are either above o or below o in the database, i.e., to nodes in the set $\{x \mid (x \succ o) \vee (x \prec o)\}$.

Example 3.28 Consider a variable v that has two children v_1 and v_2 in a query Q. Suppose that v_1 and v_2 are mapped by a semiflexible matching μ to the objects o_1 and o_2, respectively, where $o_1 \not\succeq o_2$ and $o_2 \not\succeq o_1$. Let o be the least common ancestor of o_1 and o_2 in the database. Then μ maps v to an object in the set $\{x \mid x \succ o\}$.

SF-properties are defined by *abstract mappings.* Note that the following definition of an abstract mapping is recursive.

Definition 3.29 (Abstract Mapping) *Let Q be an acyclic query, D be a tree database and v be a variable in Q. Consider a pair (o, S), where o is an object of D and S is a set of objects of D. The pair (o, S) is an* abstract mapping *of v w.r.t. D if the following three conditions hold.*

1. *$label(v) = label(o)$.*
2. *All the objects in S are on a path with o.*
3. *Suppose that v has children $v_1, \ldots, v_k$ in Q. Then there are abstract mappings $(o_1, S_1), \ldots, (o_k, S_k)$ of $v_1, \ldots, v_k$, respectively, such that, for each $1 \leq i \leq k$, $S \subseteq S_i$, and $o \in S_i$.*

Proposition 3.30 (Completeness) *Suppose that Q is an acyclic query and D is a tree database. Let μ be a semiflexible matching of Q w.r.t. D and let v be a variable in Q. Then, $(\mu(v), S)$, where $S = \{\mu(u) \mid u \succ v\}$, is an abstract mapping of v w.r.t. D.*

Proof. We need to show that $(\mu(v), S)$ satisfies the three conditions of Definition 3.29. According to Corollary 3.13, μ is a connectivity-preserving mapping (Definition 3.10) and, thus, Condition 1 and Condition 2 are satisfied.

Satisfaction of Condition 3 is shown by induction on the structure of Q. If v is a leaf, then Condition 3 holds in a trivial way since v has no children. The inductive hypothesis is that $(\mu(v'), \{\mu(u) \mid u \succ v'\})$ satisfies Condition 3 of Definition 3.29, for every variable v' that has less than n descendents.

Consider a variable v that has n descendents in Q. Let $v_1, \ldots, v_k$ be the children of v. For each $1 \leq i \leq k$, the variable v_i has less then n descendents. Thus, according to the inductive hypothesis, $(\mu(v_i), S_i)$, where $S_i = \{\mu(u) \mid u \succ v_i\}$, is an abstract mapping of v_i w.r.t. D. Then, for each $1 \leq i \leq k$, $S \subseteq S_i$ and $\mu(v) \in S_i$. Hence, $(\mu(v), \{\mu(u) \mid u \succ v\})$ satisfies Condition 3 of Definition 3.29. □

Consider an acyclic query Q, a tree database D and a variable v of Q. Let o be an object in D such that $label(o) = label(v)$. We say that an abstract mapping (o, S) of v is *maximal* if $S' \subseteq S$ holds for every abstract matching (o, S') of v. The goal of the next proposition is to show that if a variable v has an abstract matching then v also has a maximal abstract matching.

Proposition 3.31 (Applying Union) *Suppose that Q is an acyclic query, D is a tree database and v is a variable of Q. If (o, S^1) and (o, S^2) are two abstract mappings of v, then also $(o, S^1 \cup S^2)$ is an abstract mapping of v.*

Proof. Because (o, S^1) and (o, S^2) are abstract mappings of v, Condition 1 in Definition 3.29 is satisfied. Condition 2 is satisfied w.r.t. S^1 and S^2. Thus, it is satisfied w.r.t. $S^1 \cup S^2$.

We complete the proof of the proposition by induction on the structure of Q. If v is a leaf then Condition 3 is satisfied in a trivial way, i.e., v has no children. Thus the proposition holds for v.

The inductive hypothesis is that the proposition holds for all the nodes v in Q that have a height of at most $n - 1$. In the following we show that if the inductive hypothesis is true then the proposition holds for nodes that have a height of n.

Consider a node v in Q that has a height of n and let (o, S^1) and (o, S^2) be abstract mappings of v. We need to show that Condition 3 in Definition 3.29 is satisfied w.r.t. $(o, S^1 \cup S^2)$.

Suppose that $v_1, \ldots, v_k$ are the children of v. Since (o, S^j) $(j = 1, 2)$ is an abstract mappings of v, there are abstract mappings $(o, S^j_1), \ldots, (o, S^j_k)$ of $v_1, \ldots, v_k$, respectively, such that $S^j \subseteq S^j_i$, for each $1 \leq i \leq k$. By the inductive hypothesis, $(o, S^1_1 \cup S^2_1), \ldots, (o, S^1_k \cup S^2_k)$ are abstract mappings of $v_1, \ldots, v_k$, respectively. These abstract mappings show that Condition 3 is satisfied w.r.t. $(o, S^1 \cup S^2)$. □

Suppose that $(o, S_1), \ldots, (o, S_m)$ are all the abstract mappings of v in which the first element is o. Let $S^{max} = \cup_{i=1}^{m} S_i$. Then, from Proposition 3.31, it follows that

(o, S^{max}) is an abstract mapping of v. Furthermore, $S_i \subseteq S^{max}$, for each $1 \leq i \leq m$. Thus, S^{max} is a maximal abstract mapping of v.

Next, we show that a maximal abstract mapping (o, S) of v defines an SF-property $P = (v, o, \{x \mid x \succ v\}, S)$ w.r.t. Q and D. Note that $\{x \mid x \succ v\}$ are the ancestors of v in Q.

Proposition 3.32 (Abstract Mappings Define SF-Properties) *Suppose that Q is an acyclic query, D is a tree database and v is a variable of Q. If (o, S) is a maximal abstract mapping of v w.r.t. D then $P = (v, o, \{x \mid x \succ v\}, S)$ is an SF-property w.r.t. Q and D.*

Proof. We need to show that P is an SF-property w.r.t. Q and D. That is, if a semiflexible matching of Q w.r.t. D maps v to o then it also maps each ancestor of v to an object of S.

Consider a semiflexible matchings μ of Q w.r.t. D. Suppose that μ maps v to o and $\hat{v} \succ v$, i.e., $\hat{v}$ is a variable above v. Then, we need to show that $\mu(\hat{v})$ is in S.

Consider the pair $(\mu(v), S_v)$, where $S_v = \{\mu(w) \mid w \succ v\})$. According to Proposition 3.30, $(\mu(v), S_v)$ is an abstract mapping of v. In addition, S_v contains $\mu(\hat{v})$, because $\hat{v}$ is above v. Since (o, S) is a maximal abstract mapping w.r.t. v and $\mu(v) = o$, it holds that $S_v \subseteq S$. Hence, $\mu(\hat{v}) \in S$. $\square$

Computing Abstract Mappings

Now, we will show how to compute abstract mappings in polynomial time in the size of the query and the database. The next proposition is a first step towards computing abstract mappings.

Proposition 3.33 (Bottom-Up Evaluation) *Suppose that Q is an acyclic query, D is a tree database, v is a variable of Q and $v_1, \dots, v_k$ are the children of v (if there are any). Then (o, S) is an abstract mapping of v if and only if the following two conditions hold.*

1. *$label(o) = label(v)$.*

2. *There are abstract mappings* $(o_1, S_1), \ldots, (o_k, S_k)$ *of* $v_1, \ldots, v_k$*, respectively, such that*

   (a) $o \in S_i$*, for each* $1 \leq i \leq k$*, and*

   (b) $S \subseteq S_1 \cap S_2 \cap \cdots S_k \cap \{x \mid x \preceq o \vee x \succeq o\}$*. (If* v *is a leaf of* Q*, then* $S \subseteq \{x \mid x \preceq o \vee x \succeq o\}$*.)*

Proof. Suppose that Conditions 1 and 2 of the proposition hold. We need to show that (o, S) is an abstract mapping. We show that the three conditions in Definition 3.29 hold. Condition 1 of Definition 3.29 is the same as Condition 1 in the proposition. Thus, it holds.

We show that Condition 2 of Definition 3.29 holds. The set S is contained in the set $S_p = \{x \mid x \preceq o \vee x \succeq o\}$, because S is the intersection of S_p with other sets. The set S_p is the set of objects that are on a path with o. Thus, all the objects in S are on a path with o, which is what we needed to show. To prove that Condition 3 of Definition 3.29 holds, we need to show that $S \subseteq S_i$, for each $1 \leq i \leq k$. Since S is an intersection of S_i with other sets, obviously, $S \subseteq S_i$. From Condition 2 it follows that $o \in S_i$, for each $1 \leq i \leq k$.

For the other direction, suppose that (o, S) is an abstract mapping of v. We need to show that Conditions 1 and 2 of the proposition hold. Condition 1 of the proposition is equal to Condition 1 in Definition 3.29. Thus, it holds.

We claim that Condition 2 of the proposition follows from Condition 2 and Condition 3 of Definition 3.29. Since (o, S) is an abstract mapping w.r.t. v, there are abstract mappings $(o_1, S_1), \ldots, (o_k, S_k)$ of $v_1, \ldots, v_k$, respectively, such that for each $1 \leq i \leq k$, $S \subseteq S_i$ and $o \in S_i$. Thus, $S \subseteq S_1 \cap S_2 \cap \ldots S_k$. The set $S_p = \{x \mid x \preceq o \vee x \succeq o\}$ contains all the objects that are on a path with o. Thus, because of Condition 2 in Definition 3.29, $S \subseteq S_p$. That is, $S \subseteq S_1 \cap S_2 \cap \ldots S_k \cap S_p$. This proves our claim. □

Corollary 3.34 (Maximal Elements) *In Proposition 3.33, if* (o, S) *is a maximal abstract mapping of* v *then there are* $(o_1, S_1), \ldots, (o_k, S_k)$ *that satisfy the conditions of the proposition and are maximal abstract mappings of* $v_1, \ldots, v_k$*, respectively. In* *addition, the equality $S = S_1 \cap S_2 \cap \cdots S_k \cap \{x \mid x \preceq o \vee x \succeq o\}$ holds (if v is a leaf, $S = \{x \mid x \preceq o \vee x \succeq o\}$).*

Proof. If v is a leaf then $(o, \{x \mid x \preceq o \vee x \succeq o\})$ is a maximal abstract mapping of v and the conditions of Proposition 3.33 are satisfied. Suppose that v is not a leaf. Let $v_1, \ldots, v_k$ be the children of v. Consider a maximal abstract mapping (o, S). According to Proposition 3.33, there are abstract mappings $(o_1, S_1), \ldots, (o_k, S_k)$ of $v_1, \ldots, v_k$, respectively, such that $o \in S_i$, for each $1 \leq i \leq k$, and $S \subseteq S_1 \cap S_2 \cap \cdots S_k \cap \{x \mid x \preceq o \vee x \succeq o\}$.

Let $(o_1, S_1^m), \ldots, (o_k, S_k^m)$ be maximal abstract mappings. Let S^m be $S_1^m \cap S_2^m \cap \cdots S_k^m \cap \{x \mid x \preceq o \vee x \succeq o\}$. Then, $S_i \subseteq S_i^m$, for each $1 \leq i \leq k$ and, hence, $S \subseteq S^m$.

According to Proposition 3.33, (o, S^m) is an abstract mapping of v. Because (o, S) is a maximal abstract mapping of v, it holds that $S \supseteq S^m$. Therefore, $S = S^m$, i.e., S is the intersection of sets in maximal abstract mappings. $\square$

Consider an abstract mapping (o, S) w.r.t. some variable v in a query. The set S has $O(|D|)$ size, where D is the size of the database. However, we will show that it is not necessary to actually compute S. Instead, for each set S in an abstract mapping, we compute an *abstract set* which is a notation of a definition of S. Such a notation is essentially a constraint such that S contains all the database objects that satisfy this constraint.

Definition 3.35 (Abstract Sets) *Let o be a database object. A* path abstract set (pas) *is a set notation of the form $\{x \mid x \succeq o\}$. A* tree abstract set (tas) *is a set notation of the form $\{x \mid x \preceq o \vee x \succeq o\}$. A basic abstract set is a set notation that is either a pas or a tas.*

Two abstract sets are equivalent (denoted $\equiv$) if the sets that they represent are equal. The following proposition shows that over a tree database, every conjunction of basic abstract sets can be written as an equivalent basic abstract set.

Proposition 3.36 (Equivalence Rules for Conjunction) *The following equivalences hold in a tree database D.*

```
Procedure combine($S_1, S_2$);
Input Two basic abstract sets $S_1$ and $S_2$;
Output A basic abstract set that is equivalent to $S_1 \cap S_2$;

if $S_1$ is a pas and $S_2$ is a tas then swap $S_1$ and $S_2$;
let $o_1$ and $o_2$ be the objects that appear in $S_1$ and $S_2$, respectively;
if $o_1 \succeq o_2$ or $o_2 \succeq o_1$ then
    let $o_h$ be the highest of $o_1$ and $o_2$ ($o_h = o_1$ if $o_1 = o_2$);
    let $o_l$ be the lowest of $o_1$ and $o_2$ ($o_l = o_2$ if $o_1 = o_2$);
    case $S_1 = \{x \mid x \preceq o_1 \vee x \succeq o_1\}$ and $S_2 = \{x \mid x \preceq o_2 \vee x \succeq o_2\}$:
        $\hat{S} = \{x \mid \preceq o_l \vee \succeq o_l\}$;
    case $S_1 = \{x \mid x \succeq o_1\}$ and $S_2 = \{x \mid x \succeq o_2\}$:
        $\hat{S} = \{x \mid x \succeq o_h\}$;
    case $S_1 = \{x \mid x \preceq o_l \vee x \succeq o_l\}$ and $S_2 = \{x \mid x \succeq o_h\}$:
        $\hat{S} = \{x \mid x \succeq o_h\}$;
    case $S_1 = \{x \mid x \preceq o_h \vee x \succeq o_h\}$ and $S_2 = \{x \mid x \succeq o_l\}$:
        $\hat{S} = \{x \mid x \succeq o_l\}$;
else (* $o_1 \not\succeq o_2$ and $o_2 \not\succeq o_1$ *)
    let $o_c$ be the least common ancestor of $o_1$ and $o_2$;
    $\hat{S} = \{x \mid x \succeq o_c\}$
return $\hat{S}$;
```

Figure 3.2: For a conjunction of two basic abstract sets, *combine* computes an equivalent basic abstract set (without conjunction), according to the seven rules of Proposition 3.36.

1. *If* $o_2 \succeq o_1$ *then*

   (a) $\{x \mid (x \preceq o_1 \vee x \succeq o_1) \wedge (x \preceq o_2 \vee x \succeq o_2)\} \equiv \{x \mid x \preceq o_1 \vee x \succeq o_1\}$

   (b) $\{x \mid x \succeq o_1 \wedge x \succeq o_2\} \equiv \{x \mid x \succeq o_2\}$

   (c) $\{x \mid (x \preceq o_1 \vee x \succeq o_1) \wedge x \succeq o_2\} \equiv \{x \mid x \succeq o_2\}$

   (d) $\{x \mid (x \preceq o_2 \vee x \succeq o_2) \wedge x \succeq o_1\} \equiv \{x \mid x \succeq o_1\}$

2. *If o is the least common ancestor of o_1 and o_2 and $(o_2 \not\succeq o_1) \wedge (o_2 \not\preceq o_1)$, i.e., o_1 and o_2 are not on the same path, then*
   (a) $\{x \mid (x \preceq o_1 \vee x \succeq o_1) \wedge (x \preceq o_2 \vee x \succeq o_2)\} \equiv \{x \mid x \succeq o\}$
   (b) $\{x \mid x \succeq o_1 \wedge x \succeq o_2\} \equiv \{x \mid x \succeq o\}$
   (c) $\{x \mid x \succeq o_1 \wedge (x \preceq o_2 \vee x \succeq o_2)\} \equiv \{x \mid x \succeq o\}$

Proof. All the equivalences directly follow from the definition of the partial order $\succeq$ and the properties of D as a tree. □

We use the equivalence rules of Proposition 3.36 to write a conjunction of basic abstract sets as a basic abstract set. A function *combine* that applies the rules of Proposition 3.36 to a conjunction of two basic abstract sets is presented in Figure 3.2.

Our goal is to compute maximal abstract matchings. This requires discarding the non-maximal elements from the sets of abstract matchings that the algorithm computes.

Let D be a tree database. We say that an abstract set S_2 *contains* an abstract set S_1 w.r.t. D, denoted $S_1 \sqsubseteq S_2$, if the set of objects that S_1 defines is contained in the set of objects that S_2 defines. The following proposition presents containment rules.

Proposition 3.37 (Containment Rules) *Let D be a tree database and let o_1 and o_2 be objects in D such that $o_2 \succeq o_1$. The following containment rules hold.*

1. $\{x \mid x \succeq o_1\} \sqsubseteq \{x \mid x \preceq o_2 \vee x \succeq o_2\}$
2. $\{x \mid x \preceq o_1 \vee x \succeq o_1\} \sqsubseteq \{x \mid x \preceq o_2 \vee x \succeq o_2\}$
3. $\{x \mid x \succeq o_2\} \sqsubseteq \{x \mid x \succeq o_1\}$
4. $\{x \mid x \succeq o_2\} \sqsubseteq \{x \mid x \preceq o_1 \vee x \succeq o_1\}$

Proof. All the containment rules directly follow from the definition of the partial order $\succeq$ and the properties of D as a tree. □

The following proposition shows that the set of containment rules in Proposition 3.37 is complete.

Proposition 3.38 (Containment Rules, Completeness) *Consider a tree database D. Let S_1 and S_2 be basic abstract sets such that $S_1 \sqsubseteq S_2$. Let o_1 and o_2 be the objects that appear in the definitions of S_1 and S_2, respectively. Then, one of the following two cases is true.*

- *It holds that $o_2 \succeq o_1$ and the containment $S_1 \sqsubseteq S_2$ has the form of Rule 1 or Rule 2 in Proposition 3.37.*
- *It holds that $o_1 \succeq o_2$ and the containment $S_1 \sqsubseteq S_2$ has the form of Rule 3 or Rule 4 in Proposition 3.37.*

Proof. If $o_2 \not\succeq o_1$ and $o_1 \not\succeq o_2$, then the containment $S_1 \sqsubseteq S_2$ does not hold. When $o_2 \succeq o_1$ or $o_1 \succeq o_2$, a simple case analysis proves the claim. □

In Figure 3.3, a procedure *remove-non-maximal* is presented. The procedure receives a database D and a set A of abstract mappings of a variable v w.r.t. D. In the procedure, if A contains two abstract matchings (o, S_1) and (o, S_2) such that $S_1 \sqsubseteq S_2$, according to one of the rules in Proposition 3.37, then (o, S_1) is removed from A.

Computing Abstract Mappings

Now, we show how to compute the maximal abstract mappings of variables in an acyclic query Q w.r.t. a tree database D. Procedure *MAMA* computes the set of maximal abstract mappings of each variable in Q. The computation is performed in a bottom-up fashion, starting with the leaves of the query and rising towards the root (i.e., an order that is inverted to the topological order of Q). For each node v, *MAMA* computes a set A_v that consists of all the maximal abstract mappings of v.

In the following, we describe how to compute the sets A_v. The computation is divided into several cases. The cases are according to the the number of children that v has.

Case 1 Suppose that v is a leaf of Q. Then, for each object o in D such that $label(o) = label(v)$, the abstract mapping $(o, \{x \mid x \preceq o \vee x \succeq o\})$ is added to A_v. Note that there are no non-maximal elements that should be removed.

```
Procedure remove-non-maximal($A, D$);
Input  A database $D$ and a set of abstract mappings $A$
       of a variable $v$ w.r.t. $D$;
Output The set $A$ after removing abstract matchings $(o, S)$ for which
       there is another abstract matching $(o, S')$ in $A$ such that $S \sqsubseteq S'$.

for each two abstract mappings $(o_1, S_1)$ and $(o_2, S_2)$ in $A$ such that $o_1 = o_2$
    let $o_{c_1}$ and $o_{c_2}$ be the objects in $S_1$ and $S_2$, respectively;
    if $o_{c_1} \preceq o_{c_2}$ then
        if $S_2$ has the form $\{x \mid x \preceq o_{c_2} \vee x \succeq o_{c_2}\}$ then
            remove $(o_1, S_1)$ from $A$;
        else if $S_2$ has the form $\{x \mid x \succeq o_{c_2}\}$ then
            remove $(o_2, S_2)$ from $A$;
    else if $o_{c_2} \prec o_{c_1}$ then
        if $S_1$ has the form $\{x \mid x \preceq o_{c_1} \vee x \succeq o_{c_1}\}$ then
            remove $(o_2, S_2)$ from $A$;
        else if $S_1$ has the form $\{x \mid x \succeq o_{c_1}\}$ then
            remove $(o_1, S_1)$ from $A$;
```

Figure 3.3: The procedure receives a set of abstract mappings and removes abstract mappings that are not maximal. The removal is according to the four containment rules of Proposition 3.37.

Case 2 Suppose that v has exactly one child v_1 and A_{v_1} has already been computed. Initially, the algorithm finds all the objects o in D such that $label(o) = label(v)$. If there is a pair (o_1, S_1) in A_{v_1}, such that $o \in S_1$ and $label(o) = label(v)$, then the abstract mapping $(o, combine(S_1, S_p))$, where $S_p = \{x \mid x \preceq o \vee x \succeq o\}$, is added to A_v. By executing *remove-non-maximal*(A_v, D), the non-maximal elements of A_v are removed.

Case 3 Suppose that v has two or more children $v_1, \ldots, v_k$ and $A_{v_1}, \ldots, A_{v_k}$ have already been computed. The computation of A_v is performed in k steps.

In the first step, a set A'_1 is computed from A_{v_1} in exactly the same way as the computation of A_v from A_{v_1} in Case 2. In the ith step ($2 \leq i \leq k$), A'_i is computed from A'_{i-1} and A_{v_i}, as described next.

1. Iteratively, all the elements (o, S) of A'_{i-1} are considered.

2. For each (o_i, S_i) in A_{v_i} such that $o \in S_i$ and $label(o) = label(v)$, the abstract mapping $(o, combine(S, S_i))$ is added to A'_i.

3. By executing *remove-non-maximal*(A'_i, D), the non-maximal elements of A'_i are removed.

The set A_v is the set A'_k.

Proposition 3.39 (Correctness) *Consider a dag query and a tree database. For each variable v of the query, the procedure MAMA computes the set of maximal abstract mappings of v w.r.t. D.*

Proof. Consider a variable v and the set A_v that was computed by *MAMA* for the variable v. Corollary 3.34 shows that the set A_v contains all the maximal abstract mappings of v. Proposition 3.33 shows that all the elements of A_v are abstract mappings. Finally, since non-maximal elements are removed, A_v contains only maximal abstract mappings. □

Proposition 3.40 (Complexity) *Consider a dag query Q and a tree database D. Let q and d be the number of nodes in Q and D, respectively. Each set A_v that is computed by MAMA has at most d elements. The time complexity of MAMA is in $O(q^2d^2)$.*

Proof. The set A_v consists of maximal abstract mappings of v. For an object o of D, there can be at most one maximal abstract mapping of v. Thus, the size of A_v cannot exceed the number of objects in D.

Now, we analyze the time complexity of *MAMA*. First, note that in $O(d^2)$ it is possible to construct a data structure that can tell in $O(1)$ whether two database objects are on a path. This is required for testing in $O(1)$ if a given object is an element of an abstract set, i.e., an element of the concrete set that the abstract set represents.

Consider a set A with x abstract mappings. Removing the non-maximal elements from A with the procedure *remove-non-maximal* is in $O(x)$ runtime. This is because, in one pass over A the elements of A can be inserted into a hash table, where the search keys of the hash table are the objects of the abstract mappings. Then, in each step, a pair of abstract mappings (o, S_1), (o, S_2) are picked. For each such pair, one of the two abstract mapping is removed, according to the test in the procedure *remove-non-maximal*. Since in each removal step, a single element is removed, there can be at most x removal steps. Thus, the runtime for the removal of the non-maximal elements is in $O(x)$.

Consider a node v in Q. *MAMA* computes A_v, according to three different cases. In all the three cases, searching D for objects that have the same label as v is in $O(d)$ runtime.

We now consider the three cases in *MAMA* for constructing A_v. If v is a leaf (i.e., Case 1 of *MAMA*) then at most d abstract mappings are added to A_v and the evaluation is in $O(d)$.

Suppose that v has a single child v_1 (i.e., Case 2 of *MAMA*). In the algorithm there is a loop over the objects of D and a nested loop over the objects of A_{v_1}. In each inner loop, there is a test that checks if an object o is contained in a set S_1. There are at most d objects in D with the same label as v. The set A_{v_1} has at most d elements. Thus, there are at most d^2 tests. Testing that o is in S_1 is in $O(1)$, using the data structure we described above. Thus, the evaluation of this case is in $O(d^2)$ runtime.

Suppose that v has $k > 1$ children. Then A_v is computed in k steps as described in Case 3 of *MAMA*. Each intermediate set A'_{i-1} has at most d objects because non-maximal elements are removed at the end of each step. Thus, in each step, two sets A'_{i-1} and A_{v_i}, with at most d elements, are combined. Combining the two sets is in $O(d^2)$ because of the same reasons that were given in the analysis of the runtime of Case 2. Since $k < q$, the evaluation of this case is in $O(qd^2)$ runtime.

All together, there are q sets to compute. Each set is computed in $O(qd^2)$

runtime. Thus, the whole computation is in $O(q^2d^2)$ runtime. □

Algorithm *SFTT*

So far, we described how to compute the maximal abstract mappings of variables in a tree query w.r.t. a tree database. Now, we will present Algorithm *SFTT* that actually computes semiflexible matchings of a plain tree query w.r.t. a tree database. We will also show that the algorithm has polynomial time complexity, in the size of the input and the output.

Before presenting the algorithm, we need to introduce a new definition. In Definition 3.26, matching properties were defined. We now define *compliance* with matching properties.

Definition 3.41 (Compliance) *Suppose that Q is a query and D is a database. Consider a matching property $P = (v, o, V_P, S_p)$ w.r.t. Q and D. Let μ be an assignment to V_P w.r.t. D. We say that μ* complies *with P if $\mu(u) \in S_p$ for each u in V_P.*

We now provide a description of *SFTT*. *SFTT* computes the semiflexible matchings using a top-down traversal over the query. Matchings are computed by a series of extension steps. In each step, the algorithm descends to a new variable and matchings that were computed in a previous step are extended w.r.t. the new variable. Maximal abstract mappings are being used as SF-properties and they make sure that the created matchings are semiflexible matchings.

Consider a tree query Q and a tree database D. Algorithm *SFTT* computes a set $\mathcal{M}$ of the semiflexible matchings of Q w.r.t. D as follows.

1. In the first phase of the algorithm, the sets of maximal abstract mappings A_v are computed by *MAMA* for all the variables v of Q.

2. The variables of Q are sorted topologically. Let $v_0, v_1, \ldots, v_n$ be the variables of Q ordered according to the topological sort. Note that v_0 is the root of Q.

3. Iteratively, $n + 1$ sets $\mathcal{M}^0, \ldots, \mathcal{M}^n$ are computed as described next.

4. If A_{v_0} is not empty, then the set $\mathcal{M}^0$ consists of a single mapping $\mu_0 = \{(v_0, r_D)\}$, i.e., μ_0 is the mapping of the query root (v_0) to the database root. If A_{v_0} is empty, then the algorithm stops and returns an empty set $\mathcal{M}$.

5. In step i, the set $\mathcal{M}^i$ is constructed from $\mathcal{M}^{i-1}$ by extending the matchings in $\mathcal{M}^{i-1}$ w.r.t. v_i, as follows.

   (a) Iteratively, each matching μ_{i-1} in $\mathcal{M}^{i-1}$ and each abstract mapping (o, S) in A_{v_i} are considered.

   (b) Let $P = (v_i, o, \{u \mid u \succ v_i\}, S)$ be a matching property that the abstract mapping (o, S) defines. If μ_{i-1} complies with P then $\mu_i = \mu_{i-1} \cup \{(v_i, o)\}$, i.e., μ_{i-1} is extended by mapping v_i to o, and μ_i is added to $\mathcal{M}^i$.

6. Let $\mathcal{M}$ be the set $\mathcal{M}^n$. The set $\mathcal{M}$ is returned.

The following propositions prove the correctness of *SFTT* and analyze its time complexity.

Proposition 3.42 (Correctness) *Suppose that Q is a plain tree query and D is a tree database. The set $\mathcal{M}$ that SFTT returns is the set of all the semiflexible matchings of Q w.r.t. D.*

Proof. There are two things to show. One thing to show is that all the matchings in $\mathcal{M}$ are semiflexible matchings of Q w.r.t. D. The second thing to show is that all the semiflexible matchings of Q w.r.t. D are in $\mathcal{M}$.

First, we show that all the matchings in $\mathcal{M}$ are semiflexible matchings of Q w.r.t. D. We show this by showing that all the matchings in $\mathcal{M}$ are connectivity-preserving mappings (Definition 3.10). This will prove what we want, according to Corollary 3.13 and the fact that in all the mappings in $\mathcal{M}$ the query root is mapped to the database root.

Consider a matching μ in $\mathcal{M}$. We show that the first condition of Definition 3.10 holds. Let v_i be a variable of Q. Suppose that Q has a simple path of the form $v_0{*}lv_i$. Then, the object $\mu(v_i)$ is reachable from the root and has a single incoming label. This label must be l because all the objects in A_{v_i} have an incoming label that is equal to the incoming label of v. Thus, there is a path in D of the form $r_D{*}l\mu(v_i)$.

We show that the second condition of Definition 3.10 holds. Let v_i and v_j be two variables of Q such that there is a path in Q of the form $v_h l_i v_i {*} l_j v_j$. Consider step j in the algorithm, where the assignments of $\mathcal{M}^{j-1}$ are extended w.r.t. v_j.

Because μ is created by the algorithm the following three assertions must be true. First, there is a matching μ_{j-1} in $\mathcal{M}^{j-1}$ such that $\mu_{j-1}(v_l) = \mu(v_l)$, for each $1 \leq l < j$. Second, there is an abstract mapping $(\mu(v_j), S)$ in A_{v_j}. Third, μ_{j-1} complies with the matching property $P = (v_j, \mu(v_j), \{u \mid u \succ v_j\}, S)$.

The variable v_i is above v_j in Q. Hence, v_i is in the set $\{u \mid u \succ v_j\}$. From the compliance of μ_{j-1} with P, it follows that $\mu_{j-1}(v_i)$ is in S. We showed earlier that $\mu_{j-1}(v_l) = \mu(v_l)$, for each $1 \leq l < j$. In particular, $\mu_{j-1}(v_i) = \mu(v_i)$. So, $\mu(v_i) \in S$.

The pair $(\mu(v_j), S)$ is an abstract mapping of v_j. According to the definition of abstract mappings (Definition 3.29), all the object in S are on a path with $\mu(v_j)$. Since $\mu(v_i)$ is in S, $\mu(v_i)$ and $\mu(v_j)$ are on a path in D. The label that enters $\mu(v_i)$ and the label that enters $\mu(v_j)$ are l_i and l_j, respectively, because $\mu(v_i)$ and $\mu(v_j)$ are in abstract mappings of A_{v_i} and A_{v_j}, respectively. Thus, D either has a path of the form $\mu(v_i){*}l_j\mu(v_j)$ or a path of the form $\mu(v_j){*}l_i\mu(v_i)$. This shows that the second condition of Definition 3.10 holds and μ is a semiflexible matching.

Now, we show that all the semiflexible matchings of Q w.r.t. D are in M. First, we show that the abstract mappings are not too restrictive.

Suppose that μ is a semiflexible matching of Q w.r.t. D. Let v be a variable of Q. Then, according to Proposition 3.30, there is an abstract mapping of $(\mu(v), S)$ is A_v such that if u is an ancestor of v then $\mu(u) \in S$.

Let μ_i $(0 \leq i \leq n)$ denote a matching that is defined for the variables $v_0, \ldots, v_i$, where $\mu_i(v_j) = \mu(v_j)$ for each $0 \leq j \leq i$. We complete the proof by induction on i.

The inductive hypothesis is that μ_i is in $\mathcal{M}^i$. For $i = 0$ the claim holds, since Proposition 3.30 implies that A_{v_0} is empty only if there are no semiflexible matchings of Q w.r.t. D.

Suppose that the inductive hypothesis holds for i. We want to show that it also holds for $i+1$, i.e., μ_{i+1} is in $\mathcal{M}^{i+1}$.

According to the inductive hypothesis, μ_i is in $\mathcal{M}^i$. As was shown above, there is an abstract mapping $(\mu(v_{i+1}), S)$ in $A_{v_{i+1}}$, such that μ_i complies with the matching property $P = (v_{i+1}, \mu(v_{i+1}), \{u \mid u \succ v_{i+1}\}, S)$. Thus, μ_i is extended by adding to it $(v_{i+1}, \mu(v_{i+1}))$. Hence, μ_{i+1} is in $\mathcal{M}^{i+1}$. This completes the proof of the induction. and completes the proof of the proposition. $\square$

In the following, we analyze the time complexity of *SFTT*. First, we show that in the second phase of the algorithm there are no *dangling matchings*. (A matchings μ_i in $\mathcal{M}^i$ is considered dangling if in the $i+1$st step of the algorithm it is not extended with any assignment to v_i.)

Lemma 3.43 (No Dangling Matchings) *In SFTT,* $|\mathcal{M}^i| \leq |\mathcal{M}^{i+1}|$ *for each* $0 \leq i \leq n-1$.

Proof. We prove the lemma by showing that for every matching μ_i in $\mathcal{M}^i$ there is a matching μ_{i+1} that extends μ_i and belongs to $\mathcal{M}^{i+1}$.

Suppose that μ_i is in $\mathcal{M}^i$. Consider the variable v_{i+1}. Let v_h be the parent of v_{i+1} in Q.

In Step h of the algorithm, the mapping μ_{h-1} was extended by assigning $\mu_i(v_h)$ to v_h. Thus, there is an abstract mapping $(\mu_i(v_h), S_h)$ in A_{v_h}, such that $\mu(u) \in S_h$ holds for each variable u above v_h.

As was claimed, $(\mu_i(v_h), S_h)$ is an abstract mapping w.r.t. v_h. In addition, v_{i+1} is a child of v_h. Hence, according to the definition of the abstract mapping (Definition 3.29), there is an abstract mapping (o, S) of v_{i+1}, such that $S_h \subseteq S$ and $\mu_i(v_h) \in S$. Thus, in $A_{v_{i+1}}$ there is (o', S') that is either equal to or contains (o, S).

Let u be a variable above v_{i+1} in Q. If u is v_h then $\mu_i(u) \in S$. If u is above v_h, then $\mu_i(u) \in S_h$ and $S_h \subseteq S$, so again $\mu_i(u) \in S$. Consider the matching property $P = (v_{i+1}, o', \{u \mid u \succ v_{i+1}\}, S')$. Then, μ_i complies with P and, hence, μ_i is extended in the $i+1$st step of the algorithm. That is, $\mu_i \cup \{(v_{i+1}, o')\}$ is in $\mathcal{M}^{i+1}$.

This shows that there are no dangling matchings. Thus, the size of the sets $\mathcal{M}^i$ monotonically increases. $\square$

Proposition 3.44 (Complexity) *Suppose that SFTT is computed w.r.t. a plain tree query Q and a tree database D. SFTT has $O(|Q|^2|D|(|D|+|\mathcal{M}|))$ time complexity, where $|Q|$, $|D|$ and $|\mathcal{M}|$ are the sizes of the query, the database and the result, respectively.*

Proof. The first phase of the algorithm is in $O(|Q|^2|D|^2)$, according to Proposition 3.40.

Consider extension step i in the second phase of the algorithm. In this step, there is an iteration over the $|\mathcal{M}^i|$ matchings of $\mathcal{M}^i$ and the abstract mappings of A_{v_i}. By Lemma 3.43, $|\mathcal{M}^i| \leq |\mathcal{M}|$. By Proposition 3.40, the size of A_{v_i} is at most $|D|$. Thus there are $O(|\mathcal{M}||D|)$ iterations. It each iteration, an abstract mapping (o, S) is tested w.r.t. the assignment to the $O(|Q|)$ variables that are above v_i in Q. Recall that it is in $O(1)$ runtime to test if an object is in a set that is defined by a basic abstract set (assuming that a suitable data structure is initially constructed).

In the second phase of the algorithm there are $|Q|$ extension steps. According to the above arguments, each step is in $O(|\mathcal{M}||D||Q|)$. Thus, the second phase of the algorithm is in $O(|Q|^2|D||\mathcal{M}|)$ runtime. The two phases together have $O(|Q|^2|D|(|D| + |\mathcal{M}|))$ time complexity. □

Evaluating a DAG query over a Tree Database

Consider a plain dag query Q and a tree database D. In this section, we show how to compute the semiflexible matchings of Q w.r.t. D in polynomial time, under input-output complexity.

SFTT was presented in the previous section as an algorithm for computing matchings of a semiflexible tree query w.r.t. a tree database. *SFTT* remains sound and complete when the query is a dag. That is, *SFTT* can compute the semiflexible matchings of a dag query Q w.r.t. a tree database D. However, for dag queries, *SFTT* does not have a polynomial-time input-output complexity. The cause of this is that in extension steps w.r.t. variables with more than one incoming edge, there could be dangling matchings.

Next, we describe how to modify *SFTT* so that the input-output time complexity will be polynomial for plain dag queries. We start by presenting some required notations and definitions.

Definition 3.45 (Intersection Node, Origin, Middlenode) *A variable v, in a dag query Q, is an* intersection node *if there are two or more edges in Q that enter v. A node u_s is a* source *of v if all the paths from the root to v go through u_s. The* origin *of v is the lowest source of v, i.e., a source of v that has no source of v below it. A node w is a* middlenode *of v if it is between u_s and v. That is, there is a path from u_s to w and a path from w to v. The origin of v is also considered a middlenode of v.*

It is easy to see that in a dag query, an intersection node has exactly one origin but a node can be the origin of several intersection nodes.

The following observation shows why evaluation of dag queries w.r.t. a tree database is not computationally hard. In a semiflexible matching of a dag w.r.t. a tree, for each intersection node v, either all the nodes above v or all the nodes below v are mapped to a single database path. This is proved in the next proposition.

Proposition 3.46 (Mapping Intersection Nodes) *Let Q be a dag query, D be a tree database and μ be a semiflexible matching of Q w.r.t. D. If v is an intersection node of Q then one of the next two statements must be true.*

- *For each pair of nodes u_1 and u_2 above v, their images $\mu(u_1)$ and $\mu(u_2)$ are connected by a path.*
- *For each pair of nodes w_1 and w_2 below v, their images $\mu(w_1)$ and $\mu(w_2)$ are connected by a path.*

Proof. Consider a case where u_1 and u_2 are nodes above v such that $\mu(u_1)$ and $\mu(u_2)$ are not connected by a path. In addition, w_1 and w_2 are nodes below v such that $\mu(w_1)$ and $\mu(w_2)$ are not connected by a path. We show that the assumptions of this case lead to a contradiction.

In the query, there is a path from u_1 to v and a path from v to w_1. Thus, there is a path, in Q, from u_1 to w_1. Similarly, we can show that Q has a path from u_1 to w_2, a path from u_2 to w_1 and a path from u_2 to w_2.

Because there is a path in Q from u_1 to w_1, either D has a path from $\mu(u_1)$ to $\mu(w_1)$ or vice versa. Similarly, there must be a path from $\mu(u_1)$ to $\mu(w_2)$ or vice versa. We assumed that $\mu(w_1)$ and $\mu(w_2)$ are not connected by a path. Hence, $\mu(w_1)$ and $\mu(w_2)$ must be below $\mu(u_1)$, in D.

By replacing the roles of u_1 and u_2 with w_1 and w_2, we can show, in the same way, that $\mu(u_1)$ and $\mu(u_2)$ must be below $\mu(w_1)$, in D. However, D is a tree, so it impossible that, simultaneously, $\mu(w_1)$ is below $\mu(u_1)$ and $\mu(u_1)$ is below $\mu(w_1)$. This contradiction completes the proof. □

We will show how to construct mappings of Q to D according to the restriction that Proposition 3.46 imposes. Consider a dag query Q and a tree database D. Let $v_0, \ldots, v_n$ be the variables of Q and $A_{v_0}, \ldots, A_{v_n}$ be the sets of maximal abstract mappings that the procedure *MAMA* computes.

Definition 3.47 (Restriction of Abstract Mappings) *Suppose that (o_i, S_i) is an abstract mapping of an intersection node v_i. The* restriction *of $A_{v_0}, \ldots, A_{v_n}$ to (o_i, S_i) and v_i are the sets $A'_{v_0}, \ldots, A'_{v_n}$ such that the following holds.*

1. *The set A'_{v_i} contains (o_i, S_i) and no other element.*

2. *Consider a middlenode v_j of v_i. The set A'_{v_j} satisfies the next three conditions.*

   (a) *$A'_{v_j} \subseteq A_{v_j}$.*

   (b) *For every (o_j, S_j) in A'_{v_j}, it holds that $o_j \in S_i$.*

   (c) *Suppose that v_j has children $v_1^j, \ldots, v_k^j$ in Q. Then there are abstract mappings $(o_1^j, S_1^j), \ldots, (o_k^j, S_k^j)$ of $v_1^j, \ldots, v_k^j$, respectively, such that, for each $1 \leq l \leq k$, $S_j \subseteq S_l^j$, $o_j \in S_l^j$ and (o_l^j, S_l^j) is in $A'_{v_l^j}$.*

3. *For v_j that is not a middlenode of v_i, A'_{v_j} is equal to A_{v_j}.*

Consider the case where in Definition 3.47, S_i has the form of $\{x \mid x \preceq o' \vee x \succeq o'\}$. Then, the sets of abstract mappings in the restriction correspond to mappings in which the middlenodes of v are mapped to database objects on the path from the database root to o'. Suppose that S_i has the form of $\{x \mid x \succeq o'\}$. Then, the sets of abstract mappings in the restriction correspond to mappings in which the nodes below v are mapped to objects of D that are above o'.

Consider a plain dag query Q, a tree database D and a semiflexible matching μ of Q w.r.t. D. Let $v_0, \ldots, v_n$ be the variables of Q and v_i be an intersection node. In addition, let $v_{i_1}, \ldots, v_{i_m}$ be the middlenodes of v_i.

Proposition 3.48 (Completeness) *Suppose that $(\mu(v_i), S_i)$ is an element of A_{v_i}. Let $A'_{v_0}, \ldots, A'_{v_n}$ be the* restriction *of $A_{v_0}, \ldots, A_{v_n}$ to $(\mu(v_i), S_i)$. Then, for each $1 \leq l \leq n$, there is $(\mu(v_l), S_l)$ in A'_{v_l} such that for every variable u above v_l, $\mu(u)$ is in S_l.*

Proof. If v_l is not an intersection node of v_i, then A'_{v_l} is equal to A_{v_l}. In this case, the claim follows from Proposition 3.30.

If v_l is an intersection node of v_i, then $\mu(v_l)$ satisfies S_i. Thus, a simple induction shows that each maximal abstract mappings $(\mu(v_l)S_l)$ of A_{v_l} is also in A'_{v_l}. □

Definition 3.49 (Dangling Abstract Sets) *Suppose that (o_i, S_i) is an abstract mapping of an intersection node v_i. We say that (o_i, S_i) is* dangling *if in the restriction of $A_{v_0}, \ldots, A_{v_n}$ to (o_i, S_i) and v_i, one (or more) of the sets is empty.*

In order to avoid dangling assignments, the extensions of assignments in the algorithm should be computed w.r.t. abstract mappings that are not dangling, as will be explained when we will describe the algorithm.

Consider a dag query Q and a tree database D. Let $v_0, \ldots, v_n$ be the variables of Q ordered topologically. Let v_i be an intersection node and $v_{i_1}, \ldots, v_{i_m}$ be the middlenodes of v_i. Let $A_{v_0}, \ldots, A_{v_n}$ be the sets of maximal abstract mappings of $v_0, \ldots, v_n$, respectively. Finally, let (o_i, S_i) be an abstract mapping in A_{v_i} and let $A'_{v_0}, \ldots, A'_{v_n}$ be the restriction of $A_{v_0}, \ldots, A_{v_n}$ to (o_i, S_i) and v_i.

Definition 3.50 (Compliance) *Suppose that μ is a mapping of $v_{i_1}, \ldots, v_{i_l}$ ($l \leq m$) to D. We say that $\mu(v_{i_1}), \ldots, \mu(v_{i_l})$* comply *with (o_i, S_i) and v_i if there are $(\mu(v_{i_1}), S_1), \ldots, (\mu(v_{i_l}), S_l)$ in $A'_{v_0}, \ldots, A'_{v_n}$, respectively.*

Now, we show how to modify algorithm *SFTT* for dealing with dag queries. We call the modified algorithm *SFDT*. (Notice that the difference in the names of the algorithms is in the third letter, where D replaces T.)

Consider a plain dag query. Let $v_0, \dots, v_n$ be the variables of Q. Let D be a tree database. *SFDT* performs the following steps.

1. As a preliminary step, for each intersection node v_i, it is verified that v_i does not have two incoming edges with different labels. In addition, the origin of v_i, denoted v_o, is discovered and the set M_{v_i} of the middlenodes of v_i is computed.

2. A topological order $v_0, \dots, v_n$ of the nodes of Q is computed.

3. The sets $A_{v_0}, \dots, A_{v_n}$ are computed using *MAMA*.

4. Consider an intersection node v_i and an abstract mapping (o_i, S_i) of v_i, in A_{v_i}. The restriction of $A_{v_0}, \dots, A_{v_n}$ to (o_i, S_i) and v_i is computed. The computation is similar to the computation of the sets $A_{v_0}, \dots, A_{v_n}$, except that for the middlenodes of v_i, abstract mappings are computed merely w.r.t. nodes that are in S_i. If, at the end of the computation, (o_i, S_i) is dangling then it is discarded from A_{v_i}.

5. The sets $\mathcal{M}^0, \dots, \mathcal{M}^n$ are computed as in *SFTT* with the following difference. Consider an assignment μ_{j-1} in $\mathcal{M}^{j-1}$ and an abstract mapping (o, S) in A_{v_j}.

   (a) Suppose that v_j is a middlenode of an intersection node v_i. In addition, let $v_{i_1}, \dots, v_{i_l}$ be the middlenodes of v_i that appear before v_j in a topological order of Step 2. Then, μ_{j-1} is extended by mapping v_j to o only if μ_{j-1} complies with the matching property that (o, S) defines and, in addition, $\mu(v_{i_1}), \dots, \mu(v_{i_l})$ and $\mu(v_j)$ comply with (o, S) and v_i.

   (b) If v_j is not a middlenode of any node, then the extension is computed as in *SFTT*.

Proposition 3.51 (Correctness) *Suppose that Q is a plain tree query and D is a tree database. Algorithm SFDT returns the set of semiflexible matchings of Q w.r.t. D.*

Proof. First, we claim that all the matchings returned by the algorithm are semiflexible matchings. This is proved in a similar way to the proof of Proposition 3.42. Essentially, the proof shows that in each extension step the assignments comply with the SF-Property that the abstract matchings define. Thus, the created assignments are semiflexible matchings.

Secondly, we claim that all the semiflexible matchings of Q w.r.t. D are computed by the algorithm. The difference between *SFDT* and *SFTT* is in adding a condition of compliance with the abstract matchings of the intersection nodes. Proposition 3.48 shows that this condition does not discard semiflexible matchings. The rest of the proof is similar to the proof of Proposition 3.42. □

Before we can show that *SFDT* has polynomial input-output time complexity, we need to show that there are no dangling matchings, and thus, each intermediate set $\mathcal{M}^i$ is not larger than the output of the algorithm.

Proposition 3.52 (Monotonic Increase) *In Algorithm SFDT,* $|\mathcal{M}^i| \leq |\mathcal{M}^{i+1}|$, *for each* $0 \leq i \leq n$.

Proof. We show that the algorithm does not produce dangling matchings. Consider a matching μ_i in $\mathcal{M}^i$. We need to show that there is a matching μ_{i+1} in $\mathcal{M}^{i+1}$ such that $\mu_i(v_h) = \mu_{i+1}(v_h)$, for each $1 \leq h \leq n$.

If v_{i+1} is not an intersection node then the proof is similar to the proof of 3.43, with the next addition. The condition of compliance with the abstract matchings of the intersection nodes does not prevent extending μ_i with an assignment of v_{i+1} to $\mu_{i+1}(v_{i+1})$ due to Proposition 3.48.

Suppose that v_{i+1} is an intersection node. Then μ_i assigns the middlenodes of v_{i+1} to objects that comply with some abstract mapping (o, S) of $A_{v_{i+1}}$. In this case, μ_i can be extended by assigning o to v_{i+1}. □

Finally, we show that the runtime of *SFDT* is polynomial in the size of the input and the output.

Theorem 3.53 (Complexity) *Let Q be a plain dag query and let D be a tree database. Algorithm SFDT computes the set of semiflexible matchings* $\mathcal{M}_{sf}(Q, D)$

in polynomial time, in the size of the input and the output.

Proof. Let q, d and m be the sizes of the query, the database and the result, respectively.

Computing the sets $A_{v_0}, \ldots, A_{v_n}$ is in $O(q^2d^2)$, according to Proposition 3.40. Computing the restrictions of $A_{v_0}, \ldots, A_{v_n}$ to abstract mappings of intersection nodes is in $O(q^3d^3)$. This is because there are at most q intersection nodes. There are at most d abstract mappings in A_{v_i}, for each intersection node v_i. Thus, there are $O(qd)$ computations of restriction and each computation is in $O(q^2d^2)$, i.e., has complexity that is similar to the complexity of computing $A_{v_0}, \ldots, A_{v_n}$.

There are $O(q)$ extension steps in the algorithm. In each extension step, at most m matchings are extended, according to Proposition 3.52. Each extension step requires checking two things. First, compliance with the matching property. This is tested in $O(q)$, since there are at most q variables above each variable of the query. The second thing to check is compliance with an abstract mapping of an intersection node. This is tested in $O(qd)$ because there are d possible abstract mapping to comply with and there are at most q middlenodes for each intersection variable.

All together, *SFDT* has $O(q^3d^3 + dm(q + d))$ time complexity. Therefore, computing the semiflexible matchings of Q w.r.t. D has polynomial time complexity, in the size of the input and the output. □

3.2.3 Cyclic Databases and Cyclic Queries

For the case of a cyclic database, deciding if there exists a semiflexible matching of a given query w.r.t. the database is NP-Complete. NP-Hardness follows from Theorem 3.23. For acyclic queries, Membership in NP follows from Theorem 3.19. For cyclic queries, since the result of evaluating a cyclic query over a dag database is always empty, NP-Hardness does not follow from Theorem 3.23. However, a reduction similar to that of Theorem 3.23 shows that nonemptiness of a cyclic query over a cyclic database is NP-Hard.

The complexity results for the various cases are shown in Table 3.2. Note that the

Database / Query	path query	tree query	dag query	cyclic query
path database	PTIME	PTIME	PTIME	the result is always empty
tree database	PTIME	PTIME	PTIME	the result is always empty
dag database	NP-Complete	NP-Complete	NP-Complete	the result is always empty
cyclic database	NP-Complete	NP-Complete	NP-Complete	NP-Complete

Table 3.2: The complexity of checking nonemptiness under the semiflexible semantics.

cases for which it is said that testing emptiness is in polynomial time (in the size of the query and the data), their query-evaluation has a polynomial-time input-output complexity. When testing emptiness is not polynomial, obviously, query-evaluation cannot have a polynomial-time input-output complexity either.

3.3 Evaluating Flexible Queries

In this section, we discuss query evaluation under the flexible semantics. First, we will show that query evaluation under the flexible semantics can be reduced to query evaluation under the rigid semantics.

Definition 3.54 (Reachability Graph) *Let $D = (O, E, r_D, \alpha)$ be a database over the set of nodes O. The* reachability graph *of D is a rooted labeled directed graph, denoted RG(D) $= (O, E_R, r_D, \alpha)$, that is obtained from D by adding edges as follows. If object o' of D has an incoming edge labeled with l and there is either a path from o' to another object o or vice-versa, then E_R has an edge labeled with l from o to o'.*

If a database D contains n nodes and the ingoing degree of D is k (the ingoing degree of D is the maximal number of edges that enter a node in D) then *RG(D)* has at most $n^2 \cdot 2k$ edges. The reason for this is that between each pair of nodes in *RG(D)* there can be at most $2k$ edges with different labels. Hence, the size of *RG(D)* is polynomial in the size of D. Note that the reachability graph can be stored in a data structure whose size is $n^2 + n \cdot k$. In the data structure, for each pair of nodes there should be a bit that indicates whether there is a path between these nodes. In addition, each node should have a list with the labels that enter it.

Theorem 3.55 (Reduction to the Rigid Semantics) *Consider a query Q and a database D. Let RG(D) be the reachability graph of D. The following two sets are equal.*

- *The set of flexible matchings of Q w.r.t. D.*
- *The set of rigid matchings of Q w.r.t. RG(D).*

Proof. First, we show that every flexible matching of Q w.r.t. D is also a rigid matching of Q w.r.t. *RG(D)*. According to Definition 3.5, if μ is a flexible matching of Q w.r.t. D, then it satisfies the rc, all the wec's and all the qec's of Q. We need to show that μ satisfies all the ec's of Q w.r.t. *RG(D)*.

Let ulv be an ec of Q. The matching μ satisfies the wec lv w.r.t. D. Thus, the object $\mu(v)$ has an incoming label l in D. In addition, μ satisfies the qec ulv w.r.t. D. So, either there is a path in D from $\mu(v)$ to $\mu(u)$ or vice-versa. According to Definition 3.54, *RG(D)* has an edge $\mu(u)l\mu(v)$, and hence the ec ulv is satisfied w.r.t. *RG(D)*. From the above, it follows that μ is a rigid matching of Q w.r.t. *RG(D)*. This is because, w.r.t. *RG(D)*, μ satisfies the rc and all the ec's of Q.

For the other direction, we show that every rigid matching of Q w.r.t. *RG(D)* is also a flexible matching of Q w.r.t. D. Let μ be a rigid matching of Q w.r.t. *RG(D)*. The matching μ satisfies the rc and all the ec's of Q w.r.t. *RG(D)*. We need to show that μ satisfies all the wec's and all the qec's of Q w.r.t. D.

Let lv be a wec in Q. The wec lv is part of an ec ulv for some node u. The matching μ satisfies the ec ulv w.r.t. *RG(D)*. Thus, there is an edge $\mu(u)l\mu(v)$ in *RG(D)*. It follows that $\mu(v)$ has an incoming edge labeled with l. So, μ satisfies the wec lv w.r.t. D. This shows that μ satisfies all the wec's of Q.

We now show that, w.r.t. D, μ satisfies all the qec's of Q. Let ulv be a qec of Q. The matching μ satisfies the ec ulv of Q w.r.t. *RG(D)*. So, *RG(D)* contains the edge $\mu(u)l\mu(v)$. This means that either there is a path in D from $\mu(u)$ to $\mu(v)$ or vice-versa. Thus, μ satisfies the qec ulv w.r.t. D. This shows that μ satisfies all the qec's of Q. Since, w.r.t. D, μ satisfies the rc, all wec's and all the qec's of Q, it is a flexible matching of Q w.r.t. D. □

Evaluating all the rigid matching of Q w.r.t. *RG(D)* can be done as follows. For each edge $e = ulv$ of Q, we create a binary *edge relation* r_e that has the attributes u and v, and the following set of tuples: $\{(o, o') \mid$ *RG(D)* has an edge $olo'\}$. In other words, r_e contains all edges of *RG(D)* that have the same label as e. The join of all the edge relations yields all the rigid matchings of *RG(D)*. When the join is acyclic, Yannakakis's algorithm [65] can be applied and, hence, we get the following corollary.

Corollary 3.56 (Tree Evaluation is in Polynomial Time) *When the query is a tree and the database is any graph, query evaluation under the flexible semantics has a polynomial-time input-output complexity.*

Let $\mathcal{M}_f(Q, D)$ denote the set of flexible matchings of a query Q w.r.t. a database D. The next lemma shows that if Q is not a tree then it is NP-hard to decide whether $\mathcal{M}_f(Q, D)$ is not empty. We use a reduction of 3SAT.

Lemma 3.57 (Reduction of 3SAT) *For a given 3CNF formula φ over a set of propositional letters P, one can construct in polynomial time a dag query Q and a database D such that the following are equivalent.*

1. *There is a flexible matching of Q w.r.t. D;*

2. *There is an assignment for the propositional letters in P that satisfies φ.*

The proof of Lemma 3.57 is given in Appendix C.

The next theorem shows that if Q is not a tree, then a query-evaluation algorithm with a polynomial-time input-output complexity is not likely to exist.

Theorem 3.58 (DAG Evaluation is NP-Complete) *Given a dag query Q and a database D, deciding whether the flexible semantics yields a nonempty result is* NP*-complete.*

Proof. NP-hardness follows from Lemma 3.57. Membership in NP follows because a flexible matching can be guessed and verified in polynomial time by checking the satisfaction of the constraints with respect to the reachability graph of D. Recall that the size of the reachability graph of D is polynomial in the size of D. □

3.4 Query Containment and Query Equivalence

In the case of relational conjunctive queries, the final step of evaluation is a projection of the matchings onto the distinguished variables. Consequently, containment is defined in terms of those projections. In the case of queries over semistructured data, the final step is a construction of the result from the matchings. A discussion of this step, however, is beyond the scope of this work. Thus, in this work, the semantics of queries is defined in terms of matchings over all the variables, and containment (equivalence) is defined as containment (equality) of the corresponding sets of matchings.

Definition 3.59 (Query Containment, Query Equivalence) *Let Q_1 and Q_2 be two queries over the same set of node variables and with the same root. We say that Q_1 is* contained *in Q_2 under the semantics s, denoted $Q_1 \subseteq_s Q_2$, if for all database D, $\mathcal{M}_s(Q_1, D) \subseteq \mathcal{M}_s(Q_2, D)$. The queries Q_1 and Q_2 are* equivalent *if for all database D, $\mathcal{M}_s(Q_1, D) = \mathcal{M}_s(Q_2, D)$.*

Deciding equivalence and containment of queries is useful for optimization techniques. The next theorem provides a characterization of containment for semiflexible queries.

Theorem 3.60 (Semiflexible Containment) *Let Q_1 and Q_2 be queries over the same set of variables V. $Q_1 \subseteq_{sf} Q_2$ if and only if the identity mapping ν over V is a semiflexible matching of Q_2 w.r.t. Q_1.*

Proof. Consider the case where the identity mapping ν is a semiflexible matching of Q_2 w.r.t. Q_1. Let D be a database, and let $\mu \in \mathcal{M}_{sf}(Q_1, D)$ be a semiflexible matching of Q_1 w.r.t. D. We want to show that μ is a semiflexible matching of Q_2 w.r.t. D.

Obviously, μ maps the root of the Q_2 to the root of D, since the root of Q_2 and the root of Q_1 are the same node.

Let $\pi_2 = v_0 l_1 v_1 \cdots l_n v_n$ be a path in Q_2. Since π_2 satisfies the SF-Condition w.r.t. ν, there is a permutation σ of $0, 1, \ldots, n$ such that $\sigma(0) = 0$ and Q_1 has a path π_1 of the form $\nu(v_{\sigma(0)})*l_{\sigma(1)}\nu(v_{\sigma(1)})\cdots*l_{\sigma(n)}\nu(v_{\sigma(n)})$. But ν is the identity mapping so the form of π_1 is actually $v_{\sigma(0)}*l_{\sigma(1)}v_{\sigma(1)}\cdots*l_{\sigma(n)}v_{\sigma(n)}$. The path π_1 satisfies the SF-Condition w.r.t. μ, because μ is a semiflexible matching of Q_1 w.r.t. D. So, there is a permutation σ' such that D includes a path with the form $\mu(v_{\sigma'(0)})*l_{\sigma'(1)}\mu(v_{\sigma'(1)})\cdots*l_{\sigma'(n)}\mu(v_{\sigma'(n)})$. This path shows that π_2 satisfies the SF-Condition w.r.t. μ.

Given a strongly connected component C in Q_2, C is also a strongly connected component in Q_1 since ν is a semiflexible matching. Furthermore, the image of C is a strongly connected component in D since μ maps strongly connected components of Q_1 to strongly connected components of D. We conclude that μ is a semiflexible matching of Q_2 w.r.t. D.

For the other direction, assume that $Q_1 \subseteq_{sf} Q_2$. We create a database from Q_1 by adding to Q_1 some arbitrary function α that maps the atomic nodes of Q_1 to values (see Definition 2.1 and Definition 2.3). The identity mapping ν over V is a semiflexible matching of Q_1 w.r.t. Q_1. Thus, ν is also a semiflexible mapping of Q_2 w.r.t. Q_1. □

Corollary 3.61 (Polynomial Time Complexity) *Deciding if $Q_1 \subseteq_{sf} Q_2$ is in polynomial time.*

Proof. According to Theorem 3.19, the time complexity for verifying that a matching is a semiflexible matching is polynomial. □

A characterization of containment for flexible queries is given in the next theorem.

Theorem 3.62 (Flexible Containment) *Let Q_1 and Q_2 be queries over the same set of variables. $Q_1 \subseteq_f Q_2$ if and only if the following two conditions hold.*

1. *For each wec lv in Q_2, there is a wec lv in Q_1.*

2. *For each qec ulv in Q_2, where $u \neq r_{Q_2}$ and $v \neq r_{Q_2}$ (r_{Q_2} is the root of Q_2), the query Q_1 contains either the qec $ul'v$ or the qec $vl'u$ for some label l'.*

Proof. Assume that Condition 1 and Condition 2 hold for Q_1 and Q_2. Let D be a database and let μ be a flexible matching of Q_1 w.r.t. D. We show that μ is a flexible matching of Q_2.

The root of Q_1 and the root of Q_2 are the same node. Thus, satisfaction of the rc of Q_1 is the same as satisfaction of the rc of Q_2. Let lv be a wec in Q_2. According to Condition 1, lv is also a wec in Q_1. Thus, μ satisfies lv. Let ulv be a qec of Q_2 such that both u and v are not r_{Q_2}. By Condition 2, Q_1 contains either a qec ulv or a qec vlu. In both cases, from satisfaction of the qec's of Q_1 it follows that there is a path in D from $\mu(v)$ to $\mu(u)$ or vice-versa. This shows satisfaction of the qec ulv of Q_2. For a qec $r_{Q_2}lv$ of Q_2, $\mu(r_{Q_2})$ is the root of D and obviously, there is a path from the root of D to any node of D, including $\mu(v)$. Thus, μ satisfies this qec. For a qec ulr_{Q_2} of Q_2, the object $\mu(r_{Q_2})$ is the root of D and obviously, there is a path from the root of D to any node of D, including $\mu(u)$. Thus, μ satisfies this qec. To conclude, μ in a flexible matching of Q_2 since it satisfies the rc, the qec's and the wec's of Q_2.

For the other direction, we assume that $Q_1 \subseteq_f Q_2$ and we show that the above two conditions hold.

To show that the first condition holds, we create a database from Q_1 by adding to Q_1 some arbitrary function α that maps the atomic nodes of Q_1 to values (see Definition 2.1 and Definition 2.3). Obviously, the identity mapping ν is a flexible matching of Q_1 w.r.t. Q_1. Since $Q_1 \subseteq_f Q_2$, ν is a flexible matching of Q_2 w.r.t. Q_1. Let lv be a wec in Q_2. Since ν satisfies lv w.r.t. Q_1, there must be an edge in Q_1 that enters v and its label is l. This means that there is a wec lv in Q_1 and Condition 1 holds.

To show that the second condition holds, consider a qec ulv in Q_2, where $u \neq r_{Q_2}$ and $v \neq r_{Q_2}$. We create a database D whose nodes are the variables of Q_1 and its root is the root of Q_1. For each label l in Q_1 and every two nodes w_1, w_2, such that $w_i \neq u$ and $w_i \neq v$ (for $i = 1, 2$), D includes an edge w_1lw_2. In addition, for each label l and node w such that $w \neq u$ and $w \neq v$, D consists of the edges wlv and wlu.

The database D has the following three properties. First, there is a path between each pair of nodes, except for the pair u and v. Second, from each node there is a path to u (v) but there is no path from u (v) to any node of D. Third, for each label l and each variable w of Q_1, D has an edge, with the label l, that enters w. Note that in the construction of D we assumed that neither v nor u is the root of Q_1. Thus, it is not necessary to have paths from from either u or v to all the other nodes.

Since D has no path from u to v or vice-versa, there is no flexible matching of Q_2 w.r.t. D. We assumed that $Q_1 \subseteq_f Q_2$, so there is no flexible matching of Q_1 w.r.t. D. According to the construction of D the rc and all the wec's of Q_1 are satisfied by the identity mapping. Furthermore, every qec, except for a qec $ul'v$ or a qec $vl'u$, is satisfied by the identity mapping. Since the identity mapping is not a flexible matching of Q_1 w.r.t. D, Q_1 must include either the qec $ul'v$ or the qec $vl'u$, for some label l'. $\square$

To decide containment of Q_1 in Q_2, under the flexible semantics, we just need to check that every wec (qec) in Q_1 has a suitable wec (qec) in Q_2. Thus, we get the following result.

Corollary 3.63 (Polynomial Time Complexity) *Deciding whether $Q_1 \subseteq_f Q_2$ is in $O(|Q_1| \cdot |Q_2|)$ time.*

If we sort the ec's and wec's of Q_2, deciding containment can be done in $O(|Q_2| \cdot \log |Q_2| + |Q_1| \cdot \log |Q_2|)$.

3.5 Database Equivalence

In this section we introduce the novel notion of database equivalence and we characterize database equivalence under the different semantics.

Definition 3.64 (Database Equivalence) *Given two databases D and D' over the same set of objects O, we say that D and D' are* equivalent *under the semantics s if for every query Q, the set of s-matchings of Q w.r.t. D is equal to the set of s-matchings of Q w.r.t. D'.*

Under the classical (i.e., rigid) semantics, two databases are equivalent if and only if they are isomorphic (i.e., have the same root and the same set of labeled edges). In the case of the semiflexible and flexible semantics, however, two databases can be equivalent even if they are not isomorphic.

One reason for investigating database equivalence is that in some cases it is more efficient to evaluate queries over databases that have a certain form than over databases that have a different form. For example, we showed that it is more efficient to evaluate queries over a tree database than over a dag database. Thus, it is important to be able to characterize equivalence of databases, and to be able to transform a database of a given form (e.g., dag) to an equivalent database that has a different form (e.g., tree).

Let D and D' be two databases that have the same set of objects and the same root. We say that a path $\phi = o_0 l_1 o_1 l_2 o_2 \cdots l_n o_n$ of D, where o_0 is the root, is *included* in a path ϕ' of D' if there is a permutation σ of $1, \ldots, n$, such that ϕ' has the form $o_0 {*} l_{\sigma(1)} o_{\sigma(1)} {*} l_{\sigma(2)} o_{\sigma(2)} \cdots {*} l_{\sigma(n)} o_{\sigma(n)}$.

We say that there is a *semiflexible path inclusion* of D in D' if for every path ϕ in D that starts at the root, there is a path ϕ' in D', such that ϕ' includes ϕ.

Theorem 3.65 (Semiflexible Equivalence) *Consider two databases D and D' over the same set of objects and with the same root. D and D' are equivalent under the semiflexible semantics if and only if the following conditions hold.*

1. *There is a semiflexible path inclusion of D in D' and vice-versa.*
2. *Each strongly connected component in D is a strongly connected component in D' and vice-versa.*

Proof. Suppose that the conditions of the theorem hold. We need to show that D and D' are equivalent under the semiflexible semantics. Let Q be a query and $\mu \in \mathcal{M}_{sf}(Q, D)$ be a semiflexible matchings of Q w.r.t. D. Since D and D' have the same set of objects, μ is an assignment of Q w.r.t. D'. We show that μ is a semiflexible matching of Q w.r.t. D' by showing that it satisfies the conditions of Definition 3.4.

Since D and D' have the same root, μ maps the root of Q to the root of D', and thus, the rc of Q is satisfied.

Given a path $\pi = v_0 l_1 v_1 \cdots l_n v_n$ in Q, μ satisfies the SF-Condition w.r.t. D and hence there is a path ϕ in D of the form $\mu(v_{\sigma(0)}) {*} l_{\sigma(1)} \mu(v_{\sigma(1)}) \cdots {*} l_{\sigma(n)} \mu(v_{\sigma(n)})$. According to Condition 1, there is a semiflexible path inclusion of D in D' and, thus, D' has a path $\mu(v_{\sigma'(0)}) {*} l_{\sigma'(1)} \mu(v_{\sigma'(1)}) \cdots {*} l_{\sigma'(n)} \mu(v_{\sigma'(n)})$, for some permutation σ' of $1, \ldots, n$. This shows that π satisfies the SF-Condition w.r.t. D'.

If C is a strongly connected component in Q then it is mapped by μ to a strongly connected component of D. This strongly connected component of D is also a strongly connected component of D', according to Condition 2.

We have shown that μ satisfies the conditions of Definition 3.4 and hence it is a semiflexible matching of Q w.r.t. D'. Exchanging D with D' shows that if μ is a semiflexible matching of Q w.r.t. D' then it is also a semiflexible matching of Q w.r.t. D. To conclude, D and D' are equivalent under the semiflexible semantics.

For the other direction, suppose that D and D' are equivalent. Let Q_D be a query that is created from D by considering the objects of D as variables and removing the atomic values. Note that the set of labeled edges of D is equal to the set of labeled edges of Q_D.

The identity mapping μ_D, from the variables of Q_D to the objects of D, is a semiflexible matching of Q_D w.r.t. D. Since D and D' are equivalent, μ_D is also a semiflexible matching of Q_D w.r.t. D'.

Suppose the ϕ is a path in D. Then, ϕ is also a path in Q_D. The matching μ_D satisfies the SF-Condition w.r.t. ϕ and D'. Hence, there is a path ϕ' in D' such that ϕ' is a permutation of ϕ. That is, there is a semiflexible path inclusion of D in D'.

Because μ_D is a semiflexible matching, it satisfies Condition 3 of Definition 3.4. Thus, every strongly connected of Q_D (and hence of D) is a strongly connected component of D'.

By replacing the roles of D and D' we can show, in the same way, that there is a semiflexible path inclusion of D' in D and that every strongly connected component in D' is a strongly connected component of D. This shows that the two conditions of the theorem hold. $\square$

Conditions (1) and (2) of the above theorem are essentially equivalent to the condition that the identity mapping (of the objects of D to the objects of D') is a semiflexible matching. Therefore, we get the following corollary.

Corollary 3.66 (Time Complexity) *Under the semiflexible semantics, deciding database equivalence is in polynomial time.*

For the flexible semantics, we have the following theorem.

Theorem 3.67 (Flexible Equivalence) *Consider two databases D and D' over the same set of objects O and with the same root. The databases D and D' are equivalent under the flexible semantics if and only if their reachability graphs are isomorphic, i.e., have the same set of labeled edges.*

Proof. Consider the case where the reachability graphs of D and D' are isomorphic. Let Q be a query. Because of the isomorphism, $\mathcal{M}_r(Q, \mathit{RG(D)}) = \mathcal{M}_r(Q, \mathit{RG(D')})$. Theorem 3.55 shows that $\mathcal{M}_r(Q, \mathit{RG(D)}) = \mathcal{M}_f(Q, D)$ and $\mathcal{M}_r(Q, \mathit{RG(D')}) = \mathcal{M}_f(Q, D')$. Hence, it follows that $\mathcal{M}_f(Q, D) = \mathcal{M}_f(Q, D')$. That is, D and D' are equivalent.

For the other direction, consider the case where D and D' are equivalent. We start by showing that the set of labeled edges of *RG(D)* is contained in the set of label edges of *RG(D')*.

Let $o_1 l o_2$ be an edge of *RG(D)*. According to the definition of the reachability graph (Definition 3.54), in D there is an edge that enters o_2 and has the label l. In addition, D either has a path from o_1 to o_2 or vice-versa. There are two cases to examine, depending on whether o_1 is the root of D or not.

If o_1 is the root of D, then we create a query Q_1 as follows. The variables of Q_1 are r_{Q_1} and v_2. There is a single edge in Q_1—an edge from r_{Q_1} to v_2 with label l. The matching μ_1 that maps r_{Q_1} to o_1 and v_2 to o_2 is a flexible matching of Q_1 w.r.t. D. Since D' is equivalent to D, μ_1 is also a flexible matching of Q_1 w.r.t. D'. This means that μ_1 satisfies the qec lv_2. Thus, in D' there is an edge with label l that enters o_2. In addition, since o_1 is the root of D it is also the root of D'. Thus, there is a path in D' from o_1 to o_2, because all the nodes are reachable from the root. Consequently, *RG(D')* has an edge o_1lo_2.

If o_1 is not the root of D, then we create a query Q_2 as follows. There are three variables in Q_2: r_{Q_2}, v_1 and v_2. The edges of Q_2 are the two edge $r_{Q_2}l_1v_1$ and v_1lv_2, where l_1 is a label on an edge that enters v_1. Note that such a label exists because there is a path in D from the root to o_1. Let μ_2 be a matching that maps r_{Q_2} to the database root, v_1 to o_1 and v_2 to o_2. Obviously, μ_2 is a flexible matching of Q_2 w.r.t. D. Since D' is equivalent to D, μ_2 is also a flexible matching of Q_2 w.r.t. D'. It means that μ_2 satisfies the wec v_1lv_2 and the qec lv_2 w.r.t. D'. That is, in D' there is either a path from o_1 to o_2 or vice-versa. In addition, there is an edge with the label l that enters o_2. Hence, *RG(D')* has an edge o_1lo_2.

Thus far, we have shown that the set of edges of *RG(D)* is contained in the set of edges of *RG(D')*. By exchanging the roles of D and D', we can show that the set of edges of *RG(D')* is contained in the set of edges of *RG(D)*. This shows that *RG(D)* and *RG(D')* are isomorphic. □

It is not necessary to solve the general case of graph isomorphism in order to decide equivalence of databases under the flexible semantics. Instead, it is sufficient to check that the identity mapping is an isomorphism. Thus, deciding whether two databases D and D' are equivalent under the flexible semantics is in $O(|D|^2 \log |D| + |D'|^2 \log |D'|)$.

3.5.1 Removing Redundancies

One application of database equivalence is the removal of *redundant parts* from databases. A part of a database is redundant if removing it from the database has no effect on the result of query evaluation.

For example, given a database D, we say that an edge o_1lo_2 in D is *redundant* w.r.t. the semiflexible (or flexible) semantics if D has a path of the form $o_1{*}lo_2$ that does not include the edge o_1lo_2. If a redundant edge is removed, the result is a database that is equivalent to the original one under the semiflexible (flexible) semantics.

Proposition 3.68 (Redundant Edges) *Suppose that D' is a database that is created from another database D by removing from D a redundant edge. Then D and D' are equivalent under both the semiflexible and the flexible semantics.*

Proof. Equivalence under the semiflexible semantics follows from Theorem 3.65. This is because a removal of a redundant edge has no effect on semiflexible path inclusion nor on the strongly connected components.

Equivalence under the flexible semantics follows from Theorem 3.67. This is because a removal of a redundant edge cause no change in the reachability graph of the database. □

The next lemma shows that under the semiflexible semantics, when there are no redundant edges, the inclusion condition for database equivalence can be relaxed. That is, if two databases are equivalent then each path, from the root to a leaf, in one database has a path, in the other database, that contains exactly the same set of objects and the same set of labels.

Lemma 3.69 (Image Path Permutation) *Let D and D' be two databases over the same set of objects, with the same root and without any redundant edges. If D and D' are equivalent under the semiflexible semantics then for each path $o_0 l_1 o_1 \cdots l_n o_n$, from the root of D to a leaf, there exists a permutation σ of $1, \ldots n$, such that in D' there is a path $o_0 l_{\sigma(1)} o_{\sigma(1)} \cdots l_{\sigma(n)} o_{\sigma(n)}$.*

Proof. Let $\phi = o_0 l_1 o_1 \cdots l_n o_n$ be a path in D such that in D' no path has the form $o_0 l_{\sigma(1)} o_{\sigma(1)} \cdots l_{\sigma(n)} o_{\sigma(n)}$. We show that this leads to a contradiction.

According to Theorem 3.65, there is a semiflexible path inclusion of D in D' and vice-versa. So, there exists a permutation σ' such that D' has a path $\phi' = o_0 {*} l_{\sigma'(1)} o_{\sigma'(1)} \cdots {*} l_{\sigma'(n)} o_{\sigma'(n)}$. According to our assumption, ϕ' contains a node o' that is not among the nodes $o_0, o_{\sigma'(1)}, \ldots, o_{\sigma'(n)}$. Note that if ϕ' is not a simple path then o' could be one of the nodes $o_1, \ldots, o_n$. Since there is a semiflexible path inclusion of D' in D we have a path $\hat{\phi}$ in D that includes ϕ'. The path $\hat{\phi}$ goes through all the nodes of ϕ and through the node o' (it may, in addition, go through other nodes). Since o_0 is the root and o_n is a leaf, the node o' is neither the first nor the last node in $\hat{\phi}$. Therefore, there must be two nodes o_i and o_{i+1} in D such that in addition to the edge $o_i l_{i+1} o_{i+1}$ there is a path $o_i {*} o' {*} l_{i+1} o_{i+1}$. The edge $o_i l_{i+1} o_{i+1}$ is redundant, in contradiction to the our assumption that D and D' are redundancy free. □

3.5.2 Transforming a Database to an Equivalent Tree

In addition to testing database equivalence, it is also possible to transform a database D to a tree, provided that D is indeed equivalent to some tree database. Note that under the semiflexible semantics, a cyclic database cannot be equivalent to a tree database. However, there are dag databases that are equivalent to tree databases. Under the flexible semantics, even a cyclic database could be equivalent to a tree database.

Transforming a database to an equivalent tree is important for several reasons. First, evaluation of queries is more efficient when the database is a tree, as was shown earlier. Secondly, in a graphical user interface, presenting trees is easier than presenting dags. Thirdly, storing data as a document (e.g., in XML) is easier when the data is a tree, since references and identifiers need not be used.

Definition 3.70 (Tree Transformable Database) *Given a database D, we say that D is* tree transformable *under the semantics s if there exists a tree database T that is equivalent to D under the semantics s.*

In this section, we characterize tree transformable databases under the semiflexible and the flexible semantics. In addition, we give algorithms for the transformations.

Tree Transformable Databases under the Semiflexible Semantics

We discuss transformation to a tree under the semiflexible semantics. Given a database D that has no redundant edges, the following are necessary conditions for D to be tree transformable under the semiflexible semantics.

The first necessary condition is that D does not include a node that has incoming edges with different labels. The reason for this condition is that in a tree database only one edge enters each node (except for the root that has no incoming edge) and this edge holds exactly one label. The second necessary condition is that D is acyclic. The reason for the second condition is that in a finite tree all paths are finite, while a cyclic graph contains an infinite path. The third necessary condition is that the number of root-to-leaf paths in D is not greater than the number of nodes in D. Recall that in a tree, the number of paths, from the root to a leaf, never exceeds the number of nodes.

In order to have conditions that are both necessary and sufficient, we need additional definitions. A *path-nodes set* of a database D is a set of all the nodes on a path in D from the root to a leaf. For a given database, the *path hypergraph* is a hypergraph whose hyperedges are path-nodes sets.

Definition 3.71 (Path Hypergraph) *Given a database D, the* path hypergraph *of D is the hypergraph $\mathcal{H}$ such that:*

1. *The nodes of $\mathcal{H}$ are the objects of D.*
2. *The hyperedges of $\mathcal{H}$ are the path-nodes sets of D.*

Definition 3.72 (Bachman Diagram) *Let $\mathcal{E}$ be the set that contains all the hyperedges of $\mathcal{H}$ and all the nonempty intersections of two or more hyperedges of $\mathcal{H}$. The* Bachman diagram *of $\mathcal{H}$, denoted $BD(\mathcal{H})$, is the following graph.*

1. *For each element of $\mathcal{E}$, there is a node in $BD(\mathcal{H})$.*
2. *There is an edge between E_1 and E_2 in $BD(\mathcal{H})$ if*
   (a) *$E_1 \subseteq E_2$, and*
   (b) *there is no element E' in $\mathcal{E}$, such that $E_1 \subseteq E' \subseteq E_2$ and E' is different from both E_1 and E_2.*

We say that $BD(\mathcal{H})$ is acyclic if it contains no cycle (as an undirected graph). In Appendix A we provide a definition of a γ-acyclic hypergraph. A discussion of acyclic Bachman diagrams and their usage in the characterization of γ-acyclic hypergraphs is given in [31]. In [55], Bachman diagrams are used to characterize full disjunctions.

Lemma 3.73 (Acyclicity is a Necessary Condition) *Let D be a database with no redundant edges and $\mathcal{H}$ be the path hypergraph of D. If D is tree transformable under the semiflexible semantics then $BD(\mathcal{H})$, the Bachman diagram of $\mathcal{H}$, is acyclic.*

Proof. To prove the lemma, we assume that the Bachman diagram of $\mathcal{H}$ is cyclic and we show that D is not tree transformable.

In [31], Fagin showed that for a hypergraph $\mathcal{H}$, the Bachman diagram of $\mathcal{H}$ is acyclic if and only if $\mathcal{H}$ is γ-acyclic. We have assumed that the Bachman diagram of $\mathcal{H}$ is cyclic and, hence, $\mathcal{H}$ itself is not γ-acyclic. This means that $\mathcal{H}$ either has a pure cycle or a γ-3-cycle.

All the hyperedges of $\mathcal{H}$ contain the root of D. Thus, there cannot be a pure cycle in $\mathcal{H}$. So, $\mathcal{H}$ must have a γ-3-cycle. That is, $\mathcal{H}$ includes three hyperedges X, Y, Z and three nodes n_0, n_1, n_2 such that:

1. n_0 is an element of all the three sets X, Y and Z;
2. n_1 is an element of X and Z and not an element of Y; and
3. n_2 is an element of Y and Z and not an element of X.

If n_0 is not the root of D, we replace it with the root. This can be done since the root of D is an element of every hyperedge.

To derive a contradiction we assume that D is tree transformable, i.e., there is a tree database T such that T is equivalent to D under the semiflexible semantics. One of the following three options must hold in T:

1. No path in T goes trough both n_1 and n_2.
2. There is a path in T from n_1 to n_2.
3. There is a path in T from n_2 to n_1.

In Case 1, the path in D whose nodes are the nodes of Z is not included in any path of T. This is because Z contains both n_1 and n_2. In Case 2, the path ϕ_Y, in D, whose nodes are the nodes of Y includes n_0 and n_2 and does not include n_1. According to Lemma 3.69, T should have a path that includes the same nodes as

ϕ_Y. However, such a path cannot exist, since T is a tree and every path in T that includes n_2 also include n_1. In Case 3, each path in T that includes n_1 must include n_2. Thus, there is no path in T that includes exactly the nodes of X, i.e., includes n_0 and n_1 and does not include n_2. Since all three cases lead to a contradiction, an equivalent tree database of D does not exist. $\square$

Next we explain why the test of Lemma 3.73 can be performed in polynomial time in the size of the given database.

Consider a hypergraph $\mathcal{H}$ with n hyperedges. It is possible to check in polynomial time in the size of $\mathcal{H}$ that the Bachman diagram of $\mathcal{H}$ is acyclic. This is based on the following two observations. First, in an acyclic diagram, the number of paths from the least node (the node that is contained in all the other nodes) to the leaves (i.e., the hyperedges of $\mathcal{H}$) is at most n. Second, there are at most n sets in each path from the root to a leaf. To see why the second observation is true, consider two nodes in the diagram E_1 and E_2 that have an edge between them, where $E_1 \subseteq E_2$. Let S_1 (S_2) be the set of hyperedges that E_1 (E_2) is their intersection. Then $S_1 \supset S_2$.

Thus, there are at most n^2 nodes in the Bachman diagram of $\mathcal{H}$. It is possible to construct the diagram by creating all the intersection of hyperedges of $\mathcal{H}$. If at some point there are more than n^2 nodes then the test stops with the answer that the diagram is cyclic. After constructing the diagram, it is possible to check in linear time, in the size of the diagram, whether there are cycles. This can be done using a DFS traversal.

The test of Lemma 3.73 can be performed in polynomial time in the size of the given database. This is based on the observation that in a tree database the number of paths cannot exceed the number of nodes. In the test, we construct all the hyperedges of the path hypergraph. If at some point, during the construction, the number of hyperedges exceeds the number of database nodes, then the test stops with the answer that the database is not tree transformable. Otherwise, the Bachman diagram is constructed and tested for acyclicity.

Thus far, we discussed necessary conditions for the existence of an equivalent tree database. Next, we show that the conditions that were presented as necessary conditions are also sufficient conditions for performing a transformation. In addition, we present a transformation algorithm.

We say that a database D is a *tree-transformable candidate* if (1) D has no redundant edges; (2) D is acyclic; (3) there is no node in D that has two different incoming labels; and (4) the Bachman diagram of the path hypergraph of D is acyclic. The algorithm that we present next can transform a tree-transformable candidate to an equivalent tree. Thus, it shows that tree-transformable candidates are actually tree transformable.

Figure 3.4 presents a polynomial-time algorithm that transforms a given database D to a tree, provided that D is tree transformable. The algorithm starts by creating the path hypergraph $\mathcal{H}$ of D. Note that in a tree transformable database, the number of paths from the root to the leaves cannot exceeds the number of nodes, and thus, the size of the path hypergraph is polynomial in the size of the database.

The second step of the algorithm is the creation of the Bachman diagram $\mathcal{B} = BD(\mathcal{H})$. This can be done in polynomial time in the size of $\mathcal{H}$. Recall that the nodes of $\mathcal{B}$ are sets of objects of the database D. The intersection of all the nodes of $\mathcal{B}$ always contains the root of D. Thus, there is a *least node* of $\mathcal{B}$ that is contained in all the other nodes. In the algorithm, we view $\mathcal{B}$ as a tree whose root is the least node.

The algorithm creates a tree database T by visiting the nodes of $\mathcal{B}$ in a topological order, starting with the least node. Initially, T is empty. For each visited node E of $\mathcal{B}$, the algorithm adds to T the objects of $E - E'$, where E' is the parent of E. The newly added objects are connected by new edges to form a simple path. The order of the objects along this path is not important, except in the case of the least node of $\mathcal{B}$, since the root of D must be the first object on the path created for the least node. Thus, each node E of $\mathcal{B}$ is associated with some path in T, and this path has a first object and a last object. A new edge is also added from the last object of the parent of E to the first object of $E - E'$.

The final step of the algorithm is to label the edges of T. Each edge is labeled with the unique label associated with the object that it enters; that is, if an edge

```
Algorithm TransformingDatabaseToTree($D$);
Input a tree-transformable database $D$;
Output a tree database $T$ that is equivalent to $D$
        under the semiflexible semantics;

let $\mathcal{H}$ be the path hypergraph of $D$;
let $\mathcal{B} = BD(\mathcal{H})$ be the Bachman diagram of $\mathcal{H}$;
$\mathcal{S} \leftarrow \emptyset$;
let $S_0$ be the least node of $\mathcal{B}$ ($S_0$ is the intersection of all nodes of $\mathcal{B}$);
(* by choosing $S_0$ to be the root, $\mathcal{B}$ can be viewed as a tree *)
create from the objects of $S_0$ a path $P_0$ with $r_D$ (the root of $D$) as the first node;
add to $\mathcal{S}$ all the children of $\mathcal{S}_0$ in $\mathcal{B}$;
while $\mathcal{S}$ is not empty
    remove some node $S_j$ from $\mathcal{S}$;
    add to $\mathcal{S}$ all the children of $\mathcal{S}_j$ in $\mathcal{B}$;
    let $S_i$ be the parent of $S_j$ in $\mathcal{B}$;
    create a simple path $P_j$ from the objects of $S_j$ that are not in $S_i$;
    (* Since $S_i$ is the parent of $S_j$, a path $P_i$ has already been created for $S_i$ *)
    add an edge from the last node of $P_i$ to the first node of $P_j$;
let $T$ be the database that was produced in the previous steps;
label $T$ by attaching to each edge the label that corresponds to its target in $D$;
return $T$;
```

Figure 3.4: Creating a tree database T from a database D.

enters an object o, then in the original database D, all the edges that enter o are labeled with the same label l, and l is also the label of the single edge that enters object o in the tree database T. Note that the root has no incoming edges in T. It is easy to see that the returned graph is indeed a tree.

Lemma 3.74 (Correctness of the Transformation Algorithm) *If a database*

D is a tree-transformable candidate, then applying the algorithm TransformingDatabaseToTree *to D produces a tree that is equivalent to D under the semiflexible semantics. Furthermore, the runtime of the transformation is polynomial in the size of D.*

Proof. Let T be the graph that the algorithm returns. We claim that T contains exactly the same set of objects as D. Note that the creation of T is done by traversing the Bachman diagram $\mathcal{B}$ and visiting all the nodes in $\mathcal{B}$. Since $\mathcal{B}$ is connected and has no cycles, each node is visited exactly once. In the traversal, an object is added to T if it is an element of a node S_j and not an element of the parent of S_j. It is easy to see that all the objects of T are objects of D. This is because the nodes of $\mathcal{B}$ are sets of objects of D. Secondly, every node of D appears in at least one node of $\mathcal{B}$ and thus appears in T. Thirdly, none of the objects is added twice to T. To see this, consider the case where an object o is added to T twice. Let S_j and S_k be the nodes of $\mathcal{B}$ for which o was added to T. Then the intersection of S_j and S_k is non-empty and contains o. This means that in $\mathcal{B}$ there is an ancestor S_i of S_j and S_k and S_i contains o. Hence, the parent of S_j contains o in contradiction to S_j being a node for which o was added to T.

The graph T is a tree because the construction of T is in such a way that the root has no incoming edge and every object, other than the root, has exactly one incoming edge. In addition, due to the way edges receive their labels, an object in T has an incoming edge with the label l if and only if it has an incoming edge in D with label l.

Next, we show that D and T are equivalent under the semiflexible semantics. The equivalence is proved by showing that there is a semiflexible path inclusion in both directions, according to Theorem 3.65.

We start by showing a semiflexible path inclusion of D in T. Consider a path $\phi = o_0 l_1 o_1 \cdots l_n o_n$ in D. We assume that ϕ is a path from the root to a leaf. This assumption can be done because every path is contained in a path from the root to a leaf, so inclusion can be shown w.r.t. the container. The path ϕ is represented by an hyperedge in $\mathcal{H}$ and thus $\mathcal{B}$ has a node S that contains all the objects on

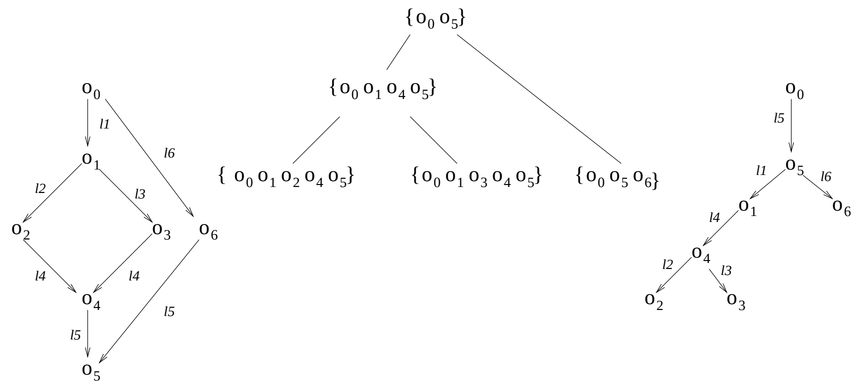
ϕ. Let $S_0, S_1, \ldots, S_k$ be a path in $\mathcal{B}$, such that S_0 is the least node and S_k is S. The algorithm creates a path from the objects of each node S_i and connects the last node on each path to the first node of the next path. The result is a path ϕ' in T that contains precisely the objects of S_k. All the objects of ϕ are included in ϕ'. In addition, according to the way labels are given, for each $1 \leq i \leq n$, the label on the edge that enters o_i, in ϕ', is l_i. Hence, there is a semiflexible path inclusion of ϕ in ϕ'.

To show inclusion in the opposite direction, consider a path $\phi' = o_0 l_1 o_1 \cdots l_n o_n$ in T, from the root to a leaf. Let S_i be the node in $\mathcal{B}$ that contains o_n. Obviously, S_i is a leaf in $\mathcal{B}$ and the set of objects S_i is equal to the set of objects on ϕ'. Since S_i has no children it is not an intersection of two other nodes in $\mathcal{B}$. Thus, S_i is an hyperedge of $\mathcal{H}$. This means that there is a path ϕ in D such that ϕ is a path from the root to a leaf and the set of nodes on ϕ is equal to S_i. The two paths ϕ and ϕ' consists of the same objects. Furthermore, if l is the label on the edge that enter o in ϕ' then l is also the label on the edge that enters o in ϕ. To conclude, there is a semiflexible path inclusion of ϕ' in ϕ.

In the algorithm, each node of $\mathcal{B}$ is traversed exactly once. For each node, a single pass over the objects of the node is needed since the algorithm does not require to connect the objects in any particular order. Thus, the construction of T is linear in the size of $\mathcal{B}$. The number of leaves in $\mathcal{B}$ is at most the number of paths, from the root to a leaf, in D. If D is tree transformable, it cannot have more paths, from the root to a leaf, than nodes. Hence, the size of $\mathcal{B}$ is linear in the size of D. Finally, since the construction of B is polynomial in the size of D, the algorithm has a polynomial runtime in the size of D. □

We conclude the discussion on tree transformable databases, under the semiflexible semantics, with the following theorem.

Theorem 3.75 (Semiflexible Equivalence to a Tree) *Let D be a database with no redundant edges and with the path hypergraph $\mathcal{H}$. The database D is tree transformable under the semiflexible semantics if and only if the following three conditions hold.*



(1) A given database D. (2) The Bacman diagram of D. (3) The equivalent tree database.

Figure 3.5: Transformation to a tree under the semiflexible semantics.

1. D is acyclic;

2. There is no node in D that has two different incoming labels; and

3. BD($\mathcal{H}$), the Bachman diagram of $\mathcal{H}$, is acyclic.

Deciding if D is tree transformable under the semiflexible semantics is in polynomial runtime in the size of D.

Proof. Correctness of the theorem follows from Lemma 3.73 and Lemma 3.74. □

Example 3.76 In Figure 3.5, a tree transformable database D is depicted. In addition, the Bachman diagram of D and an equivalent tree database of D are shown. When applying the transformation algorithm to D, the result is either the shown tree database or a tree database that is equivalent to the one that is shown.

Tree Transformable Databases under the Flexible Semantics

We now consider transformation to a tree under the flexible semantics. A necessary condition for the existence of a transformation under the flexible semantics is as follows. The database should not have a node o, such that two edges with different labels enter o. Moreover, there should not be any edge that enters the root.

Next, we give some additional definitions. The *augmented reachability graph* of a database D is the reachability graph of D augmented with an unlabeled edge from each node to the root. An edge between two nodes, in the augmented reachability graph, reflects the existence of a path in D between these nodes. Since each node in the database is reachable from the root, there is an edge in the augmented reachability graph from each node to the root.

The *maximal-clique hypergraph* of a database D is the hypergraph that has the same nodes as D and has, as hyperedges, the maximal cliques in the augmented reachability graph of D. Note that in a directed graph, a clique is a set of nodes, such that every two nodes are connected by edges in both directions. A clique is maximal if it is not contained in any other clique.

Given that a database D is tree transformable under the flexible semantics, the creation of the equivalent tree T is the same as in the case of the semiflexible semantics, except for the following. In the algorithm of the transformation (Figure 3.4), the hypergraph $\mathcal{H}$ that is used is the maximal-clique hypergraph of the augmented reachability graph of D (instead of the path hypergraph of D).

Theorem 3.77 (Flexible Equivalence to a Tree) *Consider a database D. Let $\mathcal{H}$ be the maximal-clique hypergraph that is created from the augmented reachability graph of D. The database D is tree transformable under the flexible semantics if and only if the following three conditions hold.*

1. *If there is an edge in D that enters the root, then the label on this edge is equal to the labels on all the other edges of D;*
2. *There is no node in D that has two different incoming labels; and*
3. *The Bachman diagram $BD(\mathcal{H})$ of $\mathcal{H}$ is acyclic.*

Proof. We start by showing that the conditions are necessary. The first condition is necessary because in a tree no edge enters the root and only a single edge enters each other node. Thus, in the reachability graph of a tree, the root and every other node are connected by a single edge and this edge carries a single label. Yet, if in D there is an edge labeled with l that enters the root r_D while an edge labeled with l' enters a different node o, then in the reachability graph there should be two edges between r_D and o—one that is labeled with l and another that is labeled with l'.

The necessity of the second condition is obvious—a node in a tree cannot have two incoming edges and thus cannot have two incoming labels.

The proof for the necessity of the third condition is similar to the proof of Lemma 3.73—we assume that $BD(\mathcal{H})$ contains a cycle and show that D has no equivalent tree, under the flexible semantics.

If $BD(\mathcal{H})$ contains a cycle then $\mathcal{H}$ is not γ-acyclic. Since all the hyperedges of $\mathcal{H}$ contain the root of D there is no pure cycle in $\mathcal{H}$. Thus, $\mathcal{H}$ must have a γ-3-cycle. The γ-3-cycle is created by three hyperedges X, Y and Z and three nodes r_D, n_1 and n_2 such that (1) r_D is an element of all the three hyperedges; (2) n_1 is an element of X and Y and not an element of Z; and (3) n_2 is an element of X and Z and not an element of Y.

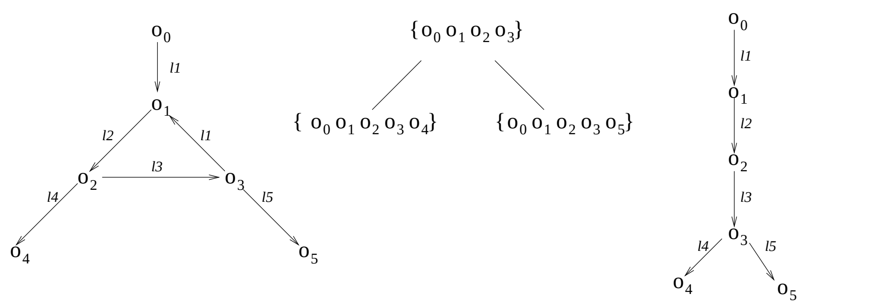
Consider a tree T that is equivalent to D under the flexible semantics. From the equivalence it follows that the reachability graph of T should be isomorphic to the reachability graph of D. We examine three possible cases:

1. There is no path in T that contains both n_1 and n_2.
2. In T there is a path from n_1 to n_2.
3. In T there is a path from n_2 to n_1.

If in T there is no path that contains both n_1 and n_2, then there is no edge between n_1 and n_2 in the reachability graph of T. This means that there is no edge between n_1 and n_2 in the reachability graph of D, in contradiction to having the hyperedge X in $\mathcal{H}$, since the nodes in X are a clique.

If in T there is a path from n_1 to n_2, then every path that goes through both the root and n_2 must also go through n_1. However, according to the hyperedge Z there is a clique in the augmented reachability graph of D (and thus also in the augmented reachability graph of T) that contains n_2 and does not contain n_1. This situation requires the existence of a node o in T, such that there is a path that goes through both o and n_2 and there is no path that goes through both o and n_1. However, since T is a tree no such object o can exist.

For the third case, where in T there is a path from n_2 to n_1, showing that it is impossible is done similarly to the second case above.



(1) A given database D. (2) The Bacman diagram. (3) The equivalent tree database.

Figure 3.6: Transformation to a tree under the flexible semantics.

To prove that the conditions are sufficient, we need to show that when the conditions hold, for a database D, the transformation algorithm returns a tree database that is equivalent to D under the flexible semantics. This is shown by showing that the reachability graph of D is isomorphic to the reachability graph of T, where T is returned by the algorithm. Since all the labels that enter a node o in D are equal to the labels on the edge that enters o in T, it is sufficient to show that a pair of nodes in D is connected by a path if and only if these two nodes are connected by a path in T.

If there are two nodes o_1 and o_2 such that D has a path from o_1 to o_2, then there is a maximal clique in the augmented reachability graph of D that contains both o_1 and o_2. It follows that $\mathcal{H}$ contains a node such that o_1 and o_2 are elements of this node and thus, in T, there is either a path from o_1 to o_2 or vice-versa.

For the other direction, if there are two nodes o_1 and o_2 such that in T there is path from o_1 to o_2, then there is a leaf of $BD(\mathcal{H})$ that contains both o_1 and o_2. This means that there is a maximal clique in $\mathcal{H}$ that contains both o_1 and o_2. Thus, in D there is either a path from o_1 to o_2 or vice-versa. □

The creation of the maximal-clique hypergraph is not in polynomial time. Thus, the transformation to a tree under the flexible semantics is not in polynomial runtime.

Example 3.78 An example of a transformation is depicted in Figure 3.6. Note that Figure 3.6 does not show the augmented reachability graph of the database

D. However, it is easy to see that the augmented reachability graph contains two maximal cliques. One clique is the set $\{o_0, o_1, o_2, o_3, o_4\}$ and the other clique is the set $\{o_0, o_1, o_2, o_3, o_5\}$. These cliques are the leaves of the Bachman diagram that is created from the maximal-clique hypergraph of the augmented reachability graph.

3.6 Summary of Contributions

In this chapter, we introduced novel query semantics, namely, the semiflexible and the flexible semantics. These semantics facilitate querying of data when the schema is unknown, complicated or changes frequently. Meaningful queries can be formulated even when the user is oblivious to the structural details of the data and is only familiar with the ontology. Moreover, queries are insensitive to common variations in the schemas of semantically similar data instances.

We investigated query evaluation under the new semantics. Under the semiflexible semantics, a polynomial-time algorithm, under input-output complexity, was introduced for the case of a DAG query and a tree database, provided that labels are not repeated in query paths. The case where the database is a tree is important since XML documents are usually trees. It is also shown that when the database is cyclic, there is no polynomial-time algorithm, in the size of the input and the output, assuming that P $\neq$ NP. This last claim holds even when the query is a path.

Under the flexible semantics, evaluation has the same complexity as under the conventional rigid semantics. Thus, for a tree query, evaluation can be done in polynomial time, under input-output complexity. For DAG queries, there is no polynomial time algorithm, in the size of the input and the output, assuming that P $\neq$ NP.

We characterized query containment and query equivalence, under both the semiflexible and the flexible semantics. The provided characterizations are a first step towards developing optimization techniques. It is shown that equivalence of queries is decidable in polynomial time.

An additional contribution presented in this chapter is in introducing and investigating the novel concept of database equivalence. Under the conventional rigid semantics, two database are equivalent only if they are isomorphic. This is not the case under the semiflexible and the flexible semantics. Database equivalence is defined and characterized, under the two semantics.

A novel feature of the new semantics is the possibility of transforming a given database D to a tree database that is equivalent to D. For the semiflexible semantics, testing whether D is equivalent to a tree database and actually transforming D to a tree database (if the test is positive) are both polynomial. Since queries can be evaluated more efficiently over tree databases, this result is of practical importance. In addition, the transformation allows to represent data in XML without using references. This is important since, for some parsers, references are problematic.

Chapter 4

Queries with Maximal Answers

This chapter introduces two new query semantics. The new semantics facilitate querying of incomplete data. The outline of the chapter is as follows. Section 4.1 describes two query semantics that allow maximal rather than complete answers, namely the OR-*semantics* and the weak semantics. In Section 4.2, an algorithm for computing maximal OR-matchings is presented, along with a proof of correctness and an analysis of the time complexity. In addition, it is shown how to modify the algorithm in order to compute maximal weak matchings. Finally, in Section 4.3, we discuss the contribution of this chapter.

4.1 Query Semantics

In the rigid semantics, matchings are *complete assignments*, i.e., all the query variables are mapped to database objects. We now define matchings that are *partial assignments*, i.e., some query variables may not be mapped to a database object and, instead, are assigned a *null value.*

4.1.1 Partial Matchings

We start with a formal definition of partial matchings.

Definition 4.1 (Partial Assignment) *Let* $Q = (V, E_Q, r_Q)$ *be a query and* $D = (O, E_D, r_D, \alpha)$ *be a database. A* partial assignment *of* Q *w.r.t. the database* D *is a mapping* $\mu \colon V \to O \cup \{\perp\}$*, such that each variable of* Q *is mapped either to an object of* D *or to a special value that is called* null *and is denoted as* $\perp$*.*

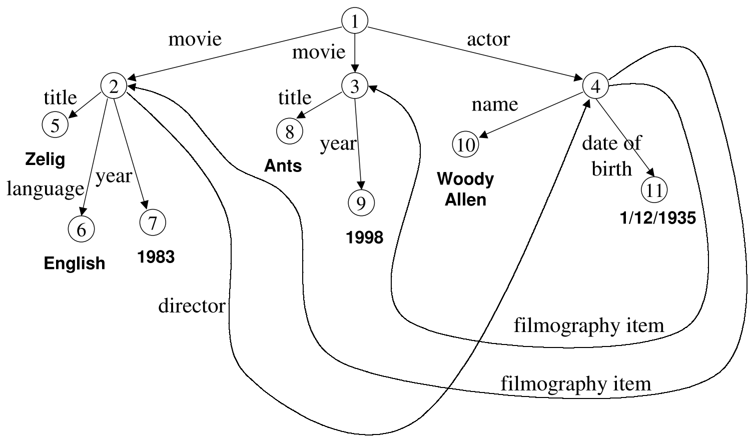


Figure 4.1: Another fragment of the movie database.

Each partial assignment can be represented as a graph. In the graph the nodes are pairs consisting of a variable and an object. Each pair denotes a mapping of an object to a variable. This way of representing partial assignments is formally specified in the next definition.

Definition 4.2 (Assignment Graph) *An* assignment graph *G of a query $Q = (V, E_Q, r_Q)$ w.r.t. a database $D = (O, E_D, r_D, \alpha)$ consists of nodes of the form (v, o), where v is a variable of Q and o is an object of D. The assignment graph G cannot have any* repeated variable, *i.e., it cannot have any variable v that appears in two distinct nodes (v, o_1) and (v, o_2), where $o_1 \neq o_2$. The assignment graph G may (but does not necessarily) have an edge from (v_1, o_1) to (v_2, o_2), denoted $(v_1, o_1)l(v_2, o_2)$, provided that v_1lv_2 is an edge of Q and o_1lo_2 is an edge of D. We say that $(v_1, o_1)l(v_2, o_2)$ is a* potential edge *of G if v_1lv_2 is in Q and o_1lo_2 is in D. If the edge $(v_1, o_1)l(v_2, o_2)$ appears in G, then it is called an* actual edge *of G.*

An assignment graph G corresponds to an assignment μ that is defined as follows. If (v, o) is a node of G, then $\mu(v) = o$; and if G has no node with v as its first component (i.e., no node of the form (v, o) for some o), then $\mu(v) = \perp$. The assignment μ is well defined, since G does not have any repeated variable.

Conversely, if μ is an assignment, then an assignment graph for μ comprises all nodes of the form $(v, \mu(v))$, where $\mu(v)$ is non-null. The assignment μ may have several assignment graphs. All those graphs have the same set of nodes, but they may have different sets of edges. However, the set of actual edges is always a subset of the set of potential edges.

Let G be an assignment graph of a query Q with respect to a database D. The *closure* of G is obtained by adding to G all potential edges that are not already in G. We say that G is *closed* if G is equal to its closure.

A matching is an assignment that satisfies some additional conditions (recall Definition 2.4). In this chapter, we define two types of matchings. The definitions are equivalent to those in [41]. Note that [41] presents three sematics for queries with maximal answers. We discuss only two of the three semantics, since in the semantics that is not presented here, evaluation of cyclic queries is NP-hard.

Definition 4.3 (OR-Matching) *An assignment μ of a query $Q = (V, E_Q, r_Q)$ w.r.t. a database $D = (O, E_D, r_D, \alpha)$ is an* OR-matching *if μ has an assignment graph M that satisfies the following conditions.*

1. *The graph M has the node (r_Q, r_D) (i.e., $\mu(r_P) = r_D$) and this node is designated as the root of M.*
2. *Each node of M is reachable from the root.*

An assignment graph M of a query Q w.r.t. a database D *preserves* the edges of Q if it satisfies the following condition. For every two nodes (v_1, o_1) and (v_2, o_2) of M, if $v_1 l v_2$ is an edge of Q, then $(v_1, o_1)l(v_2, o_2)$ is an edge of M (and thus, $o_1 l o_2$ is an edge of D).

Definition 4.4 (Weak Matching) *An assignment μ is a* weak matching *of Q if it has an assignment graph M that satisfies the conditions of an OR-matching and, in addition, M preserves the edges of Q.*

An assignment graph that satisfies the conditions of an OR-matching is called an OR-*matching graph*. Similarly, an assignment graph that satisfies the conditions of a weak matching is called a *weak-matching graph*. Note that a weak-matching graph is always closed.

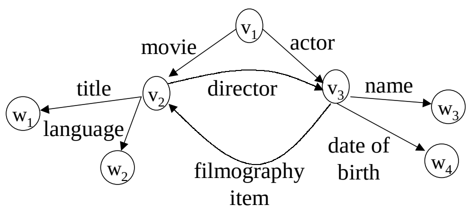


Figure 4.2: A query over the movie database.

Essentially, the result of posing a query to a database is a set of matchings. Under the OR-semantics, this set comprises all the OR-matchings. Under the weak semantics, this set comprises all the weak matchings.

4.1.2 Subsumption and Maximal Matchings

Let $\mathcal{A}$ be a set of assignments of a query Q w.r.t. a database D. Given two assignments μ and μ' of $\mathcal{A}$, we say that μ *subsumes* μ', denoted $\mu' \sqsubseteq \mu$, if $\mu(x) = \mu'(x)$ whenever $\mu'(x)$ is non-null. The assignment $\mu \in \mathcal{A}$ is a *maximal* element of $\mathcal{A}$ if no element in $\mathcal{A}$, other than μ itself, subsumes μ. Intuitively, if $\mu' \sqsubseteq \mu$, then μ has more information than μ'. Therefore, all non-maximal elements can be discarded from $\mathcal{A}$ without any loss of information.[1]

Subsumption is defined for assignment graphs in the natural way. Let G_1 and G_2 be assignment graphs. We say that G_1 subsumes G_2 if G_2 is a subgraph of G_1. That is, the set of nodes (edges) of G_2 is a subset or equal to of the set of nodes (edges) of G_1. An assignment graph is maximal, in a set of assignment graphs, if it is not subsumed by any other graph in the set.

Consider two assignment graphs G_1 and G_2 that correspond to the assignments μ_1 and μ_2, respectively. It is easy to see that if G_1 subsumes G_2 then μ_1 subsumes μ_2. Similarly, if $\mathcal{M}$ is a set of assignment graphs and G is a maximal element of $\mathcal{M}$, then the assignment that corresponds to G is maximal among the assignments that correspond to the elements of $\mathcal{M}$.

Given a query Q and a database D, the set of all maximal OR-matchings of Q

[1]This may not be true if bag semantics has to be assumed (e.g., in order to evaluate aggregate queries). However, the issue of bag semantics is beyond the scope of this work.

Semantics	v_1	v_2	v_3	w_1	w_2	w_3	w_4
OR	1	2	4	5 (Zelig)	6 (English)	10 (Woody Allen)	11 (1/12/1935)
	1	3	4	8 (Antz)	$\perp$	10 (Woody Allen)	11 (1/12/1935)
Weak	1	2	4	5 (Zelig)	6 (English)	10 (Woody Allen)	11 (1/12/1935)
	1	3	$\perp$	8 (Antz)	$\perp$	$\perp$	$\perp$

Figure 4.3: $\mathcal{M}_{or}(Q, D)$ and $\mathcal{M}_w(Q, D)$ for the Q and D of Figures 4.1 and 4.2.

w.r.t. D is denoted as $\mathcal{M}_{or}(Q, D)$. Similarly, the set of all maximal weak matchings of Q w.r.t. D is denoted as $\mathcal{M}_w(Q, D)$. Note that both $\mathcal{M}_{or}(Q, D)$ and $\mathcal{M}_w(Q, D)$ can be viewed as relations with columns that are labeled with the variables of Q and tuples that are filled with oid's and nulls.

Example 4.5 Figure 4.2 shows a query over the movie database of Figure 4.1. Intuitively, the query looks for an actor (along with her name and date of birth) and a movie (along with its title and language). The query specifies two relationships between the movie and the actor. First, the edge labeled with "filmography item" requires that the actor indeed acted in the movie. Secondly, the edge labeled with "director" requires that the actor was also the director of the movie.

The answers to the query differ according to the semantics that is used. The set of maximal OR-matchings and the set of maximal weak matchings are shown in Figure 4.3. Each matching is shown as a tuple of oid's and nulls. For atomic nodes, the value is shown next to the oid, inside parenthesis. Formally, however, a matching is always an assignment of oid's and not of values.

In a maximal weak matching, the edges of the query are preserved. Thus, an answer to the query includes a movie and an actor only if the actor acted in the movie and was also the director of the movie. The first weak matching in Figure 4.3 finds Woody Allen as both an actor and the director of the movie Zelig. An actor will be in an answer without a movie (i.e., null in v_2) if none of the movies of the actor was directed by her. A movie will be in an answer without an actor (i.e., null in v_3) if none of the actors in the movie was also a director of the movie. The second weak matching in Figure 4.3 includes the movie Antz without an actor, since no actor in Antz was also the director of this movie.

In a maximal OR-matching, the edges of the query are not necessarily preserved. Thus, an OR-matching can include an actor and a movie even if the actor neither acted in the movie nor was the director of the movie. The second OR-matching in Figure 4.3 gives a movie and an actor, such that the actor acted in the movie but was not the director. There are no other maximal OR-matchings in this example, since a movie always has an actor and an actor (or a director) always has a movie in the given database.

4.2 Computing Maximal Matchings

Algorithms for computing the sets $\mathcal{M}_{or}(Q, D)$ and $\mathcal{M}_w(Q, D)$ were given in [42] for queries that are acyclic graphs. The time complexity of these algorithms was shown to be polynomial in the size of the input (i.e., the query and the database) and the output (i.e., the set of all maximal matchings). In this section, we generalize this result to queries that may have cycles.

4.2.1 The Product Graph

We define a graph that combines the query and the database. This graph will be used for computing OR-matchings and weak matchings.

Definition 4.6 (Product) *Consider a query $Q = (V, E_Q, r_Q)$ and a database $D = (O, E_D, r_D, \alpha)$. The* product *of Q and D, denoted $Q \times D$, is the graph $\mathcal{P} = (\mathcal{V}, \mathcal{E}, r_\mathcal{P})$, that satisfies the following conditions.*

1. *The root of $\mathcal{P}$, denoted $r_\mathcal{P}$, is the pair (r_Q, r_D);*
2. *The set of nodes of $\mathcal{P}$, denoted $\mathcal{V}$, consists of all nodes of the form (v, o), where v is a variable of Q and o is an object of D, i.e., $\mathcal{V} = V \times O$; and*
3. *The set of edges of $\mathcal{P}$, denoted $\mathcal{E}$, is a set that consists of all the edges of the form $(v_1, o_1)l(v_2, o_2)$, where $v_1 l v_2$ is an edge of Q and $o_1 l o_2$ is an edge of D.*

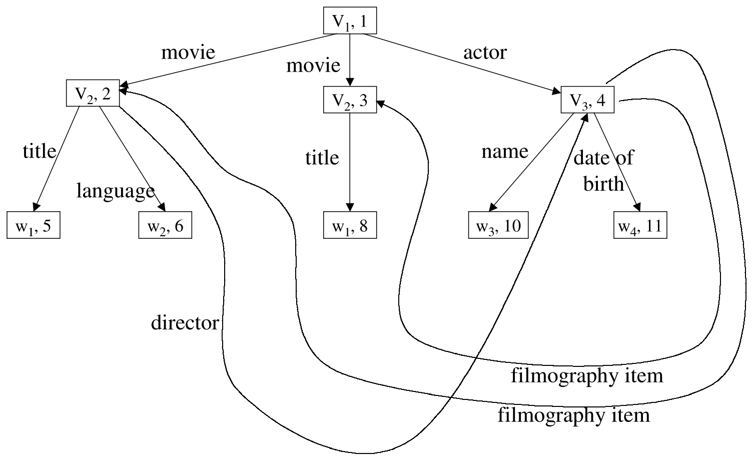


Figure 4.4: The product of the query that is presented in Figure 4.2 and the database that is presented in Figure 4.1.

Example 4.7 Figure 4.4 shows the product of the query that is presented in Figure 4.2 and the database that is depicted in Figure 4.1. Note that only nodes that are reachable from the root are shown.

The graph G is a *subgraph* of $Q{\times}D$ if G comprises some subset of the nodes of $Q{\times}D$. The edges of G may be some or all the edges of $Q{\times}D$ that connect nodes appearing in G. It is rather obvious that if M is either an OR-matching graph or a weak-matching graph, then M is a subgraph of $Q{\times}D$. The next proposition states when the converse is also true.

Proposition 4.8 *Consider the following four conditions on a subgraph G of $Q{\times}D$.*

1. *G has no repeated variables; that is, G does not have two nodes (v, o_1) and (v, o_2), such that $o_1 \neq o_2$.*
2. *G contains the root of $Q{\times}D$.*
3. *In G, each node is reachable from the root.*
4. *G preserves the edges of Q.*

A subgraph G of $Q{\times}D$ is an OR-matching graph if and only if it satisfies the first three conditions. G is a weak-matching graph if and only if it satisfies all four conditions.

Proof. Condition 1 is needed so that G will be the assignment graph of an assignment μ that is defined as follows. If (v, o) is a node of G, then $\mu(v) = o$; otherwise, $\mu(v) = \perp$. Conditions 2 and 3 are identical to the two conditions in the definition of an OR-matching. Condition 4 is the additional condition in the definition of a weak matching. □

4.2.2 Computing Maximal OR-Matchings

In this section, we will show how to compute the maximal OR-matchings of a query Q w.r.t. a database D. The main idea is to compute all the maximal subgraphs G of $Q{\times}D$, such that G satisfies Conditions 1–3 of Proposition 4.8.

The computation is incremental; it starts with small subgraphs and expands them by adding edges. Initially, the root is the only current subgraph. In every iteration, each current subgraph is expanded by adding one more edge from $Q{\times}D$. The algorithm works efficiently if the edges are added in the "right order." If Q is a DAG (directed acyclic graph), a topological sort of Q gives the right order and, in this case, there is really no need to consider the product $Q{\times}D$. Instead, it is sufficient to apply topological sort just to Q; see the details in [41, 42].

The General Idea

When Q has cycles, a different approach is needed. Instead of Q itself, the product $Q{\times}D$ should be considered. Furthermore, when adding edges in the "right order," it is not sufficient to consider each edge just once. To formalize the notion of the "right order," we say that an edge $m_1 l m_2$ of $Q{\times}D$ has a *depth* d if there is a path consisting of d edges that starts at the root of $Q{\times}D$ and ends with the edge $m_1 l m_2$. Two observations should be noted. First, an edge can have more than one depth when $Q{\times}D$ is not a tree. Second, an edge with depth 1 emanates from the root of $Q{\times}D$.

We divide the edges of $Q{\times}D$ into strata, such that stratum k contains all the edges at depth k (and k is called the *depth* of the stratum). Note that an edge may belong to more than one stratum. A *stratum traversal* of $Q{\times}D$ is an ordered list T

that starts with the edges of stratum 1, followed by the edges of stratum 2, and so on. Notice that the order of the edges within each stratum is unimportant and can be determined arbitrarily, but the edges of stratum k must appear before the edges of stratum $k+1$. Clearly, a stratum traversal may have multiple occurrences of the same edge, but in each stratum a given edge may appear at most once.

An important part of the algorithm for computing maximal OR-matchings is to create a stratum traversal T in polynomial time. However, as defined, a stratum traversal could be infinite. We will now show how to construct a finite stratum traversal T in polynomial time.

An occurrence of an edge e in stratum k is *extraneous* if e is not the kth edge of any path that emanates from the root and consists of distinct edges. In principle, all extraneous occurrences of edges can be removed from a stratum traversal, but doing so is computationally hard. We will use a stratum traversal with some extraneous occurrences of edges. However, these extraneous occurrences neither affect the correctness of the algorithm nor add an exponential blowup to the time complexity.

Obviously, if there are m edges in $Q \times D$, then only the first m strata may have non-extraneous occurrences of edges. Actually, it is sufficient for a stratum traversal to include only the first n strata, where n is the number of nodes in Q. The reason for this is that a matching graph cannot include two nodes (v, o_1) and (v, o_2), such that $o_1 \neq o_2$. Therefore, the number of nodes in a matching graph cannot exceed the number of nodes in Q.

The algorithm for computing maximal OR-matchings uses a stratum traversal T that consists of the first n strata, where n is the number of nodes in the query Q, and is constructed as follows. Stratum 1 contains all the edges that emanate from the root of $Q \times D$. Stratum k contains all edges $(v_1, o_1)l_1(v_2, o_2)$ of $Q \times D$, such that stratum $k-1$ has an edge that enters node (v_1, o_1). The size of the stratum traversal T is $O(q^2 d)$, where q is the size of the query Q and d is the size of the database D. The stratum traversal T can be constructed in $O(q^2 d)$ time.

Let T be the stratum traversal that was constructed as described above. The maximal OR-matchings are generated by an incremental algorithm that iterates over the edges in T. Initially, the set of current subgraphs of $Q \times D$ has only one member, namely, the root of $Q \times D$. In the ith iteration, each current subgraph is expanded, if possible, by adding the edge that appears in position i of T (the nodes of the edge are also added, unless they are already in the subgraph).

The Various Cases of Adding An Edge

Let G be a subgraph of $Q \times D$, such that G satisfies Conditions 1–3 of Proposition 4.8; that is,

- G has no repeated variables;
- G contains the root of $Q \times D$; and
- All the nodes of G are reachable from the root (via edges that belong to G).

Consider an attempt to add an edge $(v_1, o_1)l(v_2, o_2)$ of $Q \times D$ to G. Several cases are possible and we now consider them one by one.

Case 1 *The node (v_2, o_2) satisfies $v_2 = r_Q$ and $o_2 \neq r_D$.* In this case, the edge $(v_1, o_1)l(v_2, o_2)$ is not added to G, since the node (v_2, o_2) cannot belong to a subgraph of $Q \times D$ that satisfies Conditions 1–3 of Proposition 4.8. Thus, G remains unchanged.

Case 2 *The node (v_1, o_1) is not in G.* In this case, the edge $(v_1, o_1)l(v_2, o_2)$ cannot be added to G and G remains unchanged.

Case 3 *The nodes (v_1, o_1) and (v_2, o_2) as well as the edge that connects them are all in G.* In this case, G remains unchanged.

Case 4 *Both (v_1, o_1) and (v_2, o_2) are in G, but the edge that connects them is not in G.* In this case, the edge $(v_1, o_1)l(v_2, o_2)$ is added to G. Notice that the result is a subgraph G' of $Q \times D$ that satisfies Conditions 1–3 of Proposition 4.8. The new graph G' replaces G.

Case 5 *The node (v_1, o_1) is in G and v_2 does not appear in any node of G.* In this case, the edge $(v_1, o_1)l(v_2, o_2)$ as well as the node (v_2, o_2) are added to G. Notice that the result is a subgraph G' of $Q \times D$ that satisfies Conditions 1–3 of Proposition 4.8. The new graph G' replaces G.

Case 6 *The node (v_1, o_1) is in G, $v_2 \neq r_Q$ and G also has a node (v_2, o_3), where $o_2 \neq o_3$.* In this case, the addition of the edge $(v_1, o_1)l(v_2, o_2)$ to G forces the deletion of the node (v_2, o_3). Thus, a new subgraph G' is created from G as follows. First, the node (v_2, o_2) and the edge $(v_1, o_1)l(v_2, o_2)$ are added to G. Second, the node (v_2, o_3) is deleted from G'. Third, all the nodes that are not reachable from the root, due to the deletion of (v_2, o_3), are deleted from G'; edges that are incident on deleted nodes are also deleted. The result is a subgraph G' of $Q \times D$ that satisfies Conditions 1–3 of Proposition 4.8. Notice that G is not subsumed by G', since the former includes the node (v_2, o_3), but the latter does not. However, it could be that G subsumes G' (that may happen if the edge $(v_1, o_1)l(v_2, o_2)$ that was initially added to G is eventually deleted). In summary, both G and G' should be retained unless G' is subsumed by G.

The Algorithm

The algorithm computes a set $\mathcal{M}$ as follows.

1. Initially, $\mathcal{M}$ has one graph that consists of the root of $Q \times D$.

2. Repeat the next two steps for each edge e in the stratum traversal T, starting with the first edge of T.

3. For each graph $G \in \mathcal{M}$, replace G with the result of adding e to G, according to the above six cases. Recall that G is replaced either with G itself, with a new graph G' that subsumes G or with G and a new graph G', such that neither one subsumes the other.

4. Remove subsumed graphs from $\mathcal{M}$, i.e., a graph $G \in \mathcal{M}$ is removed if it is subsumed by some other graph of $\mathcal{M}$ (and this step is repeated until there are no subsumptions among the graphs of $\mathcal{M}$).

The next theorem shows that the algorithm is correct and gives its time complexity.

Theorem 4.9 (Correctness) *Given a query Q and a database D, the algorithm terminates with $\mathcal{M} = \mathcal{M}_{or}(Q, D)$. The running time of the algorithm is $O(q^3dm^2)$, where q is the size of Q, d is the size of D and m is the size of $\mathcal{M}_{or}(Q, D)$.*

Correctness of the Algorithm

The algorithm iterates over all the edges in the stratum traversal T. In each iteration, Steps 3 and 4 are executed once. The ith iteration is for the edge that occurs in position i of T. The value of $\mathcal{M}$ at the end of the ith iteration is denoted as $\mathcal{M}^i$.

Consider a maximal OR-subgraph M of $Q \times D$. Intuitively, we would like to prove that for all i, there is a graph $G \in \mathcal{M}^i$, such that G subsumes the restriction of M just to the edges that appear in the first i positions of T. However, the minimal depth of an edge in M may be greater than its minimal depth in $Q \times D$. Therefore, a more elaborate definition is needed.

Consider a position i in the stratum traversal T and suppose that position i occurs in the stratum that has depth d. Let M be a maximal OR-matching subgraph of $Q \times D$. The *i-portion* of M, denoted M^i, consists of all the edges m_1lm_2 of M, such that the minimal depth of m_1lm_2 in M is not greater than d and there is at least one occurrence of m_1lm_2 among the first i positions of T. Note that, by definition, the 0-portion of M consists just of the root of M. Also note that M^i is an OR-matching subgraph of $Q \times D$.

Next, we will prove the following lemma that shows the correctness of the algorithm.

Lemma 4.10 *When the algorithm completes its execution, $\mathcal{M} = \mathcal{M}_{or}(Q, D)$.*

Proof. The proof follows from the following two claims.

Claim 4.11 *During the evaluation of the algorithm, all the graphs in $\mathcal{M}$ are OR-matchings graphs.*

This claim is true, since all the graphs in $\mathcal{M}$ satisfy Conditions 1–3 of Proposition 4.8 throughout the evaluation of the algorithm.

Claim 4.12 *When the algorithm terminates, the following holds. For every maximal* OR*-matching graph M, the set $\mathcal{M}$ has a graph G, such that G subsumes M.*

Together, the two claims imply that at the end of the algorithm, $\mathcal{M}$ is exactly the set of all maximal OR-matching graphs, since the set $\mathcal{M}$ is subsumption free.

Claim 4.12 is proved by induction, using the following inductive hypothesis: If M is a maximal OR-graph, then M^i is subsumed by some $G \in \mathcal{M}^i$.

For the basis of the induction, we should consider the initialization of $\mathcal{M}$ just prior to the first iteration. Initially, $\mathcal{M}$ contains one subgraph G of $Q \times D$ that comprises the root of $Q \times D$. The 0-portion of M is just the root, so the induction hypothesis is true. Now suppose that the inductive hypothesis holds at the end of the ith iteration. We will show that it also holds at the end of the $(i+1)$st iteration.

Let M be a maximal OR-matching graph and $(v_1, o_1)l(v_2, o_2)$ be the $(i+1)$st edge in the stratum traversal T. We need to show that the induction hypothesis holds after expanding the graphs in $\mathcal{M}^i$, by adding $(v_1, o_1)l(v_2, o_2)$.

We consider two cases, depending on whether $(v_1, o_1)l(v_2, o_2)$ is an edge of M^{i+1} or not. We show that in both cases in $\mathcal{M}^{i+1}$ there is a G' that subsumes G.

Suppose that $(v_1, o_1)l(v_2, o_2)$ is not an edge of M^{i+1}. Then, M^i and M^{i+1} are identical. By the inductive hypothesis, there is a graph G in $\mathcal{M}^i$, such that M^i is subsumed by G. Since $G \in \mathcal{M}^i$, there is a $G' \in \mathcal{M}^{i+1}$, such that G is subsumed by G'. Thus, M^{i+1} is subsumed by G' and the inductive hypothesis is proven.

Now, suppose that $(v_1, o_1)l(v_2, o_2)$ is an edge of M^{i+1}. By the inductive hypothesis, there is a graph G in $\mathcal{M}^i$, such that M^i is subsumed by G.

An important observation is that (v_1, o_1) is a node of G. This observation follows from the following assertions. First, M^{i+1} is an OR-matching graph and (v_1, o_1) is reachable from the root by a path that comprises edges of M^i. Thus, (v_1, o_1) is a node of M^i. Secone, since G subsumes M^i, (v_1, o_1) is a node of G.

Next, we consider the six cases of adding the edge $(v_1, o_1)l(v_2, o_2)$ to G during the $(i+1)$st iteration, as described in Section 4.2.2. We show that in all the cases, $\mathcal{M}^{i+1}$ cotains a grapg that subsumes M^{i+1}.

The first two cases contradict the assumption that $(v_1, o_1)l(v_2, o_2)$ is an edge of

M^{i+1}. In the first case, the edge $(v_1, o_1)l(v_2, o_2)$ cannot be in M and, thus, it is not an edge of M^{i+1}. In the second case, the node (v_1, o_1) is not in G, in contradiction to the above observation.

In the third case, G already has the edge $(v_1, o_1)l(v_2, o_2)$. Since G subsumes M^i, it follows that it also subsumes M^{i+1}. Therefore, since G' subsumes G it also subsumes M^{i+1}.

The fourth and the fifth cases are similar. The graph G' has all the nodes and edges of G and also the added edge $(v_1, o_1)l(v_2, o_2)$. Since G subsumes M^i, G' subsumes M^{i+1}.

Finally, consider the sixth case. First note that M cannot include the node (v_2, o_3). This is because M^{i+1} includes the node (v_2, o_2) and, hence, M includes (v_2, o_2). M is an assignment graph so it cannot have two nodes with the same variable.

In the sixth case, a new subgraph G' is created from G by adding the edge $(v_1, o_1)l(v_2, o_2)$ and then deleting the node (v_2, o_3) as well as all the nodes that are not reachable from the root. Let $\bar{G}$ be the subgraph that is obtained immediately after the addition of $(v_1, o_1)l(v_2, o_2)$ and just before the deletion of any node. Since M does not include (v_2, o_3), all the nodes of M^{i+1} are reachable from the root via paths that do not include (v_2, o_3). By the induction hypothesis, M^i is subsumed by G and, consequently, all the nodes and edges of M^{i+1} also appear in $\bar{G}$. In $\bar{G}$, all the nodes of M^{i+1} are reachable from the root via paths that do not include (v_2, o_3). Thus, none of the nodes and none of the edges of M^{i+1} is deleted from $\bar{G}$. Therefore, G' subsumes the graph M^{i+1}.

In summary, in all of the above cases, the inductive hypothesis holds at the end of the $(i + 1)$st iteration. □

Time Complexity

In this section, we will prove the part of Theorem 4.9 that refers to the time complexity. The proof hinges on the fact that the number of graphs in $\mathcal{M}$ does not decrease from one iteration to the next, as shown by the following lemma.

Lemma 4.13 *For each $i \geq 0$, $|\mathcal{M}^i| \leq |\mathcal{M}^{i+1}|$.*

Proof. The lemma is proved by showing that each $G' \in \mathcal{M}^{i+1}$ subsumes at most one $G \in \mathcal{M}^i$. To derive a contradiction, we assume that G_1 and G_2 are two graphs in $\mathcal{M}^i$ that are subsumed by some $G' \in \mathcal{M}^{i+1}$. Suppose that position i of the stratum traversal T occurs in a stratum that has a depth d. Consider the graph $\hat{G}$ that consists of all the edges that appear in either G_1 or G_2, as well as all the nodes connected by those edges. (If there are no edges, then $\hat{G}$ consists just of the root of $Q \times D$.) Clearly, each edge of $\hat{G}$ has a depth (in $\hat{G}$) that is not greater than d and has an occurrence among the first i positions of the stratum traversal T. Furthermore, $\hat{G}$ has no repeated variables, since it is subsumed by another OR-matching subgraph, G'. Thus, $\hat{G}$ is an OR-matching subgraph of $Q \times D$, and $\hat{G}^i$ (the i-portion of $\hat{G}$) is the same as $\hat{G}$. By the inductive hypothesis in the proof of Lemma 4.10, there is a $G \in \mathcal{M}^i$, such that G subsumes $\hat{G}$. Consequently, G subsumes both G_1 and G_2, in contradiction to the fact that $\mathcal{M}^i$ is subsumption free. □

We will now complete the proof of the time complexity. Recall that q, d and m are the sizes of the query, the database and the result, respectively. By Lemma 4.13, at any time during the evaluation of the algorithm, before the removal of subsumed graphs, $\mathcal{M}$ has at most twice as many graphs as the final value of $\mathcal{M}$. After the removal of subsumed graphs, the number of graphs in $\mathcal{M}$ does not exceed the number of graphs in the result. In Step 3 of the algorithm, adding an edge to a graph $G \in \mathcal{M}$ takes $O(q)$ time. Thus, each execution of Step 3 takes $O(qm)$ time. Each execution of Step 4 takes $O(qm^2)$ time. The size of the stratum traversal is $O(q^2 d)$ and it can be constructed in $O(q^2 d)$ time. Hence, Steps 3 and 4 are repeated $O(q^2 d)$ times and the whole algorithm takes $O(q^3 d m^2)$ time.

4.2.3 Computing Maximal Weak Matchings

To modify the algorithm so that it will compute the set of all maximal weak matchings, Cases 5 and 6 of Section 4.2.2 should be changed. In each of these two cases, after creating the subgraph G', there is a need to test and possibly modify G' so that it will preserve the edges of Q. This is done as follows.

Let $(v_1, o_1)l(v_2, o_2)$ be the edge that was added to G'. For each node (v_i, o_i) in G', if there is a label h, such that $v_i h v_2$ is an edge of Q, but $o_i h o_2$ is not an edge of D, then delete node (v_i, o_i) from G'. Similarly, for each node (v_j, o_j) in G', if there is a label h, such that $v_2 h v_j$ is an edge of Q, but $o_2 h o_j$ is not an edge of D, then delete node (v_j, o_j) from G'. Nodes that become non-reachable from the root should also be deleted. Edges that are incident on deleted nodes are deleted as well. As a result of these deletions, G' may no longer subsume G and consequently, in Case 5, both G' and G should be added to $\mathcal{M}$ (or just G should be added if it subsumes G'). The same is done in Case 6.

Theorem 4.14 *Given a query Q and a database D, the set $\mathcal{M}_w(Q, D)$ of all maximal weak matchings can be computed in $O(q^3 d m^2)$ time, where q is the size of Q, d is the size of D and m is the size of $\mathcal{M}_w(Q, D)$.*

In proving the correctness of the algorithm for computing the maximal weak matchings, Claim 4.11 should be replaced with the following claim.

Claim 4.15 *During the evaluation of the algorithm, the closure of each graph in $\mathcal{M}$ is a weak matching.*

Claim 4.12 remains the same, except for changing "maximal OR-matching graph M" to "maximal weak-matching graph M." The proofs of correctness and time complexity are similar to the ones in the case of computing the maximal OR-matchings.

4.3 Summary of Contributions

Chapter 4 follows the work of [41, 42] on queries with incomplete answers over semistructured data. In [41, 42], three semantics, AND, OR and weak, were presented and the complexity of query evaluation was investigated. In particular, it was shown that for DAG queries, both the OR-semantics and the weak semantics have polynomial-time evaluation algorithms under input-output complexity. In this chapter, we have shown that this result carries over to cyclic queries.

Chapter 5

Combining Flexibility with Maximal Answers

In the previous chapters we introduced two paradigms: The paradigm of flexible queries and the paradigm of maximal answers. In the current chapter, we show how to combine the two paradigms. Combining the two paradigms facilitates, querying of databases that are both irregular and incomplete.

The outline of this chapter is as follows. In Section 5.1, four new semantics are defined, namely, the *semiflexible*-OR, the *semiflexible-weak*, the *flexible*-OR and the *flexible-weak* semantics. In Section 5.2, query evaluation under the four new semantics is investigated. Finally, in Section 5.3 we summarize the contribution of this chapter.

5.1 Queries under the Combined Semantics

In this section we define new semantics that are called *combined semantics*. The combined semantics generalize the weak and the OR-semantics and add flexibility to matchings, according to the paradigm of flexible queries.

5.1.1 Combined Semantics

We start by providing required definitions. We also provide a characterizations for weak and OR-matchings. We will use this characterization when we define the combined semantics.

Definition 5.1 (Rooted Fragment) *Given a query $Q = (V, E_Q, r_Q)$, a* rooted fragment *of Q is a rooted graph $F = (V_F, E_F, r_Q)$, where V_F is a subset of V and E_F is a subset of $E_Q \cap (V_F \times V_F)$. The rooted fragment F* preserves the edges *of Q if each edge of Q that lies on two nodes of F is contained in E_F, that is, all the edges of E_Q that connect nodes of V_F belong to E_F*[1] .

The next proposition characterizes weak and OR-matchings in terms of rooted fragments.

Proposition 5.2 (Weak-Matchings and OR-Matchings) *Let Q be a query and D be a database. A mapping μ is an* OR*-matching of Q w.r.t. D if and only if there is a rooted fragment F of Q, such that μ is a rigid mapping of F w.r.t. D and μ maps to null all the variables of Q that are not in F. The mapping μ is a weak-matching of Q w.r.t. D if and only if the rooted fragment also preserves the edges of Q.*

Proof. Suppose that F is a rooted fragment of Q and μ is a rigid mapping of F w.r.t. D. We show that μ is an OR-matching of Q w.r.t. D.

First, we create an assignment graph G. The nodes of G are pairs of the form $(v, \mu(v))$, where v is a variable of F. For each edge ulv if F there is an edge $(u, \mu(u))l(v, \mu(v))$ in G. To see why G is an assignment graph, one should observe that if ulv is an edge in F then there is an edge $\mu(v)l\mu(u)$ in D. This is because μ satisfies all the ec's of F.

Since μ maps the query root r_Q to database root r_D, the pair (r_Q, r_D) is a node of G. Because in F each node is reachable from the root, also in G each node is reachable from the root. Thus, according to Definition 4.3, μ is an OR-matching.

The correctness of the other direction follows from the following observation. If M is an OR-matching graph of μ, then the nodes and edges of Q that appear in nodes and edges of M form a rooted fragment of Q. For weak matchings, the arguments are similar. □

Essentially, according to Proposition 5.2 an assignment μ is an OR-matching (weak-matching) if there exists a rooted fragment (a rooted fragment that preserves

[1]*Note that in literature, sometimes a fragment F that preserves the edges of Q is called "the induced subgraph of Q on the nodes of V_F".*

the edges) of Q such that μ satisfies the rc and all the ec's of the fragment. In order to add flexibility to the weak and OR-semantics, we replace the query constraints in the definitions as defined next.

Definition 5.3 (Combined Semantics) *Suppose that Q is a query and D is a database. Let ξ be one of the three semantics: rigid, semiflexible and flexible. Consider the following four conditions.*

1. *F is a rooted fragment of Q.*
2. *μ is a ξ-matching of F w.r.t. D.*
3. *μ maps to null all the variables of Q that are not in F.*
4. *F preserves the edges of Q.*

*A mapping μ is a ξ-*OR-matching *of Q w.r.t. D if the first three conditions hold. It is a ξ-*weak-matching *of Q w.r.t. D if all four conditions are true.*

For tree queries there are actually only three combined semantics, as pointed out by the following proposition.

Proposition 5.4 (Tree Queries) *Suppose that Q is a tree query, D is a database and ξ is one of the three semantics: rigid, semiflexible and flexible. An assignment μ is a ξ-*OR-matching *of Q w.r.t. D if and only if μ is a ξ-*weak-matching *of Q w.r.t. D.*

Proof. In a tree, every rooted fragment preserves the edges. □

For queries that are arbitrary graphs, the following holds.

Proposition 5.5 (Weak and OR-Matchings) *Suppose that Q is a query, D is a database and ξ is one of the three semantics: rigid, semiflexible and flexible. If μ is a ξ-*weak-matching *of Q w.r.t. D then μ is a ξ-*OR-matching *of Q w.r.t. D.*

Proof. If μ satisfies all the four conditions of Definition 5.3 then it satisfies the first three conditions of the definition. □

Query	Root (r)	Movie (x)	Title (y)	Year (z)	Actor (u)	Name (v)
Query 1	1	11	Star Wars (23)	1977 (24)	21	Mark Hamill (30)
Query 2	1	11	Star Wars (23)	1977 (24)	21	Harrison Ford (31)
and	1	12	Leon (26)	$\perp$	25	Natalie Portman (32)
Query 3	1	29	Dune (35)	1984 (36)	14	Kyle MacLachlan (27)

Table 5.1: The flexible-weak matchings of Query 1, Query 2 and Query 3 of Figure 2.2 w.r.t. the movie database of Figure 2.1. Only matchings in which both x and u are not null are presented.

Query results, under the combined semantics, consist only of maximal answers. If Q is a query and D is a database, the set of maximal flexible-OR-matchings of Q w.r.t. D is denoted as $\mathcal{M}_{f\text{–}or}(Q, D)$. The set of maximal flexible-weak-matchings of Q w.r.t. D is denoted as $\mathcal{M}_{f\text{–}w}(Q, D)$. The notations for the combined semantics in which the flexible semantics is replaced with the semiflexible semantics are similar (i.e., $\mathcal{M}_{sf\text{–}or}(Q, D)$ and $\mathcal{M}_{sf\text{–}w}(Q, D)$).

5.1.2 Examples

Example 5.6 Recall the three queries that are depicted in Figure 2.2. All three queries look for actor-movie pairs in which the actor acted in the movie. For actors, the queries return their name. For movies, the queries return their title and the year of production.

We examine the case where the three queries are posed, under the flexible-weak semantics, to the movie database that is shown in Figure 2.1. Table 5.1 presents the flexible-weak matchings in which x and u are not assigned a null value, i.e., matchings that consist of both a movie and an actor. Note that with the requirement that x and u should not be null, all three queries have the same set of matchings. For Query 1 and Query 3, there is an additional flexible-weak matching where r, x and y are mapped to 1, 13 and 33, respectively, and all the other variables are mapped to null. This matching is not shown in Table 5.1 because it maps u to null. In the table, for atomic objects, their value is presented in addition to the object identifier.

Consider a user that looks for related actor-movie pairs but is familiar only with

Query	Root (r)	Movie (x)	Title (y)	Year (z)	Actor (u)	Name (v)
Query 1	1	11	Star Wars (23)	1977 (24)	21	Mark Hamill (30)
	1	11	Star Wars (23)	1977 (24)	21	Harrison Ford (31)
	1	12	Leon (26)	$\perp$	25	Natalie Portman (32)
	1	13	Magnolia (33)	$\perp$	$\perp$	$\perp$
	1	29	Dune (35)	1984 (36)	14	$\perp$
Query 2	1	11	$\perp$	$\perp$	21	Mark Hamill (30)
	1	11	$\perp$	$\perp$	21	Harrison Ford (31)
	1	12	$\perp$	$\perp$	25	Natalie Portman (32)
	1	29	Dune (35)	1984 (36)	14	Kyle MacLachlan (27)
Query 3	1	11	$\perp$	$\perp$	21	Mark Hamill (30)
	1	11	$\perp$	$\perp$	21	Harrison Ford (31)
	1	11	Star Wars (23)	1977 (24)	$\perp$	$\perp$
	1	12	$\perp$	$\perp$	25	Natalie Portman (32)
	1	12	Leon (26)	$\perp$	$\perp$	$\perp$
	1	13	Magnolia (33)	$\perp$	$\perp$	$\perp$
	1	29	Dune (35)	1984 (36)	14	Kyle MacLachlan (27)

Table 5.2: The semiflexible-weak matchings of Query 1, Query 2 and Query 3 of Figure 2.2 w.r.t. the movie database of Figure 2.1.

the ontology of the database and not with the structure of the database. This user may formulate one of the three queries of Figure 2.2. Under the flexible-weak semantics, all three queries will return a correct answer, that is, all the actor-movie pairs in which the actor acted in the movie. As shows in Chapter 1, this is not the case when using the traditional rigid semantics. The example also demonstrates how the flexible-weak semantics facilitates the querying of data which is both irregular and incomplete.

Under the flexible-OR semantics, Query 1 and Query 2 will return the same set of matchings as under the flexible-weak semantics. This is because the queries are trees. Query 3, however, may return, under the flexible-OR semantics, actors and movies that are not related. The reason for this is that under the flexible-OR semantics, it is not required that matchings will preserve the edges of the query and, in particular, the edge that goes from u to x.

Example 5.7 Table 5.2 shows the semiflexible-weak matchings of the three queries of Figure 2.2 w.r.t. the movie database. The semiflexible semantics is more restrictive than the flexible semantics. Therefore, for each semiflexible-weak matching in Table 5.2 there is a flexible-weak matching, of the same query, that subsumes it. This shows that queries under the semiflexible-weak semantics may extract less information than under the flexible-weak semantics. Yet, the semiflexible-weak semantics reduces the chance of having non-related objects in a matching, in comparison to the flexible-weak semantics. Most importantly, under the semiflexible-weak semantics, all three queries find all the related actor-movie pairs. Furthermore, actor-movie pairs that are not related are never returned.

5.2 Computing Maximal Matchings

In this section, we study the problem of computing maximal matchings under the combined semantics that were presented above. We describe evaluation algorithms for queries with combined semantics and analyze the input-output complexity of these algorithms.

5.2.1 Computing Semiflexible-OR-Matchings

Evaluation of maximal semiflexible-OR-matchings is discussed next. At the beginning of this section, we show that evaluation of dag queries is NP-hard. Then we provide a polynomial-time algorithm for the case where the query is either a tree or a dag in which the number of nodes, with more than one parent, is bounded. At the end of the section, we provide an algorithm for computing maximal semiflexible-OR-matchings of cyclic queries.

DAG Queries

Consider the evaluation of dag queries under the semiflexible-OR semantics. We show that it is not likely that this evaluation has an algorithm with polynomial-time input-output complexity. For a proof we use a reduction of 3SAT.

Lemma 5.8 (Reduction of 3SAT) *For a given 3CNF formula φ over a set of propositional letters P, one can construct in polynomial time a dag query Q and a dag database D, such that the following are equivalent.*

- *The set of maximal semiflexible-*OR*-matchings consists of a single matching and this matching does not map any variable of Q to null.*
- *There is an assignment, for the propositional letters in P, that satisfies φ.*

The proof of Lemma 5.8 is given in Appendix C.

Theorem 5.9 (Complexity of Semiflexible-OR-Queries) *If there exists an algorithm that computes $\mathcal{M}_{sf\text{–}or}(Q, D)$ in time polynomial in the size of the input (Q and D) and the output, then* PTIME=NP.

Proof. Let q, d and m be the sizes of the query, the database and the result, respectively. Suppose that there is an evaluation algorithm $\mathcal{A}$ and there is a polynomial $P(x_1, x_2, x_3)$ such that for every database D and query Q, the algorithm $\mathcal{A}$ computes $\mathcal{M}_{sf\text{–}or}(Q, D)$ in $O(P(q, d, m))$ time. We show that this would yield a polynomial algorithm to decide 3SAT. However, 3SAT is NP-complete.

Let φ be a 3CNF formula over a set $\mathcal{P}$ of propositional letters. Let D_φ and Q_φ be the query and the database that are constructed according to Lemma 5.8. Consider a satisfying assignment f to the propositional letters of $\mathcal{P}$. If f satisfies φ, then $\mathcal{M}_{sf\text{–}or}(Q_\varphi, D_\varphi)$ contains a single element. Furthermore, the evaluation of $\mathcal{M}_{sf\text{–}or}(Q_\varphi, D_\varphi)$ is in $O(P(|Q_\varphi|, |D_\varphi|, 1))$ time. Since the sizes of D_φ and Q_φ are polynomial in the size of φ, there exists a polynomial $P'(x)$ such that $\mathcal{A}$ terminates after time $P'(|\varphi|)$, i.e., after polynomial time in the size of φ.

In order to solve 3SAT in polynomial time, we would construct D_φ and Q_φ and then execute $\mathcal{A}$ with D_φ and Q_φ as the input. If $\mathcal{A}$ does not terminate after $P'(|\varphi|)$ time, we stop the run and know that φ is unsatisfiable. If $\mathcal{A}$ stops without our intervention, we examine the result. If the result consists of a single matching and this matching maps all the variables of Q to database objects (i.e., not to null), then we know that φ is satisfiable. Otherwise, we know that φ does not have a satisfying assignment. $\square$

The above theorem showes that there is no polynomial-time algorithm for evaluating dag queries under the semiflexible-OR-semantics, assuming that P $\neq$ NP. However, consider a query Q, a database D and a mapping μ that assigns objects of D to the variables of Q. To decide if μ is a semiflexible-OR-matching of Q w.r.t. D is in NP. This is because one can guess a rooted fragment F of Q and, according to Theorem 3.19, verify in polynomial time, in the size of Q and D, whether μ is a semiflexible matching of Q w.r.t. F.

Next, we provide an algorithm that computes maximal semiflexible-OR matchings. First we describe the algorithm for dag queries and then modify this algorithm, to deal with cyclic queries. In the general case, the input-output complexity of the algorithm is not polynimal. Yet, there are cases in which the algorithm is tractable. In particular, for tree queries, the algorithm has a polynomial-time input-output complexity. An analysis of the algorithm runtime will be given at the end of the section.

Before presenting the algorithm, we show that when computing semiflexible-OR-matchings, it is sufficient to compute the matchings w.r.t. subtrees of the query instead of examining all the rooted fragments of the query, including fragments that are not trees.

Proposition 5.10 (Tree Expansion) *Consider a query Q and a database D. An assignment μ is a semiflexible-*OR*-matching of Q w.r.t. D if and only if there is a tree T that satisfies the following three conditions.*

1. *T is a rooted fragment of Q;*
2. *μ is a semiflexible-matching of T w.r.t. D; and*
3. *μ maps to null all the variables of Q that are not in T.*

Proof. Obviuosly, if there exists a tree T that satisfies the conditions of the proposition, then μ is a semiflexible-OR-matching. This follows directly from Definition 5.3.

We show the other direction. Suppose that μ is a semiflexible-OR-matching. According to Definition 5.3, there is rooted fragment F of Q such that μ is a semiflexible-matching of F w.r.t. D and μ maps to null all the variables of Q that are not in F. If F is a tree, then the claim is correct and there is nothing to prove. Otherwise, let T be a spanning tree of F, i.e., a tree that consists of all the nodes of F. Obviously, T is a rooted tree and μ maps to null all the variables of Q that are not in T. Furthermore, μ is a semiflexible matching of F w.r.t. D. Thus, μ satisfies the SF-Condition w.r.t. every path of F. Since all the paths of T are also paths of F, μ satisfies the SF-Condition w.r.t. every path of T. Therefore, μ is a semiflexible matching of T w.r.t. D. □

Next, we present an evaluation algorithm for queries under the semiflexible-OR-semantics. Generally, the evaluation algorithm computes rooted subtrees of the query and finds the semiflexible matchings w.r.t. these trees. For trees and for dag queries, the evaluation is performed by a series of extension steps. We describe extension steps, first, and then describe how to apply these extension steps when evaluating a query.

Definition 5.11 (Extension Set) *Let $T = (V_T, E_T, r_Q)$ be a rooted subtree of a query Q and let μ be a semiflexible matching of T w.r.t. a database D. Consider a variable v of Q that is not in V_T. The set $Ext_{sf\text{-}or}(T, \mu, v)$ is the set of all pairs (T', μ') such that:*

1. *T' is a rooted subtree of Q.*
2. *The variables of T' are $V_T \cup \{v\}$ and the edges of T' consists of E_T plus one edge that connects a node in V_T to v.*
3. *The matchings μ and μ' are equal on all the nodes of T.*
4. *The matching μ' is a semiflexible matching of T' w.r.t. D.*

Intuitively, $Ext_{sf\text{-}or}(T, \mu, v)$ contains pairs of a tree T' and a matching μ', where T' is an extension of T (T' is created by adding to T an edge that connects T with v) and μ' is a semiflexible matching of T' w.r.t. D.

Computing $Ext_{sf\text{-}or}(T, \mu, v)$ is described next. First, we find all the edges of Q that connect a node of T to v. Adding each of these edges to T provides a tree T'

that extends T to include v. For an object o in D, let $\mu \cup \{(v, o)\}$ be a matching that is equal to μ on the variables of T and assigns o to v. Consider a pair of a tree T' that is the result of extending T with v and a matching $\mu' = \mu \cup \{(v, o)\}$. If μ' is a semiflexible matching of T' w.r.t. D, we add the pair (T', μ') to the result. Checking if μ' is a semiflexible matching of T' w.r.t. D is being done by testing the two conditions that are described in the next proposition.

Proposition 5.12 (Testing Extended Matchings) *Let μ' be a matching that is created by extending μ with an assignment of o to v, as described above. Then μ' is a semiflexible matching of T' w.r.t. D if the next two conditions are satisfied.*

1. *Let l be the label on the edge that enters v in T'. There is an edge in D that enters o and is labeled with l.*
2. *If v_a is an ancestor of v in T' and has an incoming label l_a, then D either has a path $\mu(v_a)*lo$ or a path $o*l_a\mu(v_a)$.*

Proof. First, we show why it is sufficient to check the conditions in the proposition only w.r.t. v and o and not w.r.t. all the variables of the query. Recall that the matching μ is a semiflexible matching of T w.r.t. D. Thus, every path of T satisfies the SF-Condition w.r.t. μ. Every path in T' that does not include v is a path in T and, hence, satisfies the SF-condition w.r.t. μ'. So, in order to check that μ' is a semiflexible matching of T' w.r.t. D, we merely need to check that paths that contain v satisfy the SF-condition. This is precisely what the conditions of the proposition do. Therefore, if the two conditions of the proposition are satisfied, then μ' is a connectivity-preserving mapping. Note that the first condition of the proposition is sufficient and necessary for satisfaction of the first condition in the definition of a connectivity-preserving mapping (Definition 3.10). According to Corollary 3.13, μ' is a semiflexible matching of T'. $\square$

Consider two pairs (T, μ) and (T', μ'), where *(1)* T and T' are subtrees of Q; and *(2)* μ and μ' are semiflexible matchings w.r.t. T and T', respectively. We say that (T', μ') *subsumes* the pair (T, μ) if *(1)* all the edges and nodes of T are also edges and nodes of T', and *(2)* for each variable v of T, $\mu(v) = \mu'(v)$. During the run of the algorithm, subsumed pairs are discarded in order to prevent redundant work.

Let Q be a dag query and D be a database. Next, we present the algorithm *SFORD* that computes the set of maximal semiflexible-OR matchings of Q w.r.t. D.

SFORD(Q, D)

1. Sort topologically the nodes of Q. Let $v_0, \ldots, v_n$ be the nodes of Q according to the topological order.

2. Create an initial set $\mathcal{M}^0$ that contains a single pair (T, μ), where T has no edges and has a single node—the query root. The matching μ is a mapping of the query root to the database root.

3. Create the sets $\mathcal{M}^1, \ldots, \mathcal{M}^n$, iteratively. For each $1 \leq i \leq n$, the set $\mathcal{M}^i$ is created by extending the matchings of $\mathcal{M}^{i-1}$ w.r.t. v_i. Extension step i is performed as follows.

   (a) Start with an empty set $\mathcal{M}^i$.

   (b) For each pair (T, μ) in $\mathcal{M}^{i-1}$, if $Ext_{\mathit{sf\text{-}or}}(T, \mu, v_i)$ is empty, add to $\mathcal{M}^i$ the pair (T, μ). Else, add to $\mathcal{M}_i$ all the elements of $Ext_{\mathit{sf\text{-}or}}(T, \mu, v_i)$.

   (c) Iteratively, remove an element of $\mathcal{M}^i$ that is subsumed by another element of $\mathcal{M}^i$, until there are no subsumptions.

4. Let $\mathcal{M}$ be the set $\{\mu \mid (T, \mu) \in \mathcal{M}^n\}$. That is, $\mathcal{M}$ consists of all the matchings μ that are part of a pair (T, μ) in $\mathcal{M}^n$.

5. For each matching μ in $\mathcal{M}$, if there are variables of Q that μ does not map to any object, then replace μ by a mapping $\tilde{\mu}$, such that $\tilde{\mu}$ is equals to μ on all the variables that μ maps to database objects. In addition, $\tilde{\mu}$ maps to null variables for which μ is not defined.

6. Remove from $\mathcal{M}$ subsumed matchings, i.e., non-maximal elements, and return $\mathcal{M}$.

The next propositions prove the correctness of the algorithm and analyze the input-output time complexity.

Proposition 5.13 (Completeness) *Suppose that Q is a dag query and D is a database. If μ is a semiflexible-*OR*-matching of Q w.r.t. D, then the result of Algorithm SFORD contains μ or contains a matching that subsumes μ.*

Proof. According to Proposition 5.10, there exists a rooted subtree T of Q, such that μ is a semiflexible matching of T w.r.t. D. We use induction to show that either $\mathcal{M}^n$ contains (T, μ) or $\mathcal{M}^n$ contains a pair (T', μ') that subsumes (T, μ).

Let $v_0, \ldots, v_i$ be the first $i+1$ variables in the topological order over the variables of Q. We denote by T_i the *restriction* of T to $v_0, \ldots, v_i$. The variables of T_i are the elements of the intersection $V_i = \{v_0, \ldots, v_i\} \cap V_T$, where V_T are the variables of T. The edges of T_i are all the edges of T that connect two variables of V_i. We denote by μ_i the restriction of μ to $v_0, \ldots, v_i$. That is, $\mu(v) = \mu_i(v)$ for each v in V_i.

The induction hypothesis is the following. In $\mathcal{M}^i$ there is a pair (T'_i, μ'_i) that subsumes or equals to (T_i, μ_i). For $i = 0$, the pair (T_0, μ_0) is in $\mathcal{M}^0$, according to Step 2 of the algorithm. Next, we assume that the hypothesis holds for $\mathcal{M}^i$ and show that it holds for $\mathcal{M}^{i+1}$.

We examine two cases. The first case is when v_{i+1} is not a node of T. In this case, (T_{i+1}, μ_{i+1}) is equal to (T_i, μ_i) because even if an edge is added to T_i it is not an edge of T. If the pair (T'_i, μ'_i) is in $\mathcal{M}^{i+1}$ then $\mathcal{M}^{i+1}$ contains an element that subsumes (T_{i+1}, μ_{i+1}). Otherwise, $\mathcal{M}^{i+1}$ contains a pair $(\tilde{T}, \tilde{\mu})$ that is created by extending T'_i and μ'_i. If so, $(\tilde{T}, \tilde{\mu})$ subsumes (T'_i, μ'_i) and, thus, subsumes (T_{i+1}, μ_{i+1}).

In the second case, v_{i+1} is a node of T. Let T'_{i+1} be a tree that is produced by adding to T'_i the node v_{i+1} and the edge of T that enters v_{i+1}. Since $v_0, v_1, \ldots, v_n$ are sorted topologically, T' is a rooted tree. Let μ'_{i+1} be $\mu'_i \cup \{(v_{i+1}, \mu(v_{i+1}))\}$, i.e., an extension of μ'_i to v_{i+1}.

Because μ is a semiflexible matching of T, all the paths of T satisfy the SF-condition w.r.t. μ. Specifically, the path to v_{i+1}, in T'_{i+1}, is also a path in T. Hence, this path satisfies the SF-Condition w.r.t. μ_{i+1}. So, (T'_{i+1}, μ'_{i+1}) is in an element of $Ext_{sf\text{-}or}(T'_i, \mu'_i, v_{i+1})$ and, consequently, it is an element of $\mathcal{M}^{i+1}$ or subsumed by an element of $\mathcal{M}^{i+1}$. Finally, the induction hypothesis holds for $\mathcal{M}^{i+1}$, because (T'_{i+1}, μ'_{i+1}) subsumes or is equal to (T_{i+1}, μ_{i+1}). $\square$

Proposition 5.14 (Soundness) *Let Q be a dag query and D be a database. When computed over Q and D, Algorithm SFORD produces only maximal semiflexible-*OR*-matchings.*

Proof. Suppose that μ is a matching in the result of Algorithm *SFORD*. In this case, according to Step 4 of the algorithm, there exists a tree T such that $\mathcal{M}^n$ contains the pair (T, μ).

For each i and every pair (T_i, μ_i) in $\mathcal{M}^i$, it holds that μ_i is a semiflexible matching of T_i w.r.t. D. This is shown by induction on i. For $i = 0$, $\mathcal{M}^0$ containts only a single element comprising a mapping of the query root to the database root. Thus, the induction hypothesis holds. For $i > 0$, according to Definition 5.11, if μ_{i-1} is a semiflexible matching of T_{i-1}, then $Ext_{sf\text{-}or}(T_{i-1}, \mu_{i-1}, v)$ consists of pairs (T_i, μ_i), such that μ_i is a semiflexible matching of T_i w.r.t. D. This shows that μ is a semiflexible matching of T w.r.t. D.

To see why μ is a maximal matching, consider the case where there exists μ'—a semiflexible-OR-matching of Q w.r.t. D that subsumes μ. In this case, according to Proposition 5.13, $\mathcal{M}$ will include either μ' or a matching that subsumes μ'. Since subsumed matching are removed, it follows that μ is not returned by the algorithm. This is a contradiction to our assumption. $\square$

We now analyze the complexity of Algorithm *SFORD*. For each variable v in Q, we denote by *fan-in*(v) the number of edges that enter v in Q.

Proposition 5.15 (Complexity) *Let Q be a dag query and D be a database. The runtime of Algorithm SFORD is $O(d^2 + q^2mr(d + mr))$, where q is the size of the query, d is the size of the database, m is the size of the result and r is the value of the multiplication $\prod_{v \in Q, v \neq r_Q}$ fan-in(v).*

Proof. Computing the topological order over the nodes of Q, in Step 1, can be done in $O(q)$. Step 2 requires a costant time. In Step 3, there are n iterations, where n

is the number of variables in Q. In each iteration, the elements of $\mathcal{M}^i$ are extended. We will show later that there can be at most mr elements in $\mathcal{M}^i$.

An extension of a pair (T, μ), by adding a mapping to a variable v, is computed in $O(dq)$. In the extension, in the worst case, all the objects of D are tested. Testing if an object o can be assigned to v is by verifying that all the ancestors of v are mapped to objects that are connected to o by a path. In Q there are, at most, n ancestors of v. We assume that it is in $O(1)$ to test if a pair of objects is connected. To make this assumption true, we create, in a preprocessing step, a suitable data structure that tells which two database objects are connected by a path. Creating such a data structure can be done in $O(d^2)$.

Removing subsumed elements from $\mathcal{M}^i$ requires $O(q(mr)^2)$, since in $\mathcal{M}^i$ there are at most mr elements and because each subtree and each matching have an $O(q)$ size. In total, the time complexity is $O(q + d^2 + q(qdmr + q(mr)^2))$.

To complete the proof, we provide a bound on the number of elements in the sets $\mathcal{M}^i$. Consider a matching μ in the result set $\mathcal{M}$. Let μ_i be the restriction of μ to $v_0, \ldots, v_i$. We say that there is a *repetition of size* k $(k \geq 1)$ w.r.t. μ, in $\mathcal{M}^i$, if there are $k+1$ pairs $(T_1, \mu_i), \ldots, (T_{k+1}, \mu_i)$ in $\mathcal{M}^i$. That is, the restriction of μ to the first i variables, in the topological sort, appears in $k+1$ different elements of $\mathcal{M}^i$.

Suppose that μ is a matching in the result set $\mathcal{M}$. In $\mathcal{M}^0$ there are no repetitions w.r.t. μ, because $\mathcal{M}^0$ contains a single element. In $\mathcal{M}^1$ there is a repetition of size *fan-in*(v_1) w.r.t. μ, because *fan-in*(v_1) edges enter v_1 in Q. We continue by induction and see that, in $\mathcal{M}^i$, there is a repetition of size *fan-in*$(v_1) \cdot$ *fan-in*$(v_2) \cdots$ *fan-in*(v_i) w.r.t. μ. This shows that for each $1 \leq i \leq n$, in $\mathcal{M}^i$, there is a repetion of size less than or equal to r w.r.t. each matching of the result. Thus, there are, at most, mr elements in $\mathcal{M}^i$. □

Corollary 5.16 (Complexity for Tree Queries) *Let Q be a tree query and D be a database. The runtime of Algorithm SFORD is $O(d^2 + q^2m(d + m))$, where q is the size of the query, d is the size of the database and m is the size of the result.*

Proof. Since in a tree each node, other than the root, has exactly one incoming edge, the value of the expression $\prod_{v \in Q, v \neq r_Q} \textit{fan-in}(v)$ is 1. □

Evaluating Cyclic Queries

Evaluation of cyclic queries is similar to the evaluation of dag queries. The main principle is to iteratively construct all the subtrees of the query and compute the semiflexible matchings w.r.t. these trees.

Next, we describe Algorithm *SFORG* that computes semiflexible-OR-matchings of queries that are arbitrary graphs. Consider a query Q with $n+1$ variables and a database D. As in Algorithm *SFORD*, which computes semiflexible-OR-matchings of dag queries, we create in *SFORG* sets of pairs $\mathcal{M}^0, \ldots, \mathcal{M}^n$. The set $\mathcal{M}^0$ consists of a pair (T_0, μ_0) where T_0 has just one node, the query root, and μ_0 is the mapping of the query root to the database root. For each i, $\mathcal{M}^i$ is the set of all pairs (T_i, μ_i) such that T_i is a subtree of Q with exactly i edges and μ_i is a semiflexible matching of T_i w.r.t. the database D. Notice that there are n sets since, for a query with $n+1$ nodes, a subtree of the query cannot have more than n edges.

The sets $\mathcal{M}^1, \ldots, \mathcal{M}^n$ are created in n iterative steps. In the ith iteration, the set $\mathcal{M}^i$ is created from the elements of $\mathcal{M}^{i-1}$ as follows. We create an empty set $\mathcal{M}^i$. Then, for each pair (T_{i-1}, μ_{i-1}) in $\mathcal{M}^{i-1}$ and for each variable v, such that v is not in T_{i-1}, the set $\textit{Ext}_{\textit{sf-or}}(T_{i-1}, \mu_{i-1}, v)$ is added to $\mathcal{M}^i$. Duplications are discarded.

During the creation of the sets $\mathcal{M}^1, \ldots, \mathcal{M}^n$, we create an additional set denoted $\mathcal{M}$. The set $\mathcal{M}$ consists of matchings that cannot be extended. At each step of creating a set $\mathcal{M}^i$, if a pair (T_{i-1}, μ_{i-1}) is such that $\textit{Ext}_{\textit{sf-or}}(T_{i-1}, \mu_{i-1}, v)$ is empty for every v, then μ_{i-1} is added to $\mathcal{M}$. After the nth iteration, for each pair (T_n, μ_n) of $\mathcal{M}^n$, μ_n is added to $\mathcal{M}$. The final step of the algorithm is the creation of the matchings from the elements of $\mathcal{M}$. In this step, each matching μ in $\mathcal{M}$ is extended by mapping the undefined variables to null. Subsumed tuples and duplications are removed from $\mathcal{M}$ and $\mathcal{M}$ is returned.

Soundness and completeness of Algorithm *SFORG* are proven in a similar way to the proof of Algorithm *SFORD*.

Proposition 5.17 (Time Complexity) *When computed over a query Q and a database D, Algorithm SFORG has a time complexity $O(d^2 + q^3dmr + q(mr)^2)$, where q is the size of the query, d is the size of the database, m is the size of the result and r is the value of the multiplication $\prod_{v \in Q, v \neq r_Q}(\textit{fan-in}(v) + 1)$.*

Proof. In $O(d^2)$ runtime, it is possible to construct a data structure that, for each pair of database objects, can answer in $O(1)$ time whether this pair of objects is connected by a path.

In the algorithm there are q iterations. In each iteration, the elements of a set $\mathcal{M}^i$ are extended. There are at most mr elements in $\mathcal{M}^i$ as will be proved later. For each pair (T_i, μ_i) in $\mathcal{M}^i$, there are at most q edges that can extend T_i. For each edge e that can be used to extend T_i, there are at most d objects that can be assigned to the target node v of e. Testing if an assignment of o to v satisfies the conditions of the semiflexible semantics requires to check that o is on a path with all the objects that are assigned to ancestors of v. There can be at most q ancestors to v. Thus, the runtime for a single iteration is $O(mrqdq)$ and q iterations are computed in $O(q^3dmr)$ time. Removing subsumed matchings, at the final step, is in $O((mr)^2q)$ time.

We show why there are at most mr elements in each set $\mathcal{M}^i$. First, note that there are no subsumed pairs in the set $\mathcal{M}^i$. This is because, for all the pairs in $\mathcal{M}^i$, all the trees in these pairs have exactly i edges. Secondly, note that $\mathcal{M}^i$ is a set and does not include duplicate elements. Thirdly, for each pair (T_i, μ_i) in $\mathcal{M}^i$, the matching μ_i is a semiflexible matching of T_i. Thus, μ_i is either equal to a matching in the result or subsumed by a matching in the result. The last three observations show that it is sufficient to count the number of of pairs (T_{i_l}, μ_{i_l}) in $\mathcal{M}^i$, such that there is a semiflexible-OR-matching of Q w.r.t. D that subsumes μ_{i_l}.

Let μ be a semiflexible-OR-matching of Q w.r.t. D. We claim that the number of pairs (T_{i_l}, μ_{i_l}) in $\mathcal{M}^i$, such that μ subsumes μ_{i_l}, is less than or equal to r. To prove this claim, let $r_Q, v_{j_1}, \ldots, v_{j_k}$ be the variables of Q that μ maps to non-null values. Consider a subset V_T of $\{r_Q, v_{j_1}, \ldots, v_{j_k}\}$ and let T be a tree whose variables are V_T. For each $1 \leq l \leq k$, in T there is at most one edge that enters v_{j_l}. There are *fan-in*(v_{j_l}) edges that enter v_{j_l} in Q. So, in T, for each $1 \leq l \leq k$, either there in no edge that enters v_{j_l} or the edge that enters v_{j_l} is choosen from the *fan-in*(v_{j_l}) edges that enter v_{j_l} in Q. Hence, there are at most $\prod_{1 \leq l \leq k}(\textit{fan-in}(v_{j_l}) + 1)$ possibilities to choose the edges for a tree over V_T. For the case where μ maps all the variables of Q to non-null values, the number of subtrees of Q is $\prod_{v \in Q, v \neq r_Q}(\textit{fan-in}(v) + 1)$, i.e., equals to r. □

5.2.2 Computing Flexible-OR Matchings and Flexible-Weak Matchings

In this section, we show how to compute flexible-OR matchings and flexible-weak matchings in polynomial runtime, in the size of the input and output. The main principle behind the methods that we provide is to pose queries, under the OR-semantics (weak semantics), to the reachability graph instead of posing the queries to the database itself.

In Theorem 3.55 we showed that the set of flexible matchings w.r.t. a database is equal to the set of rigid matchings with respect to the reachability graph. We now apply this principle to the combined semantics.

Proposition 5.18 (Computation Reduction) *Let Q be a query, D be a database and RG(D) be the reachability graph of D. The following two sets are equal.*

- *The set of flexible-*OR*-matchings of Q w.r.t. D.*
- *The set of* OR*-matchings of Q w.r.t. RG(D).*

Proof. Let μ be a matching of Q w.r.t. D. Consider the following statements.

1. The matching μ is a flexible-OR-matching of Q w.r.t. D.
2. There exists a rooted fragment F of Q, such that μ is a flexible matching of F w.r.t. D.
3. There exists a rooted fragment F of Q, such that μ is a rigid matching of F w.r.t. *RG(D)*.

4. The matching μ is an OR-matching of Q w.r.t. *RG(D)*.

According to Definition 5.3, Statement 1 is true if and only if Statement 2 is true. Statement 2 is true if and only if Statement 3 is true, due to Theorem 3.55. Statement 3 and Statement 4 are either both true or both false, according to Proposition 5.2. □

For weak queries, the next proposition describes a principle that is similar to the the principle of Proposition 5.18.

Proposition 5.19 (Computation Reduction) *Let Q be a query, D be a database and $RG(D)$ be the reachability graph of D. The following two sets are equal.*

- *The set of flexible-weak matchings of Q w.r.t. D.*
- *The set of weak-matchings of Q w.r.t. RG(D).*

The proof of Proposition 5.19 is similar to the proof of Proposition 5.18.

Let Q be a query and D be a database. Computing flexible-OR-matchings (flexible-weak matchings) of Q w.r.t. D is a two-step process. Firstly, *RG(D)*—the reachability graph of D—is constructed. Secondly, Q is evaluated under the OR-semantics (weak-semantics) w.r.t. *RG(D)*. Correctness follows from Proposition 5.18 (Proposition 5.19).

Theorem 5.20 (Polynomial Time Complexity) *Let Q be a query and D be a database. There exists an algorithm that computes the flexible-*OR*-matchings of Q w.r.t. D in polynomial time in the size of the input and output. This theorem is also true if we replace "flexible-*OR*-matchings" with "flexible-weak-matchings".*

Proof. The theorem follows from the following two facts. First, the size of the reachability graph is polynomial in the size of the database. Secondly, as shown in Chapter 3, there exists an algorithm that computes OR-matchings (weak-matchings) in polynomial time in the size of the input and output. □

Query / Semantics	Semiflexible-OR	semiflexible-weak	flexible-OR	flexible weak
path query	PTIME	PTIME	PTIME	PTIME
tree query	PTIME	PTIME	PTIME	PTIME
dag query	NP-Complete	Open Problem	PTIME	PTIME
cyclic query	NP-Complete	Open Problem	PTIME	PTIME

Table 5.3: The complexity of query evaluation under the four combined semantics. (NP-Complete results refer to checking non-emptiness.)

5.3 Summary of Contributions

In this chapter, we presented four new semantics, namely, the semiflexible-OR, the semiflexible-weak, the flexible-OR and the flexible-weak semantics. Query evaluation was investigated under these four new semantics. It was shown that under the semiflexible-OR semantics, a tree query can be evaluated in polynomial time in the size of the input and the output. For a DAG query, there is no polynomial-time algorithm, in the size of the input and the output, unless P = NP. Under the flexible-OR and the flexible-weak semantics, any arbitrary query can be evaluated in polynomial time in the size of the input and the output. When the query is a tree, the semiflexible-weak semantics and the semiflexible-OR semantics are the same. Thus, under the semiflexible-weak semantics, tree queries can be computed in polynomial time in the size of the input and the output. For DAG queries and cyclic queries it is still an open question whether such an algorithm exists.

Chapter 6

Full Disjunctions

In this chapter, we present the *full-disjunction approach* for oblivious querying. The manner of oblivious querying, in the previous chapters, required from users to provide a query in the form of a rooted graph. Since providing a rooted graph could be cumbersome, in the full disjunction approach, the semistructured database is automatically transformed into a relation and the user queries this relation using a traditional relational query language. Thus, the user is oblivious to the structure of the semistructured database.

The outline of this chapter is as follows. In Section 6.1 a short introduction is given. Section 6.2 provides some preliminary definitions and, in particular, full disjunctions are defined. Section 6.3, explains how to transform a semistructured database into a universal relation. Such a transformation facilitates the querying of semistructured databases and admits the querying of a single relation with a known schema—instead of querying a graph whose schema is unknown or is more complicated. In Section 6.4, it is shown how to reduce the evaluation of full disjunctions to the evaluation of maximal weak matchings. This reduction provides an algorithm, for full disjunctions, that has polynomial-time input-output complexity. Section 6.5 discusses projections of a full disjunction on a given set of attributes. It is shown that for the case of relations with γ-acyclic schemas (see Appendix A, for a definition of γ-acyclic hypergraphs), there is an algorithm that computes the projection of a full disjunction in polynomial time in the size of the input and the output. However, in the general case, i.e., when the relation schemas are not γ-acyclic, such algorithm does not exists, unless P = NP. Section 6.6 presents a generalization of full disjunctions that facilitates integration of relational data by means of join conditions that are more general than just equalities. Finally, in Section 6.7, we discuss the contribution of this chapter and conclude.

6.1 Motivation

For a start, we explain the motivation behind full disjunctions. The usual way to integrate data from several relations is by computing a natural join of the relations. However, in a natural join, there could be *dangling tuples.* Given a set of relations $r_1, \ldots, r_n$, a tuple t, in some relation r_i with schema R_i, is a dangling tuple if the the projection of the natural join of $r_1, \ldots, r_n$ on the attributes of R_i does not include t. Obviously, in a natural join, dangling tuples are not part of the result. That is, they do not appear as part of any tuple of the result. In this sense, natural joins cause a information loss.

Consider a set of relations $\mathcal{R} = \{r_1, \ldots, r_n\}$. Suppose that we take all the subsets of $\mathcal{R}$, compute the natural join w.r.t. each subset and add the tuples of these natural joins to a set $\mathcal{J}$. In $\mathcal{J}$, no dangling tuple is lost. However, there are two fundamental problems in $\mathcal{J}$ that prevent $\mathcal{J}$ from being useful.

One problem with J is that tuples in the result may come from disconnected relations. We consider two relations as directly connected if their schemas have a non-empty intersection. Two relations r and r' are connected if one of the following two conditions is true: *(1)* r and r' are directly connected; *(2)* there are two connected relations $\tilde{r}$ and $\tilde{r}'$ such that $\tilde{r}$ is directly connected to r and $\tilde{r}'$ is directly connected to r'. Joining disconnected tuples should be avoided. Otherwise, we would not be able to distinguish between the next two cases. First, a case where two pieces of information are in one tuple of the result because they are related. Second, a case where the two pieces of information appear in one tuple of the result because they come from disconnected tuples.

A second problem with $\mathcal{J}$ is that it contains tuples that are not maximal. More specifically, consider a tuple t that is the join of the tuples $t_{i_1}, \ldots, t_{i_k}$, where

$t_{i_1}, \ldots, t_{i_k}$ are tuples of the relations $r_{i_1}, \ldots, r_{i_k}$, respectively. If t is in $\mathcal{J}$, then for each subset of $t_{i_1}, \ldots, t_{i_k}$, the join of the tuples in this subset will also be in $\mathcal{J}$. This may cause an exponential blowup in the size of $\mathcal{J}$, without adding information.

In an integration of two relations, the *outerjoin* can give an answer to the three problem that were presented above: loss of dangling tuples, connecting unrelated pieces of data and an exponential blowup due to keeping non-maximal elements. An outerjoin of two relations r_1 and r_2 is a relation that consists of all the tuples in the natural join of r_1 and r_2. In addition, the outerjoin contains all the tuples of r_1 (r_2) that are not joined with any tuple of r_2 (r_1), padded with null values.

For integration of more than two relations, the semantics of outerjoins is problematic, since outerjoins are not associative. Galiando-Legaria [34] proposed full disjunctions, which are commutative and associative, as an alternative to outerjoins. Essentially, the full disjunction of a set of relations is obtained by taking all joins of any subsets of relations, excluding joins that involve a Cartesian product, and removing subsumed tuples.

In full disjunctions, dangling tuples are not discarded. In addition, the two problems that were discussed above are prevented—the problem of tuples that are produced by a join of disconnected tuples and the problem of causing an exponential blowup due to keeping non-maximal tuples in the result.

Rajaraman and Ullman [55] showed that the full disjunction of some given relations can be evaluated by a natural outerjoin sequence if and only if the relation schemas are connected and γ-acyclic. It is easy to see that the sequence of outerjoins has a polynomial input-output complexity. They did not provide a polynomial time algorithm, under input-output complexity, for computing full disjunction of relations with schemas that are not γ-acyclic.

In the next section, we provide a formal definition of full disjunctions. We describe how to use full disjunction for oblivious querying of semistructured data and we present an algorithm that computes full disjunctions in polynomial time in the size of the input and output. The runtime of the algorithm remains polynomial even for relations with schemas that are not γ-acyclic.

6.2 Preliminaries

Let $r_1, \ldots, r_n$ be relations with relation schemas $R_1, \ldots, R_n$, respectively. The jth tuple of r_i is denoted as t_{ij}. Given a subset $X \subseteq R_i$, the projection of the tuple t_{ij} on X is denoted as $t_{ij}[X]$.

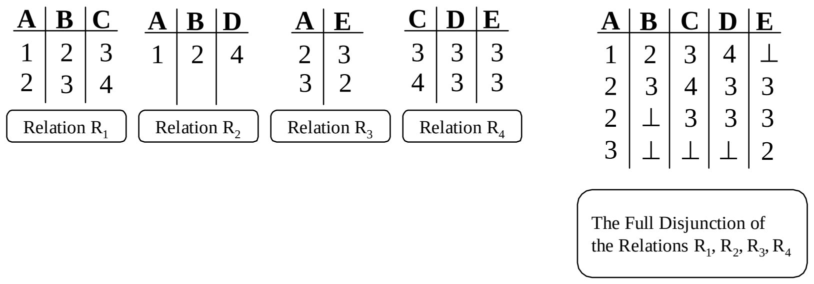
Two distinct relation schemas R_i and R_j are *connected* if $R_i \cap R_j$ is non-empty. This definition is generalized to m ($m \geq 1$) distinct relation schemas as follows. The *schema graph* of $R_{i_1}, \ldots, R_{i_m}$ consists of a node for each R_{i_j} and edges in both directions between R_{i_h} and R_{i_k} ($1 \leq h < k \leq m$) if R_{i_h} and R_{i_k} are connected, i.e., $R_{i_h} \cap R_{i_k}$ is non-empty. We say that the relation schemas $R_{i_1}, \ldots, R_{i_m}$ are connected if their schema graph is connected. (There is an equivalent definition of connectivity that is based on the notion of the hypergraph of $R_{i_1}, \ldots, R_{i_m}$.) We say that tuples $t_{i_1 j_1}, \ldots, t_{i_m j_m}$ from m ($m \geq 1$) distinct relations are connected if their relation schemas are connected. Note that a single tuple $t_{i_1 j_1}$ is connected.

Two connected tuples $t_{i_1 j_1}$ and $t_{i_2 j_2}$ ($i_1 \neq i_2$) are *join consistent* if $t_{i_1 j_1}[R_{i_1} \cap R_{i_2}] = t_{i_2 j_2}[R_{i_1} \cap R_{i_2}]$. More generally, m tuples $t_{i_1 j_1}, \ldots, t_{i_m j_m}$ from m distinct relations are join consistent if every pair of connected tuples $t_{i_h j_h}$ and $t_{i_k j_k}$ is join consistent. The natural join of $t_{i_1 j_1}, \ldots, t_{i_m j_m}$, denoted $\bowtie_{k=1}^{m} t_{i_k j_k}$, is either empty or has one tuple over the attributes of $\cup_{k=1}^{m} R_{i_k}$. It has one tuple if and only if $t_{i_1 j_1}, \ldots, t_{i_m j_m}$ are join consistent.

A *universal* tuple is defined over the set of all the attributes, namely $\cup_{i=1}^{n} R_i$, and its columns are filled with nulls as well as non-null values. If u is non-null exactly on the attributes of Z, then $u[Z]$ is called the *non-null portion* of u and is denoted as $\hat{u}$. The universal tuple u is called an *integrated* tuple if there are $m \geq 1$ connected and join-consistent tuples $t_{i_1 j_1}, \ldots, t_{i_m j_m}$, such that $\hat{u} = \bowtie_{k=1}^{m} t_{i_k j_k}$. The tuples $t_{i_1 j_1}, \ldots, t_{i_m j_m}$ are called *generators* of u.

A universal tuple v *subsumes* a universal tuple u, denoted $u \sqsubseteq v$, if $v[Z] = u[Z]$, where u is non-null exactly on Z. Given a set $\mathcal{D}$ of universal tuples, $u \in \mathcal{D}$ is *maximal* if it is not subsumed by any other tuple of $\mathcal{D}$.

Definition 6.1 (Full Disjunction) *The* full disjunction *of the relations $r_1, \ldots, r_n$ is the set of all maximal integrated tuples that can be generated from m ($1 \leq m \leq n$)*



A	B	C
1	2	3
2	3	4

Relation R_1

A	B	D
1	2	4

Relation R_2

A	E
2	3
3	2

Relation R_3

C	D	E
3	3	3
4	3	3

Relation R_4

A	B	C	D	E
1	2	3	4	⊥
2	3	4	3	3
2	⊥	3	3	3
3	⊥	⊥	⊥	2

The Full Disjunction of the Relations R_1, R_2, R_3, R_4

Figure 6.1: The full disjunction of four relations.

tuples of $r_1, \ldots, r_n$.

Example 6.2 In Figure 6.1, the full disjunction of four relations is depicted. The first tuple of the full disjunction is created as a join of the the first tuples of the relations R_1 and R_2. The second tuple of the full disjunction is the join of the second tuple of R_1, the first tuple of R_3 and the second tuple of R_4. The third tuple is a join of the first tuples of R_3 and R_4. Finally, the fourth tuple is created from the second tuple of R_3.

6.3 Transforming Semistructured Data to Relational

We now show how full disjunctions can be used for oblivious querying of semistructured databases, provided that in the database there are no repeating labels on simple paths. To facilitate querying of irregular semistructured data, the database is transformed into a relation. Intuitively, the transformation is as follows. First, for each simple database path, from the root to a leaf, a tuple is created. Secondly, the tuples are divided to relations according to the labels on the paths that define them. Finally, the full disjunction of the relations is computed and returned. Next, we provide some definitions and then introduce the transformation.

Definition 6.3 (Induced Schema and Induced Tuple) *Let D be a semistructured database and $\phi = o_0 l_1 o_1 \cdots o_{n-1} l_n o_n$ be a simple path from the root of D to a leaf o_n. Furthermore, assume that there are no repeated labels on ϕ. The* induced schema *of ϕ is the set Sch_ϕ that consists of the labels on ϕ, i.e., $l_1, \ldots, l_n$. The* induced tuple *of ϕ is a tuple t_ϕ over the schema Sch_ϕ. For $1 \le i \le n-1$, $t_\phi[l_i]$ is the oid of o_i. For the atomic node o_n, $t_\phi[l_n]$ is the atomic value that is attached to o_n, in D.*

Consider a semiflexible database D that has no repeated labels on simple paths. The method for transforming D into a relation is as follows.

Transform a Semistructured Database to a Relation

1. Let $\mathcal{R}$ be an empty set of relations.
2. Compute all the simple paths, from the root to a leaf, in D.
3. For each path ϕ that was found in Step 1, compute the induced schema of ϕ, Sch_ϕ, and the induced tuple of ϕ, t_ϕ. If there exists a relation r in $\mathcal{R}$ with the schema Sch_ϕ, then add t_ϕ to r. Else, create a new relation r with the schema Sch_ϕ, add t_ϕ to r and add r to $\mathcal{R}$.
4. Compute $\mathcal{F}$, the full disjunction of the relations of $\mathcal{R}$ and return $\mathcal{F}$.

Example 6.4 In this example we demonstrate the full-disjunction approach by applying the transformation to the movie database of Figure 2.1. The tables that are created in the first step of the transformation are presented in Appendix B. The result of the transformation is shown in Table 6.1.

Without providing any information, the user receives a table that she can query using a relational query language. Note that a projection on the attributes "Movie" and "Actor" produces all the movie-actor pairs in which the actor acted in the movie.

The full disjunction approach is less accurate than the combined semantics that were presented in Chapter 5. By this we mean that there can be tuples in $\mathcal{F}$ where the tuples indicate that two values or objects are related, while actually these values or object are not related. This happens, for example, in Line 6 of the table. In this line, the T.V. Series "Twin Peaks" is attached to the year 1984, although 1984 is actually the year when "Dune" was produced. Thus, the full disjunction approach

Movie	Title	Year	Actor	Director	Name	Filmography	T.V. Series
11	Star Wars	1977	21	⊥	Mark Hamill	⊥	⊥
11	Star Wars	1977	22	⊥	Harrison Ford	⊥	⊥
11	Star Wars	1977	⊥	41	George Lucas	⊥	⊥
12	León	⊥	25	⊥	Natalie Portman	⊥	⊥
13	Magnolia	⊥	⊥	⊥	⊥	⊥	⊥
29	Twin Peaks	1984	14	⊥	Kyle MacLachlan	15	28
29	Twin Peaks	1984	14	42	David Lynch	15	28
29	Dune	1984	14	⊥	Kyle MacLachlan	15	⊥
29	Dune	1984	14	42	David Lynch	15	⊥

Table 6.1: The result of transforming the movie database into a relation, using full disjunction.

is suitable for naive users but it may have errors that the flexible-weak semantics (or the other combined semantics) of Chapter 5 would not have.

6.4 Computing Full Disjunctions

In order to compute the full disjunction, we need to view tuples as objects and relation schemas as variables. By a slight abuse of notation, t_{ij} denotes both the tuple and its oid. Similarly, R_i denotes both the relation schema and its corresponding variable.

We construct a database $D = (O, E_D, r_D, \alpha)$ and n queries $Q_i = (V, E_{Q_i}, r_{Q_i})$ $(1 \leq i \leq n)$ as follows. D has a root r_D and an object for each tuple t_{ij}. There are edges in both directions between every pair of connected and join-consistent tuples. There is also an edge from the root to every tuple. An edge that enters a tuple of r_k $(1 \leq k \leq n)$ is labeled with l_k. The values of the atomic objects can be chosen arbitrarily.

The query $Q_i = (V, E_{Q_i}, r_{Q_i})$ is constructed from the schema graph of $R_1, \ldots, R_n$ by adding a root r_{Q_i} and an edge from the root to each R_i. An edge that enters R_k $(1 \leq k \leq n)$ is labeled with l_k. Note that only one connected component of the schema graph of $R_1, \ldots, R_n$ can be reached from the root of Q_i.

Consider a weak matching μ of some Q_i w.r.t. D. Recall that μ assigns to each variable of Q_i either a tuple or null. Let $g(\mu)$ denote the tuples in the image of μ. Since μ is a weak matching, the tuples of $g(\mu)$ are connected and join consistent. Therefore, the tuples of $g(\mu)$ generate an integrated tuple that is denoted as $\bowtie(\mu)$.

Proposition 6.5 (Soundness and Completeness) *The tuple u is an integrated tuple of $r_1, \ldots, r_n$ if and only if there is a weak matching μ of some Q_i w.r.t. D, such that $u = \bowtie(\mu)$.*

Proof. As noted earlier, if μ is a weak matching of Q_i w.r.t. D, then $\bowtie(\mu)$ is an integrated tuple. For the other direction, consider an integrated tuple u that is generated by the tuples $t_{i_1 j_1}, \ldots, t_{i_m j_m}$. Let μ be the assignment for Q_{i_1} w.r.t. D that is defined as follows: $\mu(r_{Q_{i_1}}) = r_D$, $\mu(R_{i_k}) = t_{i_k j_k}$ $(1 \leq k \leq m)$, and $\mu(R_i) = \perp$ for any R_i that is not among $R_{i_1}, \ldots, R_{i_m}$. Note that all the edges that enter R_{i_k} and all the edges that enter $t_{i_k j_k}$ $(1 \leq k \leq m)$ have the same label l_{i_k}. Moreover, $t_{i_1 j_1}, \ldots, t_{i_m j_m}$ are connected and join consistent. Thus, μ is a weak matching. By the construction of μ, we have that $u = \bowtie(\mu)$. This proves the proposition. $\square$

The full disjunction $\mathcal{F}$ of $r_1, \ldots, r_n$ can be computed as follows. First, compute the set $\mathcal{M}_w(Q_i, D)$ for $i = 1, 2, \ldots, n$. Let $\mathcal{M}$ be obtained from $\cup_{i=1}^{n} \mathcal{M}_w(Q_i, D)$ by removing subsumed matchings.[1] Finally, $\mathcal{F} = \{\bowtie(\mu) \mid \mu \in \mathcal{M}\}$.

Theorem 6.6 (Complexity) *The set $\mathcal{F}$ is the full disjunction of the relations $r_1, \ldots, r_n$ and it can be computed in $O(n^5 s^2 f^2)$ time, where n is the number of relations, s is the total size of all the relations and f is the size of $\mathcal{F}$.*

Proof. From Proposition 6.5, it follows that $\mathcal{F}$ consists of integrated tuples. To complete the proof that $\mathcal{F}$ is the full disjunction, it should be observed that if μ_1 and μ_2 are two assignments of Q_i w.r.t. D, such that $\mu_1 \sqsubseteq \mu_2$, then $\bowtie(\mu_1) \sqsubseteq \bowtie(\mu_2)$. Therefore, it is sufficient to compute only the maximal weak matchings in order to generate all the maximal integrated tuples.

[1] Technically, $\cup_{i=1}^{n} \mathcal{M}_w(Q_i, D)$ has matchings of all the Q_i, but all these matchings are for the same set of variables (ignoring the root) and therefore, subsumptions among them are well defined.

In order to show that the running time of the algorithm is polynomial in the size of the input and the output, it is necessary to show that $\mathcal{F}$ has only maximal integrated tuples. We will derive a contradiction by assuming that there are two distinct integrated tuples u_1 and u_2 in $\mathcal{F}$, such that $u_1 \sqsubseteq u_2$. Let μ_1 and μ_2 be two maximal weak matchings of Q_i and Q_j, respectively, such that $u_1 = \bowtie(\mu_1)$ and $u_2 = \bowtie(\mu_2)$. Essentially, it can be shown that there is a weak matching μ of Q_j w.r.t. D, such that $u_2 = \bowtie(\mu)$ and μ subsumes both μ_1 and μ_2. This contradicts the fact that μ_1 and μ_2 are maximal in $\mathcal{M}$.

The size of each query Q_i is $O(n^2)$ and the size of the database D that is constructed from the relations is s^2. The size of each $\mathcal{M}_w(Q_i, D)$ is no more than the size of $\mathcal{F}$. In general, the size of the product graph is $O(pd)$, where p is the size of the query and d is the size of the database. However, in the computation of the full disjunction, for each node t_{ij} of D, the product graph $Q_k \times D$ $(1 \leq k \leq n)$ has at most one node $(R_i, t_{ij}) \in Q_k \times D$ that is reachable from the root. Consequently, the size of the portion of $Q_k \times D$ that is reachable from the root is $O(d)$. Therefore, the size of the stratum traversal is $O(pd)$ and the time to compute each $\mathcal{M}_w(Q_i, D)$ is $O(n^4 s^2 f^2)$. Hence, $\mathcal{M}$ is computed in $O(n^5 s^2 f^2)$ time. Removing subsumed matchings from $\cup_{i=1}^{n} \mathcal{M}_w(Q_i, D)$ takes $O(n^3 f^2)$ time. The final step of constructing $\mathcal{F}$ from $\mathcal{M}$ takes $O(f)$ time. Thus, the running time is $O(n^5 s^2 f^2)$. □

Theorem 6.6 shows that full disjunction can be computed in polynomial runtime, under input-output complexity. However, in the polynomial expression, the exponent of n is 5, where n is the number of relations in the input. Hence, in order to efficiently compute the full disjunction in the case where the input comprises many relations, optimization techniques should be applied. Next, we present an optimization rule that allows to improve the evaluation runtime for many cases of full disjunctions. We start by providing some definitions.

Consider a connected undirected graph G. A *separation* of G w.r.t. a node v are two subgraphs of G, G_1 and G_2, such that the following holds. *(1)* v is a node in both G_1 and G_2. *(2)* Every node of G, except v, is either in G_1 or in G_2 but not in both. *(3)* For each two nodes v_1 and v_2 of G_1 and G_2, respectively, there is no edge between v_1 and v_2 in G, unless v_1 is v or v_2 is v. We say that G_1 and G_2 are a *non-trivial separation* of G if they are a separation in which each of the two graphs has at least two nodes.

An *articulation node* of G is a node whose removal will disconnect G. That is, a node v, in G, is an articulation node if removing v from G (and also removing from G all the edges that are incident on v) creates a subgraph of G that is not connected. It is easy to see that there exists a non-trivial separation of G w.r.t. v if and only if v is an articulation node of G. Computing all the articulation nodes of G is in $O(|E|)$ where $|E|$ is the number of edges in G (see [27] for details).

We provide some required notations. Consider a set of relations $r_1, \ldots, r_n$ with schemas $R_1, \ldots, R_n$, respectively. Let G be the schema graph of $R_1, \ldots, R_n$. By $FD(G)$ we denote the full disjunction of $r_1, \ldots, r_n$, i.e., the full disjunction of the relations that correspond to the schemas that are nodes of G. We use the symbol $\overset{o}{\bowtie}$ to denote the outerjoin operation. The next proposition shows how graph separations can assist in computing full disjunctions.

Proposition 6.7 (Applying an Outerjoin) *Suppose that $r_1, \ldots, r_n$ are relations with schemas $R_1, \ldots, R_n$, respectively. Let the schema graph of $R_1, \ldots, R_n$ be G and R_j be an articulation node of G. In addition, let G_1 and G_2 be a separation of G w.r.t. R_j. Then $FD(G_1) \overset{o}{\bowtie} FD(G_2)$ is equal to $FD(G)$.*

The proof of Proposition 6.7 is given in Appendix C.

Based on Proposition 6.7, we propose the *Separate-and-Join* method for computing full disjunctions. Consider a set of relations $r_1, \ldots, r_n$ with schemas $R_1, \ldots, R_n$, respectively. Separate-and-Join works recursively. First, the schema graph G of $R_1, \ldots, R_n$ is computed. If there is a separation G_1 and G_2 of G then Separate-and-Join is called with G_1 and with G_2. The results of these calls are joined using an outerjoin. The result of the outerjoin is returned. If there is no separation of G then the general evaluation algorithm (the one that is presented at the beginning of the section) is called.

Note that at the worst case, Separate-and-Join has the same runtime complexity as the general algorithm. However, in many cases, evaluation of a full disjunction by Separate-and-Join is performed as series of outerjoin operations.

6.5 Projections and Restrictions of Full Disjunctions

One way of generalizing the result of Theorem 6.6 is by considering the complexity of evaluating a projection of the full disjunction. Formally, the *projection problem* is the problem of computing the projection of the full disjunction on a given set of attributes X.

In some cases, it might be desirable to compute only those tuples of the full disjunction that are non-null on all the attributes of a given set X. Formally, the *restriction problem* is the problem of computing just those tuples of the full disjunction that are non-null on X. The following theorem shows that the restriction problem cannot be computed in polynomial time in the size of the input and the output, even if the relation schemas are α-acyclic (a definition of α-acyclic schemas is provided in Appendix A) and X has only two attributes.

Theorem 6.8 (NP-Completeness) *Let $r_1, \dots, r_n$ be relations with α-acyclic relation schemas and let X be a set of two attributes. Deciding whether the full disjunction has a tuple that is non-null on all the attributes of X is* NP*-complete.*

Proof. NP-hardness is shown by a reduction of the Hamiltonian-path problem. Let $G = (V, E)$ be a given directed graph, and let s and t be two nodes in G. The problem is to decide whether G has a directed path from s to t that goes through all the nodes of G, visiting each node exactly once.

Suppose that G has k nodes, denoted $n_1, \dots, n_k$, and m edges. We assume that $s = n_1$ and $t = n_k$. We construct the following $m + 2$ relations:

1. A relation r_s that has the relation schema $S(A, N_1)$ and contains the single tuple $(1, 1)$.
2. A relation r_t that has the relation schema $T(N_k, B)$ and contains the single tuple (k, k).

3. For each edge $e_l = (n_i, n_j)$ $(1 \leq l \leq m)$, we construct a relation r_l that has the relation schema $E_l(N_i, N_j)$ and contains the $k-1$ tuples: $(1, 2), (2, 3), \ldots, (k-1, k)$.

In addition to the above $m + 2$ relations, we also construct an empty relation r_α that has the set of all the attributes as its relation schema. The sole purpose of r_α is to ensure that the relation schemas of the $m + 3$ relations are α-acyclic.

Let $\mathcal{F}$ denote the full disjunction of the $m+3$ relations and let q be the projection of $\mathcal{F}$ on the attributes A and B.

It is easy to see that only the tuples $(1, k)$, $(1, \perp)$ and $(\perp, k)$ may appear in q. Furthermore, there is a Hamiltonian path in G from s to t if and only if the tuple $(1, k)$ is in the result of q. This shows NP-hardness.

To show membership in NP, note that we can guess an assignment of values and nulls to all the attributes of all the relation schemas, such that the attributes of X are assigned non-null values. We can then verify in polynomial time that the guess creates an integrated tuple. $\square$

The reduction that is used in the proof of Theorem 6.8 implies that the projection problem cannot be computed in polynomial time under input-output complexity if P $\neq$ NP. For the problem of deciding whether a tuple is in the projection of the full disjunction on a gives set of attributes, membership in NP is also shown as described in the proof of Theorem 6.8. Thus, we have the following corollary.

Corollary 6.9 **(NP-Completeness)** *Deciding whether a given tuple is in the projection of the full disjunction on X is* NP*-complete.*

Theorem 6.8 and Corollary 6.9 show that if $P \neq$ NP, then there cannot exists an algorithm, for either the restriction problem or the projection problem, that is polynomial under input-output complexity. It is easy to see that the projection problem and the restriction problem have polynomial-time algorithms if each relation schema either contains X or is disjoint from X. Note that, in particular, this condition is satisfied if X is a singleton.

A related problem to the projection problem is the evaluation of a projection of the natural join of n relations. Yannakakis [65] showed that this problem has a polynomial-time algorithm, in the size of the input and the output, if the relation schemas are α-acyclic. In the general case, deciding non-emptiness of the natural join of n relations is NP-Complete [49].

In [55], Rajaraman and Ullman showed that the full disjunction of $r_1, \ldots, r_n$ can be evaluated by a natural outerjoin sequence if and only if the relation schemas are connected and γ-acyclic. Their result can be generalized as follows.

Theorem 6.10 (Complexity) *The projection problem has a polynomial-time algorithm under input-output complexity if the relation schemas are γ-acyclic.*

The proof of Theorem 6.10 is given in Appendix C.

6.6 Generalizing Full Disjunctions

Full disjunction are based either on equijoins [34] or natural joins [55]. Quite frequently, however, the process of integrating information involves more general conditions than merely equality of attributes. Consider the scenario of piecing together information about people, in the absence of a unique ID for each person. Joining two records, simply because both carry the same name, might be error prone. A safer approach could be to join two records if the names are the same or similar (e.g., one has a middle initial and the other does not) and either the address or one of the phone numbers (e.g., home, office or mobile phone) are the same. A similar scenario is joining two records, such that one has a complete address while the other only contains the city and state. In this case the appropriate condition is that the city and state from the second record appear in the address of the first record.

The approach of Section 6.4 can be easily generalized to the above scenarios. We use the same queries as in Section 6.4 and use the same database, but the edges should reflect the new conditions. Thus, there are edges in both directions between R_i and R_j if some condition is specified for this pair of relation schemas. Similarly, there are edges in both directions between the tuples $t_{i_1 j_1}$ and $t_{i_2 j_2}$ if these two tuples satisfy the condition that is specified for their relation schemas, R_{i_1} and R_{i_2}.

Conditions of general types could be used for joining tuples. Essentially, we may assume that queries are formulated using a SELECT-FROM-WHERE clause and the WHERE clause is a conjunction of conditions $C_1 \wedge \ldots \wedge C_k$, where each C_h involves either one or two relations. If C_h involves just one relation r_i, then it is a selection that can be applied to r_i before starting the evaluation of the full disjunction. If C_h involves two relations r_i and r_j, then it may be any condition that can be evaluated for pairs of tuples from these two relations.

Thus, the weak semantics provides the means to generalize full disjunctions and thereby facilitating integration of information from the Web, as well as from other heterogeneous sources, in a more general manner than earlier work.

6.7 Summary of Contributions

In Chapter 6, we have shown that evaluation of full disjunctions is reducible to evaluation of queries under the weak semantics. Thus, full disjunctions can be computed in polynomial time in the size of the input and the output. Previously, the only known algorithm to compute full disjunctions in the general case was by evaluating all connected joins and removing subsumed tuples. That algorithm runs in exponential time in the size of the input and the output. Hence, our result makes full disjunctions practical for real-world applications.

For α-acyclic relation schemes, Yannakakis [65] showed that any projection of the natural join of all the relations can be computed in polynomial time in the size of the input and the output. Theorem 6.10 builds on the results of [55, 65]. We have also shown that when the relation schemes are α-acyclic, there are no polynomial-time algorithms, under input-output complexity, for computing projections and restriction of full disjunctions.

The weak semantics provides a substantial generalization of full disjunctions, by allowing general types of join conditions (provided that those conditions can be computed efficiently). This is achieved without increasing the time complexity of evaluating full disjunctions, in stark contrast to other approaches for querying incomplete information, where complex join conditions quickly lead to intractability of query evaluation.

The OR-semantics can provide another, more general form of full disjunctions. In the OR-semantics, an answer should be connected, but it does not have to satisfy all the join conditions among its parts; instead, it is sufficient to satisfy only a subset of the conditions, provided that the satisfied join conditions are connected. In traditional databases, this would not be an answer at all, but in the realm of the Web this might still be of interest to the user, although it would probably be ranked lower than answers that are obtained under the weak semantics.

Chapter 7

Conclusion

Our work introduces novel ways of querying semistructured data and XML. We propose new querying methods that are ontology-based querying—the user is given the ontology of the database, i.e., a list of labels, and is not required to have any knowledge about structural details of the data. An ontology-based querying is well suited to querying data in the following cases: (1) the data do not conform to a schema, (2) the data have a structure that changes frequently, and (3) the schema of the data is unknown. For all these cases, traditional querying methods are generally problematic since the user formulates a query having a specific schema in mind and receives a result that does not include all the expected answers.

Irregularity and incompleteness are two fundamental features of semistructured data and ontology-based querying deals with both. In irregular databases, the order of labels, in database paths, is unknown. The paradigm of flexible queries introduced in this work facilitates querying of data that are irregular. In incomplete databases, some fragments of the database are expected to includes certain labels, but some of these labels are missing. The paradigm of queries with maximal answers that was introduced in [41, 42] facilitates querying of data that are incomplete.

The main contributions of our work is by proposing methods for dealing with irregularity and incompleteness in semistructured data and XML. In the paradigm of flexible queries, we defined the semiflexible and the flexible semantics. Query evaluation, under these semantics, was investigated. We also investigated query equivalence and the novel concept of database equivalence. In the framework of queries with maximal answers, we continued the work of Kanza, Nutt and Sagiv

[41, 42] by providing a polynomial-time algorithm, in the size of the input and the output, for computing cyclic queries, under the weak and the OR-semantics. We also showed how to combine the semantics of the two paradigms. The combined semantics facilitate the querying of data that are both irregular and incomplete. The user queries the data while being oblivious to the structural details of the data and to the possibility of incompleteness.

An additional important contribution of our work is by showing that evaluation of full disjunctions is reducible to evaluation of queries under the weak semantics. Thus, full disjunctions can be computed in polynomial time in the size of the input and the output. Previously, known algorithms for computing full disjunctions of any arbitrary relations required exponential time in the size of the input and the output. Hence, our results are important for making full disjunction a practical tool in real-world applications.

Future work should address the issue of optimizing the evaluation of queries, under the different semantics that are presented in this work. Semistructured databases, in general, and XML documents, in particular, could be large. Thus, evaluation algorithms should exploit the disk and should not perform the evaluation while the entire database is in memory. Note that for full disjunctions, this problem has received an initial solution in the Separate-and-Join method, which was described in Chapter 6.

Future work should also address the problem of ranking the answers of queries. For example, a user may expect complete answers to be ranked higher than incomplete answers.

It seems that among the four combined semantics that were presented in this work, the flexible-weak semantics is the most useful, due to two reasons. First, query evaluation under this semantics is polynomial, in the size of the input and the output, for any arbitrary query and database. Secondly, it seems that this semantics captures the intended meaning of users better than the other combined semantics. However, additional work is required in order to provide means for measuring the quality of different semantics. A quality measure could assist in determining whether indeed the flexible-weak semantics is better than the other semantics.

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Appendix A

Hypergraphs and Acyclic Hypergraphs

There are many equivalent definitions for the notation of "acyclic" hypergraph. We present in this appendix the definitions that we use in the thesis. The definitions that are presented here are generally used by database researchers and are taken from [55, 60].

A *hypergraph* is a pair $\mathcal{H} = (V, E)$, where V is a set of nodes and E is a set of subsets of V. Each subset in E is called an *hyperedge* of H.

Alpha-Acyclic Hypergraphs

Let E and F be two hyperedges in an hypergraph $\mathcal{H}$. Suppose that the attributes in $E - F$ are unique to E; that is they appear in no hyperedge other than E. Then we call E an *ear*. The removal of E from $\mathcal{H}$ is called an *ear removal.* If E intersects no other hyperedge of $\mathcal{H}$, it is also considered an ear and its removal is an ear removal.

In a *GYO-reduction* of a hypergraph, ear removal is applied until there are no ears to remove. A hypergraph is α-acyclic if its GYO-reduction is the empty hypergraph. Otherwise, it is not α-acyclic.

Gamma-Acyclic Hypergraphs

We now introduce γ-acyclic hypergraphs. The definition is based of [31]. In [31], there is also a notion of β-acyclic hypergraphs that we do not present here, since β-acyclic hypergraphs are not discussed in our work.

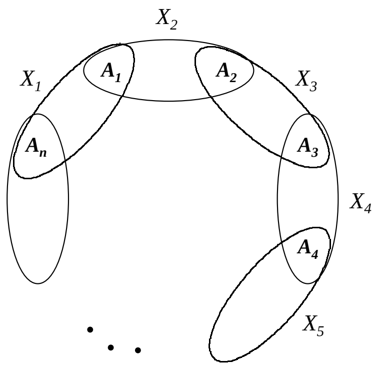


Figure 1: A pure cycle in a hypergraph

Pure Cycle A *pure cycle* in a hypergraph is a collection of three or more hyperedges and nodes in the form of a cycle as shown in Figure 1. In the pure cycle, each pair of hyperedges X_i and X_{i+1} (and also X_n and X_1) has at least one node A_i (or A_n in the case of X_n and X_1) in common. Moreover, none of the shared nodes A_j appears in more than two hyperedges of the pure cycle shown by Figure 1. These nodes, however, may appear in hyperedges that are *not* part of the pure cycle. In the pure cycle, there may be more than one node in the intersection of two adjacent hyperedges, but these nodes must not appear in any other hyperedges of the pure cycle.

Gamma-Three-Cycle A *γ-3-cycle* is a set of three hyperedges in the configuration of Figure 2. That is, the nodes A, B and C of the hypergraph must exists. Any other regions of the diagram of Figure 2 may or may not be empty, and the regions containing A, B and C may also contain other nodes. In other words, there must be some node in all three hyperedges, one that is only in X and Z, and one that is only in Y and Z. For example, the smallest γ-3-cycle consists of the three hyperedges $\{ABC, AB, BC\}$.

Gamma-Cycle A *γ-cycle* is a cycle of at least three edges, in the form of Figure 1. However, unlike a pure cycle, a γ-cycle permits A_n to appear in any of hyperedges $X_2, \ldots X_{n-1}$, as well as X_1 and X_n. All other nodes in the intersections appear only

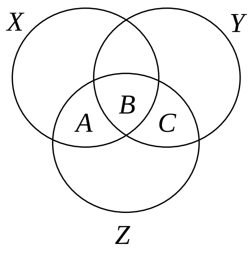


Figure 2: A γ-3-cycle in a hypergraph

in two adjacent X_i's.

Gamma-Acyclic Hypergraphs A hypergraph is γ-*acyclic* if and only if it has no γ-cycle. Equivalently, a hypergraph is γ-acyclic if and only if it contains no pure cycle and no γ-3-cycle. A hypergraph is γ-*cyclic* if and only if it is not γ-acyclic.

Appendix B

The tables that are constructed during the transformation of the movie database (Figure 2.1) to a relation, according to the full-disjunction approach shown in Example 6.4.

r_1	Movie	Actor	Name
	11	21	Mark Hamill
	11	21	Harisson Ford
	12	25	Natalie Portman

r_2	Movie	Title
	11	Star Wars
	12	León
	13	Magnolia

r_3	Movie	Year
	11	1977

r_4	Movie	Director	Name
	11	41	George Lucas

r_5	Actor	Name
	14	Kyle MacLachlan

r_6	Actor	Filmography	T.V. Series	Title
	14	15	28	Twin Peaks

r_7	Actor	Filmography	Movie	Title
	14	15	29	Dune

r_8	Actor	Filmography	Movie	Year
	14	15	29	1984

r_8	Actor	Filmography	Movie	Director	Name
	14	15	29	42	David Lynch

Appendix C

The Proof of Theorem 3.21

We present Algorithm *SFPT* that evaluates semiflexible path queries w.r.t. tree databases in $O(|Q||D| + |M|)$ runtime, where $|Q|$ is the size of the query, $|D|$ is the size of the database and $|M|$ is the size of the result.

Intuitively, *SFPT* computes the matchings as follows. A traversal is performed over the database. The correspondence sets are computed during the traversal. Matchings are computed , from the correspondence sets, each time that the traversal reaches a leaf.

We now describe Algorithm *SFPT*. First we describe three data structured that *SFPT* uses. How the data structured are being used will be described later. Consider a path query Q and a tree database D. The first data structure that *SFPT* uses is an array A of correspondence sets. In A there is a correspondence set for each variable of Q. Initially, all the correspondence sets in A are empty.

The second data structure that *SFPT* holds is an array L of positive numbers. The array L is used for counting the number of labels in D on paths from the root to a leaf. In L there is an entry for each label l of Q. The entry that corresponds to l is denoted $L(l)$. Initially, all the entries in L are equal to 0.

The third data structure in *SFPT* is a set of matchings $\mathcal{M}_T$. The set $\mathcal{M}_T$ contains, at each step of the algorithm, the matchings that where computed up to this step. Initially, $\mathcal{M}_T$ is empty.

SFPT performs a traversal over D that is similar to a DFS. The traversal is performed using the recursive procedure *visit*, that is described in Figure 3. The traversal starts by calling *visit* with the root of the database.

```
**Procedure** *visit*($o$)
**Input** a database object $o$;

$D$ is the tree database
$\mathcal{M}_T$ is a set of matching that is being constructed during the traversal;
$L$ is an array of numbers that count the number of instances
for each label on the path to $o$;
$A$ is an array of correspondence sets that is being updated during the traversal;

if $o$ is not the root of $D$ then
    let $l$ be the label that enters $o$ in $D$;
    update $L$ by adding 1 to $L(l)$;
    update $A$ by adding $o$ to all the correspondence sets
    of variables with incoming label $l$ in $Q$;
else update $A$ by adding $o$ to the correspondence set of the root;
if $o$ is a leaf of $D$ then
    if for each label $l$ in $Q$, the number of variables in $Q$ with
    incoming label $l$ is not greater than $L(l)$ then
        compute $\mathcal{M}$ according to Equation 3.1;
        add $\mathcal{M}$ to $\mathcal{M}_T$;
for each child $o'$ of $o$
    visit($o'$);
if $o$ is not the root of $D$ then
    update $L$ by subtracting 1 from $L(l)$;
    update $A$ by removing $o$ from all the correspondence sets
    of variables with incoming label $l$ in $Q$;
```

Figure 3: The recursive procedure *visit* of Algorithm *SFPT*. Algorithm *SFPT* computes the semiflexible matchings of a path query w.r.t. a tree database.

In the traversal there are steps of descending from a node to a child and steps of ascending from a node to the parent. In each step, the arrays A and L are modified according to the node that is visited. Consider a node o in D with incoming label l.

When descending to o, $L(l)$ is increased by 1. In A, o is added to the correspondence sets of variables that have incoming label l in Q. When ascending from o, $L(l)$ is decreased by 1. In A, o is removed from the correspondence sets of variables with incoming label l.

When reaching a leaf of D, the algorithm computes the set of matchings w.r.t. the path to this leaf. The computed matchings are added to $\mathcal{M}_T$. The matchings are computes using the correspondence sets of A.

Computing the matchings is as follows. First, the algorithm checks that there are "enough nodes for each label" for at least one matching. This is done by checking that for each label l in Q, if there are k variables in Q with incoming label l then $L(l) \geq k$. If this test fails, then no matching is added to $\mathcal{M}_T$ and the computation w.r.t. the current leaf is completed. Else, the set of matchings on the left side of Equation 3.1 is computed and added to $\mathcal{M}_T$.

Consider a path query Q and a tree database D. Algorithm *SFPT* computes the semiflexible matchings of Q w.r.t. D in $O(|Q||D|+|M|)$ runtime, where $|Q|$, $|D|$ and $|M|$ are the sizes of the query, the database and the result, respectively.

Proof of Theorem 3.21. We show that *SFPT* returns the semiflexible matchings of Q w.r.t. D. First we claim that the *visit* procedure computes the correspondence sets correctly. Consider a call to *visit* with an object o that is not the root. Let π be the path in the database from the root to the parent of o. Then, for each label l in the query, $L(l)$ holds the number of objects in π with incoming label l. For each variable v in Q, the correspondence set of v in A holds all the objects on π that have an incoming label l, where l is the label on the edge that enters v. The correspondence set of the root holds merely the database root. If o is the root then L and A hold the initial values. The claim can be proved by a simple induction on the number of calls to *visit*.

According to the above claim, every matching of Q w.r.t. D is a matching of Q w.r.t. a path of D from the root to a leaf. Essentially, Algorithm *SFPT* evaluates Q w.r.t. each path in D, from the root to a leaf, and performs a union on the result of the computations w.r.t. the different paths. Correctness of the computation w.r.t. the different paths follows from Proposition 3.20.

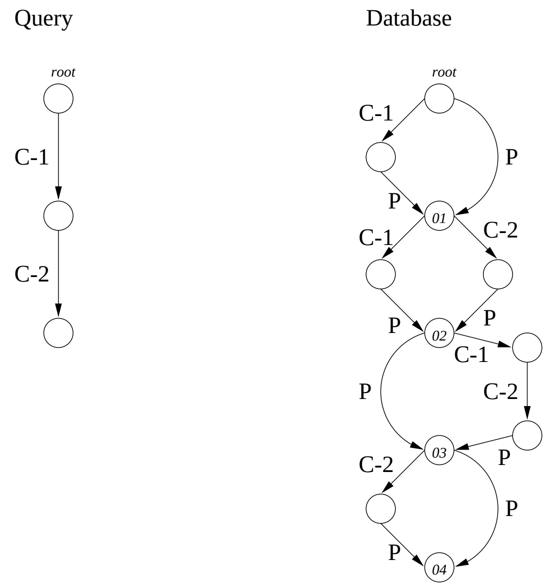


Figure 4: An illustration of how to construct a query and a database, according to Lemma 3.22, for the formula $\varphi = (p_1 \vee p_2 \vee \neg p_3) \wedge (\neg p_2 \vee \neg p_3 \vee p_4)$. The clauses of φ are denoted by *C-1* and *C-2*, respectively. For each node, the truth-assignment path is the path that comes from the left and the false-assignment path is the path that comes from the right.

We analyze the time complexity of the algorithm. In the algorithm, every node of D is visited once. The updating of L and A in each visit of a node can be done in $O(1)$. Computing the matchings has the following complexity. Testing that there are "enough object for each labels", for creating at least one matching, is in $O(|Q|)$. There can be $O(|D|)$ leafs for which this test should be performed . Finally, creating the matchings is in $O(|M|)$ runtime. Thus, the algorithm has $O(|Q||D| + |M|)$ runtime. □

The Proof of Lemma 3.22

Proof of Lemma 3.22. Let $\varphi = c_1 \wedge c_2 \wedge \ldots \wedge c_m$ be a 3CNF formula and $U = \{p_1, \ldots, p_n\}$ be the propositional letters appearing in φ. We construct from φ a path query Q and a dag database D in such a way that a semiflexible matching of Q w.r.t. D simulates a satisfying assignment for the propositional letters in U.

First, we explain how to construct Q. The query Q is a path made of m edges such that the i-th edge in Q is labeled with C-i. Thus, the query has the form $v_0, C\text{-}1, v_1, \ldots, v_{m-1}, C\text{-}(m), v_m$.

Secondly, we explain how the database is constructed. The database, has n nodes denoted by $o_{p_1}, \ldots, o_{p_n}$. These n nodes represent the propositional letters in U. In addition, the database has one more node that serves as the root of the database and is denoted by r_D. There are other nodes in the database that we do not name explicitly.

Generally, there are exactly two paths in D from the root to the node o_{p_1}. Similarly, there are exactly two paths from each node o_{p_i} to the node $o_{p_{i+1}}$ (for $1 \leq i \leq m-1$). Next, we describe these pairs of paths.

Given a propositional letter p_i, let the set $C_T = \{c_{i_1}, \ldots, c_{i_k}\}$ be the set of clauses satisfied by assignment of true to p_i. Let the set $C_F = \{c_{j_1}, \ldots, c_{j_{k'}}\}$ be the set of clauses satisfied by an assignment of false to p_i. We create a path from $o_{p_{i-1}}$ (r_D if $i = 1$) to o_{p_i} that we call the *truth-assignment path* of o_{p_i}, and a similar path from $o_{p_{i-1}}$ (r_D if $i = 1$) to o_{p_i} that we call the *false-assignment path* of o_{p_i}. The truth-assignment path of o_{p_i} is a simple path of $k+1$ edges that are labeled with the names of the clauses in C_T and an additional edge labeled with P, that is, $o_{p_{i-1}}, C\text{-}i_1, o_1, \ldots, o_{k-1}, C\text{-}i_k, o_k, P, o_{p_i}$. Except the first and the last nodes, all the nodes in the truth-assignment path are fresh, i.e., they are not among the nodes $r_D, o_{p_1}, \ldots, o_{p_n}$ and they do not appear in any other truth-assignment path or false-assignment path. The last edge is labeled with P and its role is to make sure that there is a truth-assignment path to o_{p_i} even if an assignment of true to p_i does not satisfy any clause. Similarly, the false-assignment path of o_{p_i} is a simple path of $k'+1$ edges that are labeled with the names of the clauses in C_F and an additional edge that is labeled with P, that is, $o_{p_{i-1}}, C\text{-}j_1, o'_1, \ldots, o'_{k'-1}, C\text{-}j_{k'}, o'_{k'}, P, o_{p_i}$. Except the first and the last nodes, all the nodes in the false-assignment path are fresh. Note that even if one or both of the sets C_T and C_F are empty, there are still two paths that enter the node p_i due to the "neutral" edge labeled with P.

There are no nodes and edges in the database in addition to the ones that were described above. Thus, the created database D is a dag.

We now show that there is a semiflexible matching of Q w.r.t. D if and only if there is a satisfying assignment of φ.

Assume that α is an assignment to the propositional letters of U that satisfies φ. We look at a path ϕ in D that starts at r_D and goes through the nodes $o_{p_1}, \ldots, o_{p_n}$ as follows. If α assigns true (false) to p_i then ϕ goes through the truth-assignment path (false-assignment path) of o_{p_i}.

The assignment α satisfies every clause in φ. Thus, a clause c_i must have a proposition p_j in it such that either p_j is positive in c_i and α assigns true to p_j or p_j is negative in c_i and α assigns false to p_j. In the first case, c_i is in C_T of p_j, the path ϕ goes through the truth-assignment path of o_{p_j}, and hence, ϕ goes through an edge labeled with the name of c_i. In the second case, c_i is in C_F of p_j, the path ϕ goes through the false-assignment path of o_{p_j}, and hence, ϕ goes through an edge labeled with the name of c_i. Thus, in both cases, the path ϕ goes through an edge labeled with the name of c_i, and this is true for all the clauses of φ.

The path ϕ is a path in the database, from the root to the only leaf, that goes through at least m edges that are labeled with the names of the m clauses of φ. A permutation of ϕ can transfer these m edges to the beginning of the path and this creates a prefix that can be matched with the query Q. Consequently, it shows the existence of a semiflexible matching of Q to D.

Conversely, assume that there exists a semiflexible matching μ of Q to D. The existence of μ entails the existence of a path ϕ in D that starts on the root, such that ϕ has a permutation with a prefix that matches Q. Since Q has edges that are labeled with all the names of the clauses of φ, it means that ϕ must also have edges that are labeled with all the names of clauses of φ.

We create an assignment α to the propositions of U,

$$\alpha(p_i) = \begin{cases} \text{true} & \phi \text{ goes through the truth-assignment path of } o_{p_i} \\ \text{false} & \phi \text{ goes through the false-assignment path of } o_{p_i} \end{cases}$$

We now show that the assignment α satisfies the formula φ. First we show that if the path ϕ goes through an edge labeled with the name of the clause c_j, then α satisfies the clause c_j. Let e be an edge, on the path ϕ, labeled with the name of c_j.

The edge e is either an edge in a truth-assignment path of some node o_{p_i} or an edge in a false-assignment path of some node o_{p_i}.

In the first case, the path ϕ goes through a truth-assignment path of the node o_{p_i}, so α assigns true to p_i. The edge e is in the truth-assignment path of o_{p_i}, and this means that the clause c_j is in the set C_T of p_i. The set C_T of p_i includes clauses that are satisfied by assigning true to p_i, and hence, α satisfies c_i. The second case, where the path ϕ goes through a false-assignment path of the node o_{p_i}, is similar.

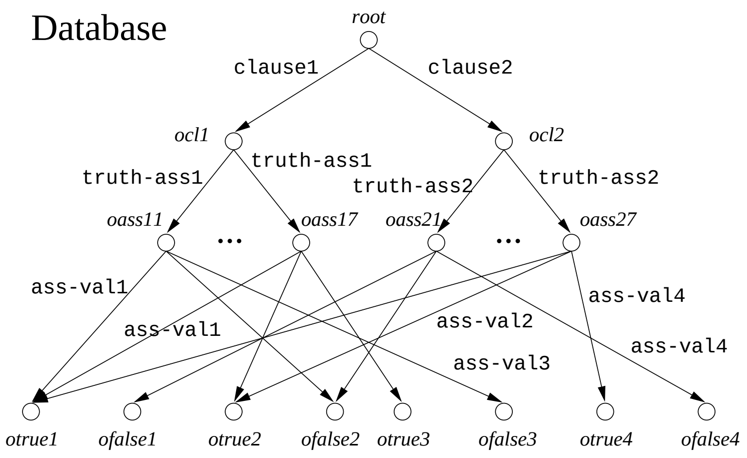
For each clause c_j of φ, the path ϕ goes through an edge labeled with the name of c_j. This means that α satisfies all the clauses of φ, and by this satisfies φ itself. Thus, the formula φ has a satisfying assignment.

The query Q and the database D that were created from φ have a polynomial size in φ. The number of edges in the query is the number of clauses in φ. In the database, there are the following edges. First there is an edge for each appearance of a propositional letter in a clause of φ. This sums to $3 \cdot m$ edges. In addition, there are two edges with the label P for each propositional letter of U. That is, there are $2 \cdot n$ edges with the label P. Thus, for a formula φ with m clauses over a set of n propositional letters, there are $3 \cdot m + 2 \cdot n$ edges in the database D. We conclude that the construction of the query and the database is polynomial in the size of the formula. □

The Proof of Lemma 3.57

Proof of Lemma 3.57. Let φ be a formula in 3CNF, let $P = \{p_1, \ldots, p_n\}$ be the set of propositional letters appearing in φ, and let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be the set of clauses in φ. We construct a database D and a query Q in a way such that evaluating the query over the database simulates the assignment of truth values to the propositional letters, and such that flexible matchings are in a one-to-one correspondence with satisfying assignments.

We construct D as follows. Let r_D be the root of D. For each clause $C_i \in \mathcal{C}$, there is an edge labeled $clause_i$ from r_D to a unique object ocl_i. The object ocl_i represents the i-th clause of the formula φ. For each propositional letter $p_j \in P$,



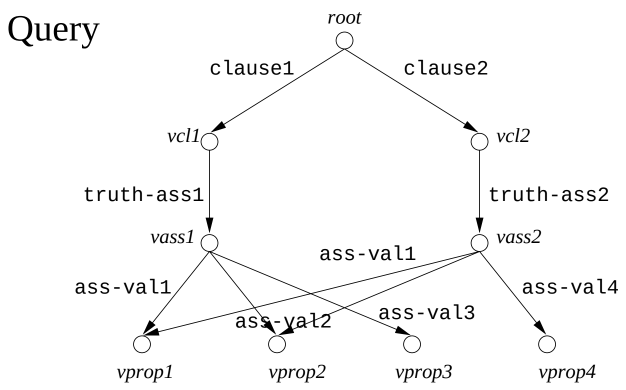


Figure 5: The construction of Lemma 3.57 for the formula $\varphi = (p_1 \vee p_2 \vee p_3) \wedge (\neg p_1 \vee p_2 \vee p_4)$.

there is an object $otrue_j$ and an object $ofalse_j$. The objects $otrue_j$ and $ofalse_j$ represent the truth values that can be assigned to the propositional letter p_j. From each object ocl_i there are seven outgoing edges labeled by $truth\text{-}ass_i$ to seven different objects $oass_{i1}, \ldots oass_{i7}$. The objects $oass_{ik}$ represent the seven possible satisfying assignments for the clause C_i. Let $p_{l_1}, p_{l_2}, p_{l_3}$ be the propositional letters that appear in the literals of the clause C_i. For each assignment to $p_{l_1}, p_{l_2}, p_{l_3}$ that satisfies C_i, we look at the object $oass_{ik}$ that represents that assignment. If in this assignment p_{l_m} is assigned true then there is an edge $ass\text{-}val_{l_m}$ to $otrue_{l_m}$ in D. If in the assignment p_{l_m} is assigned false then there is an edge $ass\text{-}val_{l_m}$ to $ofalse_{l_m}$ in D. There are no other objects and edges in D.

We construct a dag-query Q. The construction is as follows. The root of Q is r_Q. For each clause $C_i \in \mathcal{C}$, there is an edge labeled $clause_i$ from r_Q to a unique variable vcl_i. The variable vcl_i represents the i-th clause of the formula φ. For each clause C_i, there is a unique variable $vass_i$ in Q, and there is an edge, from vcl_i to $vass_i$, labeled by $truth\text{-}ass_i$. For each propositional letter $p_j \in P$, there is a variable $vprop_j$ in Q. Let $p_{l_1}, p_{l_2}, p_{l_3}$ be the propositional letters that appear in the literals of the clause C_i. There is an edge labeled $ass\text{-}val_{l_m}$ in Q from $vass_i$ to the variable $vprop_{l_m}$, for $m = 1, 2, 3$. There are no other variables and edges in Q. From the construction it is easy to see that Q is a dag.

In a flexible matching of Q to D, each variable $vprop_i$ is is mapped to either the database object $otrue_i$ or the object $ofalse_i$. That corresponds to an assignment of truth values to the propositional letters in P. A flexible matching of Q to D is an assignment that satisfies all $vass_i ass\text{-}val_j vprop_j$ constraints, by assigning to $vass_i$ and $vprop_j$ database nodes that are indirectly connected by a path in D and with appropriate incoming labels. Hence, an object $oass_{ik}$ is assigned to each variable $vass_i$ where $oass_{ik}$ stands for a satisfying assignment for the clause C_i. That means that the matching determine directly an assignment that satisfies each clause of φ, and thus satisfies φ itself.

For the other direction, we assume that there is a satisfying assignment to φ. Let this assignment be μ. The satisfying assignment determine a matching in that an object $otrue_i$ or $ofalse_i$ is assigned to the variable $vprop_i$ according to the truth value which is assigned to the propositional p_i by μ. The restriction of μ to the variables of a clause C_i is a satisfying assignment for C_i, and hence one of the seven possible satisfying assignments for C_i. That determine which object $oass_{ik}$ is assigned to $vass_i$. Finally, we get from μ a flexible matching of Q to D.

We conclude that there is a one-to-one correspondence between satisfying truth assignments for φ and flexible matchings of Q to D. $\square$

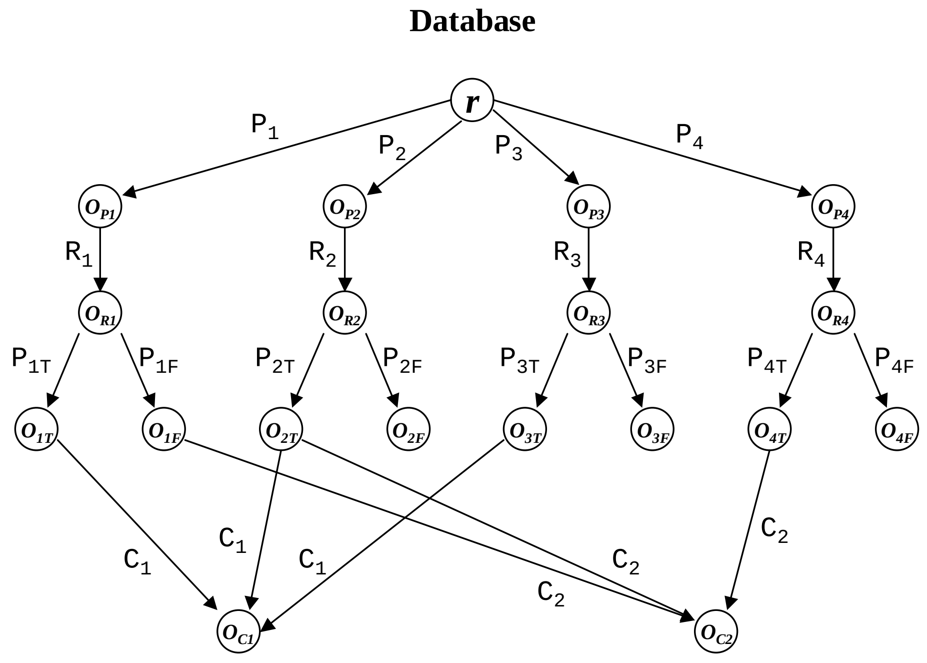


Figure 6: A database according to the construction of Lemma 5.8 for the formula $\varphi = (p_1 \vee p_2 \vee p_3) \wedge (\neg p_1 \vee p_2 \vee p_4)$.

The Proof of Lemma 5.8

Proof of Lemma 5.8. Let φ be a formula in 3CNF, let $P = \{p_1, \ldots, p_n\}$ be the set of propositional letters in φ, and let $\mathcal{C} = \{C_1, \ldots, C_m\}$ be the set of clauses of φ. We construct a query Q and a database D in a way such that if a maximal semiflexible-OR-matching maps all the variables of the query to non-null values, then this mapping indicates the existence of a satisfying assignment for φ. Without loss of generality, we assume the no clause of φ contains both a literal and its negation. Obviously, if such a clause exists, it can be removed from φ without affecting satisfaction.

We first describe how to construct the database D. In D there are four levels of objects below the root. The first level represents the propositional letters of φ. That is, there are n objects $o_{P_1}, \ldots, o_{P_n}$ and for each object o_{P_j} there is an edge with the

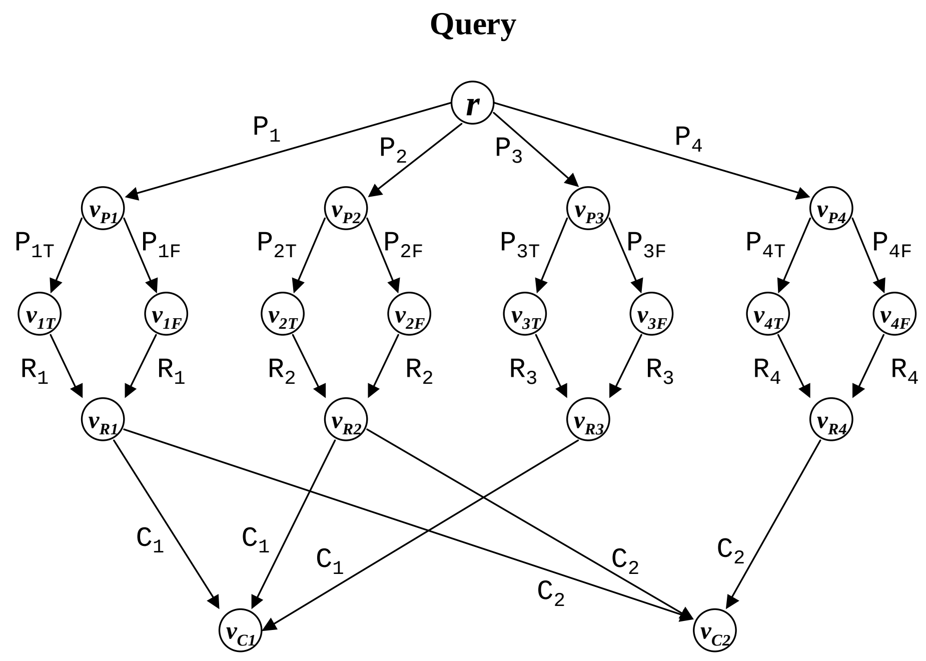


Figure 7: A query according to the construction of Lemma 5.8 for the formula $\varphi = (p_1 \lor p_2 \lor p_3) \land (\neg p_1 \lor p_2 \lor p_4)$.

label P_j from the root to o_{P_j}. The second level consists of n objects $o_{R_1}, \ldots, o_{R_n}$. For each $1 \leq j \leq n$, there is an edge with the label R_j from o_{P_j} to o_{R_j}. On the third level, for each proposition p_j, there are two objects o_{jT} and o_{jF}. There is an edge with the label P_{jT} from o_{R_j} to o_{jT} and there is an edge with the label P_{jF} from o_{R_j} to o_{jF}. On the fourth level, there are m nodes $o_{c_1}, \ldots, o_{c_m}$ that represent the m clauses of φ. There is an edge from o_{jT} to o_{c_i} if assigning true to p_j satisfies the clause C_i, i.e., p_j is a positive literal in C_i. There is an edge from o_{jF} to o_{c_i} if assigning false to p_j satisfies C_i, i.e., C_i includes $\neg p_j$. There are no additional edges or nodes in D.

We now describe how to construct the query Q. As in the database, the query also has four levels of objects below the root. The first level represents the propositional letters of φ and comprises n variables $v_{P_1}, \ldots, v_{P_n}$. Each variable v_{P_j} is connected to the root by an edge with the label P_j. In the second level, there is a pair of variables

v_{jT} and v_{jF} for each propositional letter p_j. From the node v_{P_j}, there is an edge with the label P_{jT} to v_{jT} and there is an edge with the label P_{jF} to v_{jF}. On the third level, there are n nodes $v_{R_1}, \ldots, v_{R_n}$. From each node v_{jT} and from each node v_{jF} there is an edge with the label R_j to v_{R_j}. Finally, on the fourth level there are m variables $v_{c_1}, \ldots, v_{c_m}$ that represent the m clauses of φ. There is an edge from v_{R_j} to v_{c_i} if the propositional letter p_j appears in the clause C_i, either positively or negatively. There are no additional edges or nodes in Q.

Next, we show that if there exists a semiflexible-OR-matching μ of Q w.r.t. D, such that μ map all the variable of Q to non-null, then φ has a satisfying assignment.

Suppose that μ is a semiflexible-OR-matching of Q w.r.t. D, such that μ does not map any variable of Q to null. Since μ is a semiflexible-OR-matching, there exists a rooted fragment F of Q such that F consists of all the nodes of Q and μ is a semiflexible matching of F w.r.t. D.

Claim .1 *If there is an edge in F from v_{R_i} to v_{c_k}, then F has exactly one of the two edges that enters v_{R_i}, i.e., in F there is exactly one edge with the label R_i.*

As a first step towards proving the claim we provide the next observation. Consider the two edges that are described next. First, an edge from o_{iF} to the node o_{c_k}. Second, an edge from o_{iT} to the node o_{c_k}. Only one of this edges can be in D. This is because the edge from o_{iT} to o_{c_k} is in D if the propositional letter p_i appears positively in c_k. The edge from o_{iF} to o_{c_k} is in D if the propositional letter p_i appears negatively in c_k. The claim of the observation follows from the assumption that a propositional letter cannot appear both positively and negatively in a clause.

According to the construction of D and the observation above, for each $1 \leq i \leq n$ and $1 \leq k \leq m$, only one of the objects o_{iF} and o_{iT} has a path to o_{c_k}.

Now, we continue proving the claim. It is easy to see that according to the labels on the edges of both the query and the database, μ maps the variables v_{R_i}, v_{c_k}, v_{iF} and v_{iT} to the objects o_{R_i}, o_{c_k}, o_{iF}, and o_{iT}, respectively.

Next, we assume that F has an edge from v_{R_i} to v_{c_k} and in addition has the two edges that enter v_{R_i}—an edge from v_{iF} to v_{R_i} and an edge from v_{iT} to v_{R_i}. We show that this assumption leads to a contradiction.

According to the above assumption, F has a path from v_{iF} to v_{c_k} and a path from v_{iT} to v_{c_k}. Since μ is a semiflexible matching w.r.t. F, $\mu(v_{iF})$ (i.e., o_{iF}) and $\mu(v_{c_k})$ (i.e., o_{c_k}) should be connected by a path. Similarly, $\mu(v_{iT})$ (i.e., o_{iT}) and $\mu(v_{c_k})$ (i.e., o_{c_k}) should be connected by a path. This contradicts the above observation that only one of the objects o_{iF} and o_{iT} has a path to o_{c_k}.

Next, we show how to construct an assignment f to the propositional letters of P, such that f satisfies φ. The assignment f is defined as follows. Consider $1 \leq i \leq n$.

- If there is a path in F from a node v_{iT} to a node v_{c_k}, for some $1 \leq k \leq m$, then f assigns true to p_i.
- If there is a path from the node v_{iF} to a node v_{c_k}, for some $1 \leq k \leq m$, then f assigns false to p_i.
- Otherwise, f maps p_i to true or to false arbitrarily.

We argue that f is well defined. This is because there cannot be in F both a path from the node v_{iT} to a node v_{c_k} and a path from the node v_{iF} to a node $v_{c_{k'}}$. Otherwise, F would include two edges that enters v_{R_i}, one from v_{iF} and the other from v_{iT}. This is a contradiction to Claim .1.

Finally, we show that f is a satisfying assignment of φ. In F, there is a path from the root to each variable. In particular, there is a path to all the variables $v_{c_1}, \ldots, v_{c_m}$. Consider a path π, in F, from the root to v_{c_k}. According to the construction of Q, there is $1 \leq i \leq n$ such that π goes either through v_{iT} or through v_{iF}.

If pi goes through v_{iT} (to v_{c_k}), then there is a path (actually an edge), in D, from o_{iT} to o_{c_k}. This is because μ is a semiflexible matching of F and it satisfies the SF-Condition (Definition 3.3) w.r.t. each path of F. According to the construction of D, there is an edge from o_{iT} to o_{c_k} if assigning p_i to true satisfies c_k. According to the definition of f, p_i is assigned true. Thus, f satisfies c_k. The case where pi goes through v_{iF} is similar.

For the other direction, let f be a satisfying assignment for the propositional letters of φ. We construct a fragment F of Q as follows. The fragment F contains all the edges that emanate from the root and all the edges that emanate from the variables $v_1, \ldots, v_n$. For each propositional letter p_j, if f assigns true to p_j then F has an edge from v_{jT} to v_{R_j}. Otherwise, F has an edge from v_{jF} to v_{R_j}. There is an edge from v_{R_j} to v_{c_i} if the assignment of f to p_j satisfies C_j, i.e., f assigns true (false) to p_j and p_j appears positively (negatively) in C_i. There are no other edges in F.

According to the construction of F and because f satisfies all the clauses of φ, F is rooted. We examine the assignment μ of F w.r.t. D, where μ is defined as follows. The variables $v_{P_1}, \ldots, v_{P_n}$ are mapped to the objects $o_{P_1}, \ldots, o_{P_n}$, respectively. Each variables v_{jT} is mapped to the object o_{jT} and, similarly, each variables v_{jF} is mapped to the object o_{jF}. The variables $v_{R_1}, \ldots, v_{R_n}$ are mapped to the objects $o_{R_1}, \ldots, o_{R_n}$, respectively. Finally, each variable v_{c_i} is mapped to the object o_{c_i}.

It is easy to see that μ satisfies the SF-condition w.r.t. paths of Q that do not reach a node v_{c_i}. For a path ϕ that does reach a node v_{c_i}, note that ϕ goes through a node v_{jT} only if p_j appears positively in C_i. In this case, o_{jT} is connected to o_{c_i}. The case where ϕ goes through v_{jF} is similar. Thus, μ satisfies the SF-condition w.r.t. each path of F and, therefore, μ is a semiflexible matching of F w.r.t. D.

We show that μ is the only maximal semiflexible-OR-matching of Q w.r.t. D. Consider a fragment F' of Q and a semiflexible matching μ' of F' w.r.t. D. It is easy to see that according to the labels on the edges of both the query and the database, for every variable v in F', $\mu'(v) = \mu(v)$. Thus, either μ is equal to μ' or μ subsumes μ'. Thus, any semiflexible-OR-matching of Q w.r.t. D that is different from μ is not maximal. □

The Proof of Proposition 6.7

Proof of Proposition 6.7. We start with the following claim.

Claim .2 *There is an attribute A in R_j such that A is not an attribute of any schema of G_1 (G_2) other than R_j.*

We show that the claim is true. Let R_i $(i \neq j)$ be a schema in G_2 that has a non-empty intersection with R_j. Such schema exists because G_2 is connected. If the claim was not true, then all the attributes of R_j were in schemas of G_1, including the attributes of the intersection of R_i and R_j. In this case, R_i was connected to a node of G_1 and to a node of G_2. This is a contradiction to the assumption that G_1 and G_2 are a separation of G w.r.t. R_j, and it proves the claim.

We now continue with the proof of the proposition. Suppose that t is a tuple of $FD(G)$. We show that t is also a tuple of $FD(G_1) \overset{o}{\bowtie} FD(G_2)$.

The tuple t is a maximal integrated tuple. Thus, it is generated from m $(m \leq n)$ tuples $t_{i_1}, \ldots, t_{i_m}$ of the relations $r_{i_1}, \ldots, r_{i_m}$, respectively. Without loss of generality, let $R_{i_1}, \ldots, R_{i_{k-1}}$ be nodes of G_1, $R_{i_{k+1}}, \ldots, R_{i_m}$ be nodes of G_2 and R_{i_k} be the node that separates G_1 and G_2, i.e., the only node that is in both G_1 and G_2. Note that each one of the sets $\{R_{i_1}, \ldots, R_{i_k}\}$ and $\{R_{i_k}, \ldots, R_{i_m}\}$ could be empty. This happens when the non-null portion of T consists of attribute of schemas of G_1 and does not include attributes of schemas of G_2, or vice-versa.

Consider the following three assertions, for the case where $\{R_{i_1}, \ldots, R_{i_k}\}$ is not empty. First, the tuples $t_{i_1}, \ldots, t_{i_k}$ are join consistent because they are part of the integrated tuple t. Secondly, the schemas $R_{i_1}, \ldots, R_{i_k}$ are connected. This is because $R_{i_1}, \ldots, R_{i_m}$ are connected (otherwise, t would not be in $FD(G)$) and none of the schemas $R_{i_1}, \ldots, R_{i_{k-1}}$ has an edge to one of the schemas $R_{i_{k+1}}, \ldots, R_{i_m}$. Thirdly, there are no R_l in G_1 and a tuple t_l of r_l such that *(1)* t_l is join consistent with $t_{i_1}, \ldots, t_{i_k}$, *(2)* R_l has an attribute A such that A does not belong to any of the schemas $R_{i_1}, \ldots, R_{i_k}$, and *(3)* R_l is connected to one of the relations $R_{i_1}, \ldots, R_{i_k}$. Otherwise, t would not be maximal and would be subsumed by the join of t and t_l.

The above three assertions show that the universal tuple t_1 whose generators are $t_{i_1}, \ldots, t_{i_k}$, is a maximal integrated tuple w.r.t. the schema nodes of G_1 and the corresponding relations of these schemas. Similarly, it can be shown that if $\{R_{i_k}, \ldots, R_{i_m}\}$ is not empty then the universal tuple t_2 whose generators are $t_{i_k}, \ldots, t_{i_m}$, is a maximal integrated tuple w.r.t. the schema nodes of G_2 and the corresponding relations of these schemas.

Thus, if $\{R_{i_1}, \ldots, R_{i_k}\}$ is not empty, then t_1 is an element of $FD(G_1)$. Otherwise we consider t_1 as a null tuple. If $\{R_{i_k}, \ldots, R_{i_m}\}$ is not empty, then t_2 is an element of $FD(G_2)$. Otherwise we consider t_2 as a null tuple.

Next, we show that the join of t_1 and t_2 is in $FD(G_1) \overset{o}{\bowtie} FD(G_2)$. Suppose that both t_1 and t_2 are non-null. Consider the generators of t_1 and the generators of t_2. Since all these tuples are generators of t, they are pairwise join consistent. Thus, t_1 and t_2 are join consistent.

The attributes in the intersection of the schemas of t_1 and t_2 are the attributes of R_{i_k}. We claim that both tuples do not have a null in any attribute of R_{i_k} and $t_1[R_{i_k}] = t_2[R_{i_k}]$. The claim is true because the tuple t_{i_k} is a generator of both t_1 and t_2. The reason that the tuple t_{i_k} is a generator of both t_1 and t_2 is the following. The generators of t belong to relations with schemas that form a connected graph. However, the only node that connects schemas of G_1 with schemas of G_2 is R_{i_k}. Thus, if both t_1 and t_2 are non-null then their join is in $FD(G_1) \overset{o}{\bowtie} FD(G_2)$.

Consider the case where t_2 is null. In this case, the set of generators of t_1 does not include t_{i_k}, or else, t_{i_k} would have been a generator of t_2 and t_2 would not be null. We argue that there is an attribute A in R_{i_k} such that $t_1[A]$ is null. Let A be an attribute that is belong to R_{i_k} and does not belong to any of the schemas of G_1. There is such an attribute according to Claim .2. Since t_{i_k} is not a generator of t_1, $t_1[A]$ is null.

Because $t_1[A]$ is null for some attribute A in R_{i_k}, in the outerjoin of $FD(G_1) \overset{o}{\bowtie} FD(G_2)$, t_1 cannot be joined to any tuple of $FD(G_2)$. Thus, t_1 appears in the result padded with null values.

The case where t_1 is null is similar to the case where t_2 is null. Thus, in all three cases: when t_1 is null, when t_2 is null and when neither of the tuples is null, t is in $FD(G_1) \overset{o}{\bowtie} FD(G_2)$.

For the other direction, suppose that t is a tuple in $FD(G_1) \overset{o}{\bowtie} FD(G_2)$. We show that t is in $FD(G)$.

There are three cases to consider. First, t is created by a join of two tuples t_1 and t_2 of $FD(G_1)$ and $FD(G_2)$, respectively. Second, t is created from a tuple t_1 of

$FD(G_1)$ by padding t_1 with nulls. Third, t is created from a tuple t_2 of $FD(G_2)$ by padding t_2 with nulls.

Consider the first case, i.e., t is created by a join of two tuples t_1 and t_2 of $FD(G_1)$ and $FD(G_2)$, respectively. We show that the generators of t_1 and t_2 create a maxiaml integrated tuple w.r.t. $r_1, \ldots, r_n$.

First, we show that the generators of t_1 and t_2 are join consistent. Let X_1 be the generators of t_1 and X_2 be the generators of t_2. The tuples of X_1 (X_2) are join consistent and connected because t_1 (t_2) is in $FD(G_1)$ ($FD(G_2)$). Let X be the union of X_1 and X_2. Because t_1 and t_2 are joined in an outerjoin operation, each two tuples, one from X_1 and one from X_2, are join consistent. Thus, the tuples in X are join consistent.

Secondly, we show that the generators of t_1 and t_2 are connected. Consider R_j—the articulation node that the separation is w.r.t. it (the schema that was denoted R_{i_k} in the first part of the proof). The tuples t_1 and t_2 do not have a null on any of the attributes of R_j. Otherwise, t_1 and t_2 were not joined in the outerjoin. Because of Claim .2, there are no null values in $t_1[R_j]$ only if there is a tuple t' in r_j that is a generators of t_1. Due to the same argument and because t_1 and t_2 are equal on the attributes of R_j, t' is also a generator of t_2. The tuple t' connects the generators of t_1 and the generators of t_2. Hence, the tuples of X are connected.

Thirdly, the join of the generators of t_1 and t_2 is a maximal integrated tuple. If the join was not maximal, then either t_1 was not maximal, w.r.t. the relations whose schemas are nodes of G_1, or t_2 was not maximal, w.r.t. the relations whose schemas are nodes of G_2. The above three arguments show that t is a maximal integrated tuple and it is in $FD(G)$.

Consider the second case, i.e., t is created from a tuple t_1 of $FD(G_1)$ by padding t_1 with nulls. Let X be the set of generators of t_1. Obviously, the tuples in X are join consistent and connected. Thus, they are the generators of an integrated tuple. This integrated tuple is maximal w.r.t. the relations whose schemas are in G_1. We need to show that this integrated tuple is also maximal w.r.t. the relations $r_1, \ldots, r_n$.

We claim that the generators of t_1 do not include a tuple of r_j (Recall that the separation of G is w.r.t. R_j.) We assume that the claim is incorrect and show that this leads to a contradiction.

Assume that there is a generator t_j of t_1 which is a tuple of r_j. The schema R_j is a node of G_2. According to the definition of the full disjunction, there is a tuple t_2 in $FD(G_2)$ such that t_j is one of its generators (note that it could be that t_j is the only generator of t_2). The tuple t_2 is join consistent with t_1 because t_1 and t_2 have the same values for the attributes of R_j. Hence, the join of t_1 and t_2 is a tuple in the outerjoin of $FD(G_1)$ and $FD(G_2)$. That yield a contradiction to the assumption that there is no tuple in $FD(G_2)$ which is joined with t_1. This shows that the generators of t_1 do not include a tuple of r_j.

Consider a tuple over the schema nodes of G whose generators are the generators of t_1. This tuple is equal to t, that is, it is the result of padding t_1 with null values. Since the generators of t_1 do not include a tuple of r_j, it follows from Claim .2 that there is an attribute A in R_j such that $t_1[A]$ is null. This shows that any set of generators that includes the generators of t_1 and a tuple of a relation whose schema is in G_2 is not connected and join consistent. This shows that t is maximal.

Thus, the tuple t is a maximal integrated tuple and, hence, it is in $FD(G)$. The third case is similar to the second case. Therefore, in all three cases, t is in $FD(G)$ and the proof is completed. $\square$

The Proof of Theorem 6.10

Before we give the proof of Theorem 6.10, we provide some background on the computation of the full disjunction, without a projection, for relations with connected, γ-acyclic schemas.

Consider a a set of relation schemas that form a connected, γ-acyclic hypergraph. The algorithm SOJO receives these schemas and returns a *sound outerjoin ordering* of the schemas. A sound outerjoin ordering is an expression that consists of the given schemas and outerjoin operations. In the expression, each schema appears exactly once. Furthermore, if in this expression, the schemas are replaced with their corresponding relations and the expression is computed, the result of the computation is the full disjunction of the relations. The exact definition of SOJO and the proof of its correctness are given in [55].

In Sketch, SOJO performs the following computation steps. In the first step, a Bachman diagram $\mathcal{B}$ is constructed from the given schemas. In subsequent steps, SOJO is applied recursively to $\mathcal{B}$. Recall that the nodes of a Bachman diagram are either schemas or attribute sets that are intersection of schemas. A node is minimal if it is not contained in any other node of the diagram.

The recursion is as follows. If $\mathcal{B}$ comprises a single node R then R is returned. Else, a minimal node Z is chosen among the nodes of $\mathcal{B}$. According to Z, $\mathcal{B}$ is decomposed into two sub-diagrams $\mathcal{B}_1$ and $\mathcal{B}_2$. The decomposition has the following property. The set Z is the intersection of the following two sets: the set of attributes in the nodes of $\mathcal{B}_1$ and the set of attributes in the nodes of $\mathcal{B}_2$. SOJO is executed on $\mathcal{B}_1$ and on $\mathcal{B}_2$. Suppose that E_1 and E_2 are the expressions that SOJO returns when recursively called with $\mathcal{B}_1$ and $\mathcal{B}_2$. Then $E_1 \overset{o}{\bowtie} E_2$ is returned for $\mathcal{B}$.

Next, we explain how to modify SOJO in order to solve the projection problem. Consider a set of relations $r_1, \ldots, r_n$ with schemas $R_1, \ldots, R_n$, respectively. Let X be a given set of attributes. Our goal is to compute the projection on X of the full disjunction of $r_1, \ldots, r_n$. To do so, we modify SOJO as follows. First, on each recursive call of the algorithm, the input includes a set of projection attributes, in addition to a diagram. Secondly, instead of returning an expression that comprises merely the schemas and outerjoins, the returned expression also includes projections.

The modified recursion is as follows. Let $\mathcal{B}$ be the Bachman diagram and X be the projection attributes that are given in the input. For the case where $\mathcal{B}$ comprises a single node R, the projection of R on X is returned.[1] For the case where $\mathcal{B}$ has more than one node, $\mathcal{B}$ is decomposed into two sub-diagrams $\mathcal{B}_1$ and $\mathcal{B}_2$, according to a minimal node Z. The algorithm is called recursively—once with $\mathcal{B}_1$ and then with $\mathcal{B}_2$. In both cases, the projection attributes are $X \cup Z$. Suppose that E_1 and E_2 are the expressions that the recursive calls produce. Then, the projection of $E_1 \overset{o}{\bowtie} E_2$ on X is returned.

SOJO is defined for connected schemas. It is easy to see that when the schema

[1]Suppose that R is a schema and X is a set of attributes that is not contained in R. We assume, in this case, that the projection of R on X is equal to the projection of R on $X \cap R$.

graph is not connected, SOJO can be applied to each connected component. The results for the connected components are combined using *outerunion.* In an outerunion of two relations r_1 and r_2 with schemas R_1 and R_2, respectively, the schema of the outerunion is the union of R_1 and R_2. The outerunion itself is a relation that consists of all the tuples of r_1 and r_2, where a tuple of one relation is padded with null values in the attributes of the other relation.

We can now prove Theorem 6.10.

Proof of Theorem 6.10. We prove the theorem by first showing that the modified SOJO correctly computes the projection on X of the full disjunction. Later we will show that the runtime of the modified SOJO is polynomial in the size of the input and output.

Suppose that $r_1, \ldots, r_n$ are relations with schemas $R_1, \ldots, R_n$, respectively. Let X be a given set of projection attributes. We denote by $\mathcal{F}_X$ the projection of the full disjunction of $r_1, \ldots, r_n$ on X. We denote with $\mathcal{F}_X^{\text{SOJO}}$ the result of evaluating the expression that the modified SOJO returns for $R_1, \ldots, R_n$ and X. It should be shown that $\mathcal{F}_X = \mathcal{F}_X^{\text{SOJO}}$.

First, we want to show that attributes that are needed for the join are not removed too early because of the projections. To see this, consider a pair of schemas R_i and R_j. Let E_1 and E_2 be two expressions that the algorithm computes in one of the recursive calls, such that R_1 appears in E_1 and R_2 appears in E_2. (E_1 and E_2 are unique for the given schemas, because a schema cannot appear more than once in the final expression). We define the schema of an expression to be the schema of the relation that is the result of computing the expression after replacing relation schemas with their corresponding relations.

Claim .3 *Both the schema of E_1 and the schema of E_2 contain the shared attributes of R_i and R_j, i.e., contain $R_i \cap R_j$.*

To see why Claim .3 is true, consider the decomposition that returns the expression $E_1 \overset{o}{\bowtie} E_2$. Let Z be the node of the diagram that this decomposition is with respect to it. Since Z is a node in a Bachman diagram, the set Z is the intersection of the following two sets. The set of attributes in schemas that appear in E_1 and the set of attributes in schemas that appear in E_2. Thus, Z contains $R_i \cap R_j$. Now, one should notice that at each recursive call, the set of projection attributes is only increased. So, all the projections, in sub-expressions of E_1 and E_2, are on sets that contain Z, and, consequently, these sets contain $R_i \cap R_j$.

The next claim shows that attributes of X are not discarded by projections.

Claim .4 *In each expression E that is produced during the run of the modified SOJO algorithm, E is the projection of some sub-expression of E on a set of attributes that contains X.*

Again, this claims follows from the fact that, in the recursive calls, the set of projection attributes is always increased, i.e., no attribute is removed.

Next, we prove correctness by showing containment in both directions. Suppose that t_x is a tuple in $\mathcal{F}_X^{\text{SOJO}}$. It follows from Claim .3 that the tuples that were joined to produce t_x are join consistent. Thus, in $\mathcal{F}_X$ there is a tuple that is equal to t_x or subsumes t_x.

Before showing containment in the other direction, we provide a notation. Consider an expression E than contains the schemas $R_1, \ldots, R_n$. The *evaluation result* of E is the relation that is produced by computing E after replacing each schema R_i with its corresponding relation r_i. We denote the evaluation result of E as $\mathit{eval}(E)$.

Now, we show containment in the other direction. Suppose that t_x is a tuple in $\mathcal{F}_X$. Then there is a tuple t in the full disjunction of $r_1, \ldots, r_n$, such that t_x is the projection of t on X. Let $t_{i_1}, \ldots, t_{i_m}$ be the generators of t.

Consider the expression E^u that the unmodified SOJO produces for $R_1, \ldots, R_n$. Then t is in $\mathit{eval}(E^u)$. In the evaluation of E^u, t is produced by a sequence of joins w.r.t. the tuples $t_{i_1}, \ldots, t_{i_m}$. We show that in the modified SOJO there is a similar sequence of joins, w.r.t. projections of $t_{i_1}, \ldots, t_{i_m}$, and this sequence produces t_x or a tuple that subsumes t_x.

Consider the expression E^m that the modified SOJO produces. The two expressions E^u and E^m have the same order of outerjoin operations. The difference between them is the additional projections in E^m. Given a sub-expression E_1 of E^u, we denote as *Proj*(E_1) the sub-expression of E^m that contains exactly the same schemas as E_1. Such sub-expression exists because the decomposition to sub-expressions in SOJO is the same as the decomposition in the modified SOJO.

Consider the evaluation of E^m after replacing each schema R_i with its corresponding relation r_i. The the following claim holds.

Claim .5 *Suppose that E' is a sub-expression of E^u. Let t' be a tuple in eval(E'), such that all the generators of t' are among $t_{i_1}, \ldots, t_{i_m}$. Then there is a set of attributes X' that contains X, such that $t'[X']$ is in eval(Proj(E')).*

We prove Claim .5 by induction on the structure of E'. If E' is a relation schema R_{i_j}, for some $1 \leq j \leq m$, then *Proj*(E') is a projection of R_{i_j} on a set X'. According to Claim .4, X' contains X. In this case, t_{i_j} is in *eval*(E') and $t_{i_j}[X']$ is in *eval*(*Proj*(E')). Thus, the claim holds.

Suppose that E' has the form $E_1 \overset{o}{\bowtie} E_2$. Let t' be a tuple in *eval*(E'), such that the generators of t' are among $t_{i_1}, \ldots, t_{i_m}$. Then one of the following three cases must hold. First, t' is in the projection of *eval*(E_1) on a set of attributes X'. Second, t' is in the projection of *eval*(E_2) on a set of attributes X'. Third, t' is a projection on a join of a tuple of *eval*(E_1) and a tuple of *eval*(E_2). In the first two cases, according to Claim .4, X' contains X and the claim holds.

We consider the third case. There are tuples t_1 and t_2 such that *(1)* t_1 and t_2 are in *eval*(E_1) and *eval*(E_2), respectively, *(2)* the generators of t_1 and t_2 are among $t_{i_1}, \ldots, t_{i_m}$, and *(3)* t' is the join of t_1 and t_2.

By the induction hypothesis, there are sets of attributes X_1 and X_2 that contain X, such that $t_1[X_1]$ is in *eval*(*Proj*(E_1)) and $t_2[X_2]$ is in *eval*(*Proj*(E_2)). Since t_1 and t_2 are join consistent, $t_1[X_1]$ and $t_2[X_2]$ are join consistent. Hence, there is an X' such that *eval*(*Proj*(E')) is a projection of the join of $t_1[X_1]$ and $t_2[X_2]$ on X'. According to Claim .4, X' contains X. This proves Claim .5.

From Claim .5 follows that there is a set of attributes X' that contain X such that *eval*(E^m) contains $t[X']$. The tuples $t[X']$ and t_x are created from the same generators. Hence, $t[X']$ is equal to t_x or subsumes t_x. Hence, there is a tuple in

$\mathcal{F}_X^{\text{SOJO}}$ that is either equal to t_x or subsumes t_x. This proves the other side of the containment.

So far, we showed the correctness of the algorithm. Next, we show that computing a projection of a full disjunction, using an expression that the modified SOJO produces, can be done in polynomial time in the size of the input and output.

First, note the following three simple fact. *(1)* The expression that modified SOJO produces does not include any schema more than once. *(2)* Computing a projection of a relation is in polynomial time in the size of the relation. *(3)* Computing an outerjoin of a pair of relations is in polynomial time in the size of the two relations. Thus, we only need to show that each intermediate relation has a bounded size w.r.t. the size of the input and the final result.

Suppose that E is the expression that the modified SOJO produces for $R_1, \ldots, R_n$ and X. Let E_1 be a sub expression of E that is returned from one of the recursive calls. In addition, let $\mathcal{F}_X$ be the result of evaluating E and $\mathcal{F}_1$ be the result of evaluating E_1.

Claim .6 *The number of tuples in $\mathcal{F}_1$ is less than or equal to $s_r \times f$, where s_r is the number of tuples in all the relation $r_1, \ldots, r_n$ and f is the size of the result, i.e., the number of tuples in $\mathcal{F}_X$.*

To prove Claim .6 we consider two cases. In the first case, E_1 is a projection on an input relation. Obviously, in this case, $\mathcal{F}_1$ has at most s_r tuples. Thus, the claim holds.

The second case is when E_1 is a projection of $E'_1 \overset{o}{\bowtie} E'_2$ on $X \cup Z$, where E'_1 and E'_2 are sub-expressions. The set Z is a node in the Bachman diagram. We show that there cannot be too many tuples in E_1.

Every tuple t is $\mathcal{F}_1$ has two parts (not necessarily disjoint). One part is the projection of t on X. The other part is the projection of t on Z. The projection of t on X is part of a tuple of $\mathcal{F}_X$. Thus, there are at most f tuples in the projection of $\mathcal{F}_1$ on X.

According to the way decomposition is performed in the algorithm, Z is a minimal node in the Bachman diagram. Hence, for each schema R_j that appears in E_1, the set Z is either contained in R_j or disjoint from R_j. Thus, the projection of t on Z is a part of a tuple in one of the relations that appear in E_1. In other words, the projection of t on Z cannot be created by a Cartesian product or a join of two tuples from two different relations. Otherwise, the schemas of these relations do not contain Z and are not disjoint from Z, which is a contradiction to the minimality of Z. Hence, the number of tuples in the projection of $\mathcal{F}_1$ on Z is at most s_r.

Let C be the Cartesian product of the following two relations: the projection of $\mathcal{F}_1$ on Z and the projection of $\mathcal{F}_1$ on X. The number of tuples in $\mathcal{F}_1$ is less than the number of tuples in C. Also, the number of tuples in C is at most $s_r \times f$. This proves Claim .6 and proves the theorem. $\square$