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RIGOROUS HIGH-DIMENSIONAL SHADOWING USING CONTAINMENT: THE GENERAL CASE

Carmen Young, Wayne B. Hayes, Kenneth R. Jackson

Corresponding Author:
Wayne B Hayes
Dept. of Computer Science
University of California, Irvine
Irvine, California 92697-3425

Abstract. A *shadow* is an exact solution to an iterated map that remains close to an approximate solution for a long time. An elegant geometric method for proving the existence of shadows is called *containment*, and it has been proven previously in two and three dimensions, and in some special cases in higher dimensions. This paper presents the general proof using tools from differential and algebraic topology and singular homology.

1. Introduction.

1.1. Background. An *orbit* of a continuous map φ is a finite or infinite sequence of points generated using

$$\mathbf{x}_{i+1} = \varphi(\mathbf{x}_i). \tag{1}$$

Often one point, $\mathbf{x}_0$, is given, called the *initial condition.* Consider an approximation $\hat{\varphi}$ to φ with the property that

$$\|\hat{\varphi}(\mathbf{x}) - \varphi(\mathbf{x})\| < \delta. \tag{2}$$

An orbit of $\hat{\varphi}$ generated using

$$\mathbf{y}_{i+1} = \hat{\varphi}(\mathbf{y}_i) \tag{3}$$

is called a δ*-pseudo-orbit* of φ and, from (2), has the property

$$\|\mathbf{y}_{i+1} - \varphi(\mathbf{y}_i)\| < \delta \text{ for all } i.$$

Pseudo-orbits are of interest to those studying computer-generated orbits because finite-precision arithmetic is used to compute them, with the consequence that an exact orbit and a pseudo-orbit starting at the same point can diverge exponentially away from each other. See for example Grebogi, Hammel, Yorke, and Sauer (1990), hereafter GHYS. Given a pseudo-orbit (3), the exact orbit (1) is a *shadow* of (3) if

$$\|\mathbf{y}_i - \mathbf{x}_i\| < \varepsilon \text{ for all } i.$$

Shadowing was first discussed by Anosov (1967) and Bowen (1975), in relation to *hyperbolic* systems, in which space along an orbit can be uniformly separated into *expanding* and *contracting* subspaces. Let S and φ be the invariant set and the map of a hyperbolic system, respectively. In such systems, Anosov (1967) proved that $\forall\varepsilon > 0$, $\exists\ \delta > 0$ such that every infinite-length δ-pseudo orbit remaining in S is ε-shadowed by an exact trajectory in S. Bowen (1975) proved that the same result

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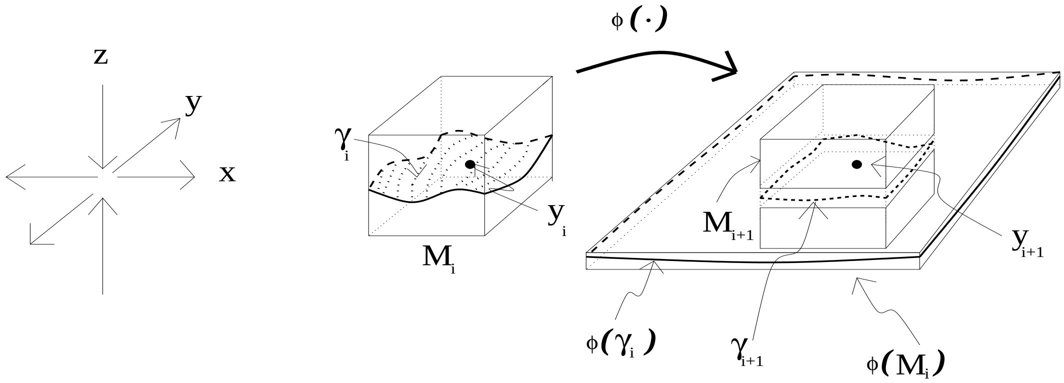

FIGURE 1. Containment in 3D with 2 expanding directions and 1 contracting.

holds if the map is required to be hyperbolic only along trajectories in the vicinity of the pseudo-orbit. Palmer (1988) proved a similar theorem along the way towards using the theory of exponential dichotomies to prove Smale's Theorem (Smale 1965, 1967).

Most systems of general interest, however, are not hyperbolic. The first studies of shadows for non-hyperbolic systems appear to be Beyn (1987) and Hammel, Yorke, and Grebogi (1987). Hammel, Yorke, and Grebogi (1988) and GHYS provide the first proof of the existence of a shadow for a two-dimensional non-hyperbolic system over a non-trivial length of time, using a method called *containment*. Here, by way of introduction, we outline a three-dimensional case that is proved in Hayes and Jackson (2003).

Let φ be a map which is not hyperbolic, but which displays *pseudo-hyperbolicity* (Hayes 2001) for a finite but non-trivial number of iterations. Let $\{\mathbf{y}_i\}_{i=a}^{b} \subset \mathbb{R}^3$ be a three-dimensional δ-pseudo-orbit of φ for integers a and b. In this case the pseudo-orbit has 1 contracting direction and two expanding directions (Figure 1), and pseudo-hyperbolicity means that as i increases, orbits separated from each other by a small distance in the expanding subspace diverge on average (but not necessarily uniformly) away from each other, while orbits separated by a small distance in the contracting subspace approach each other, on average. The three-dimensional containment process consists of building a parallelogram M_i around each point $\mathbf{y}_i$ of the pseudo-orbit such that the first pair of expanding faces $F_i^{\pm 1}$ are separated along one expanding direction (the x direction in Figure 1), the second pair of expanding faces $F_i^{\pm 2}$ are separated along the other expanding direction (the y direction in Figure 1), and the one pair of contracting faces $F_i^{\pm 3}$ are separated from each other along the contracting direction (the z direction in Figure 1). In order to prove the existence of a shadow, we require that $\varphi(M_i)$ maps over M_{i+1} so that φ flattens M_i into a thin slice, cutting M_{i+1} into 3 pieces, the middle piece of which contains a contiguous section of $\varphi(M_i)$ (as well as possibly some isolated pieces of $\varphi(M_i)$). Now, assume γ_i is a surface in M_i whose boundary connects and "wraps around" all of the expanding sides of M_i. Then there is a contiguous patch of $\varphi(\gamma_i) \cap M_{i+1} \equiv \gamma_{i+1}$ lying wholly in M_{i+1} whose boundary $\partial\gamma_{i+1}$ connects and "wraps around" the expanding sides of M_{i+1}. If this property continues for each step then, by induction, there is a $\gamma_N \neq \emptyset$ lying wholly within M_N whose boundary $\partial\gamma_N$ connects and wraps around the expanding sides of M_N. Then any point $\mathbf{x}_N \in \gamma_N$

can be traced backwards to a point $\mathbf{x}_i \in \gamma_i \subset M_i$ for $i = 0, 1, \ldots, N-1$, and the $\mathbf{x}_i$ trajectory is an exact orbit lying close the pseudo-orbit — *i.e.,* a shadow. In fact, since $\mathbf{x}_N$ can be any point in γ_N, this arguments demonstrates the existence of a 2-dimensional family of shadows. In general when there are k expanding directions, we will have a k-dimensional family of shadows.

This case, along with all other one-, two-, and three-dimensional cases, as well as some special cases in higher dimension, were proved in Hayes and Jackson (2003). The purpose of this paper is to present the general n-dimensional proof in which k directions are expanding, while $n-k$ directions are contracting.

1.2. **Overview.** The machinery that we use requires that the intersections of the manifolds $\varphi(\gamma_i) \cap M_{i+1}$ are transversal. Theorem 6 (Sard's Theorem) demonstrates that there exists γ_0 such that for every $i > 0$, $\varphi(\gamma_i)$ is transversal to M_{i+1}. In Section 2 we present the main result. Section 3 presents the background for Sard's Theorem, while Section 4 provides a brief background to singular homology and cohomology, and finally, the proof of our main result.

2. **Main Result.** Let $\varphi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism. Assume that φ displays pseudo-hyperbolicity such that there exist k directions which expand on average over time, which we will call the *nominally expanding directions*. Similarly, assume there are $(n-k)$ directions which contact on average over time, called the *nominally contracting directions*. For each $i = 0, \ldots, N$, let M_i be an n-cube in $\mathbb{R}^n$. For convenience assume that the faces of M_i are labeled so that the first $2k$ faces $F_i^{\pm j}$, $j = 1, \ldots, k$, lie transverse to the nominal expanding directions of φ, and the remaining $2(n-k)$ faces $F_i^{\pm j}$, $j = k+1, \ldots, n$, lie transverse to the nominal contracting directions of φ. We denote the union of a set of faces by listing multiple integers in the superscript. Thus the *expanding faces* of M_i are collectively denoted $\partial_X M_i \equiv F_i^{\pm 1, \ldots, \pm k}$ and the *contracting faces* of M_i are denoted $\partial_C M_i \equiv F_i^{\pm(k+1), \ldots, \pm n}$.

Let $Int(A)$ denote the interior of the set A. Refer to Figure 2. We say that M_i and M_{i+1} satisfy the (n,k)-*Inductive Containment Property* (abbreviated (n,k)-ICP), for φ if

(ICP1) $\varphi(\partial_X M_i) \cap M_{i+1} = \emptyset$ and, for all $j \in \{1, \ldots, k\}$, $\varphi(F_i^{-j})$ and $\varphi(F_i^{+j})$ lie on opposite sides of the infinite slab between the two hyperplanes containing F_{i+1}^{-j} and F_{i+1}^{+j}, respectively.

(ICP2) There is a parallelepiped $Q_{i+1} \subset \mathbb{R}^n$ with faces G_{i+1}^j parallel to the faces F_{i+1}^j of M_{i+1} for $j = \pm 1, \ldots, \pm n$ such that
 - *i)* $\varphi(M_i) \subset Int(Q_{i+1})$,
 - *ii)* $Q_{i+1} \cap \partial_C M_{i+1} = \emptyset$ and, for all $j \in \{k+1, \ldots, n\}$, F_{i+1}^{-j} and F_{i+1}^{+j} lie on opposite sides of the infinite slab between the two hyperplanes containing G_{i+1}^{-j} and G_{i+1}^{+j}.

The conditions of the Inductive Containment Property can be rigorously verified computationally (Hayes 2001).

Theorem 1 ((n,k)-Inductive Containment Theorem). *Suppose that M_i and M_{i+1} satisfy (n,k)-ICP for φ for all $i = 0, \ldots, N-1$. Then there exists a sequence of non-empty k-manifolds $\gamma_i \subset M_i$, $i = 0, \ldots, N$, such that*

$$Int(\gamma_i) \subset Int(M_i)\,, \quad \partial\gamma_i = \gamma_i \cap \partial_X M_i\,, \quad \text{and} \quad \gamma_{i+1} \subset \varphi(\gamma_i)$$

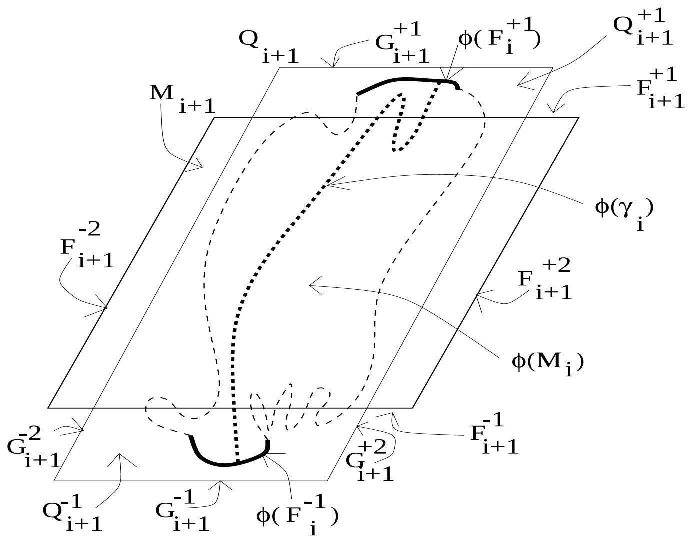

FIGURE 2. Schematic diagram of the Inductive Containment Property in 2 dimensions. The tall parallelogram is $Q_{i+1} \supset \varphi(M_i)$, the wider one is M_{i+1}. The vertical direction is expanding. The horizontal direction contracting.

for all i.

We will prove this theorem in stages. For a cleaner exposition of the proof, we will translate all objects to a *standardized* frame in the vicinity of the origin, as follows. Let $\square_n$ denote the standard unit cube in $\mathbb{R}^n$,

$$\square_n = \{(x_1, \dots, x_n) : |\, x_j| \leq 1 \text{ for } j = 1, \dots, n\},$$

and denote its faces by

$$E^{+j} = \{(x_1, \dots, x_n) : x_j = +1, |\, x_l| \leq 1 \text{ for } l \neq j\}\,, \quad j = 1, \dots, n,$$

$$E^{-j} = \{(x_1, \dots, x_n) : x_j = -1, |\, x_l| \leq 1 \text{ for } l \neq j\}\,, \quad j = 1, \dots, n.$$

We introduce the *Standardized* (n, k)*-Inductive Containment Property* by transforming both M_i and M_{i+1} to $\square_n$, as follows. For each $i = 0, \dots, N$ there is an orientation-preserving diffeomorphism (*i.e.,* change of coordinates) $\psi_i : \mathbb{R}^n \to \mathbb{R}^n$ that maps M_i to $\square_n$, and maps $F_i^{\pm j}$ to $E^{\pm j}$ for $j = 1, \dots, n$. Let $\mathcal{M}_i = \square_n$ for all i. Let $\phi_i = \psi_{i+1} \circ \varphi \circ \psi_i^{-1}$. If M_i and M_{i+1} satisfy (n, k)-ICP for φ, then by construction $\mathcal{M}_i$ and $\mathcal{M}_{i+1}$ satisfy (n, k)-ICP for ϕ_i, and we say that the Standardized (n, k)-ICP holds for ϕ_i. Note that it is easy to choose ψ_i so that the Standardized (n, k)-ICP holds for ϕ, such that there exists positive $\varepsilon < 1$ such that (ICP2) holds with

$$Q = \{(x_1, \dots, x_n) : |x_i| \leq 1/\varepsilon \text{ for } i = 1, \dots, k \text{ and } |x_i| \leq 1 - \varepsilon \text{ for } i = k + 1, \dots, n\}.$$

For our purposes, the term *manifold* will refer to a smooth manifold with boundary and corners.

Definition 1. Suppose that $\Gamma \subset \square_n$ is a k-manifold with $\partial\Gamma \subset \partial_X \square_n$. We say that $\partial\Gamma$ **wraps around** $\partial_X \square_n$ if the homology class of $[\partial\Gamma]$ in $H_{k-1}(\partial_X \square_n)$ is **not** zero.

Lemma 1. *Let $\phi : \mathbb{R}^n \to \mathbb{R}^n$ be a diffeomorphism and assume that the standardized (n,k)-ICP holds for ϕ. Let $\Gamma \subset \square_n$ be a non-empty k-manifold with boundary $\partial\Gamma \subset \partial_X \square_n$, and suppose further that $\partial\Gamma$ wraps around $\partial_X\square_n$. Finally, suppose that $\phi(\Gamma)$ is transverse to $\partial_X\square_n$, and let $\Gamma' \equiv \phi(\Gamma) \cap \square_n$. Then the following hold:*

i) Γ' is a non-empty k-manifold with boundary $\partial\Gamma' = \Gamma' \cap \partial\square_n$.
ii) $\partial\Gamma' \subset \partial_X\square_n$.
iii) $\partial\Gamma'$ wraps around $\partial_X\square_n$.

We will prove this lemma in Section 4.

Proof of Theorem 1. We will prove the theorem by induction. Let $\psi_i : \mathbb{R}^n \to \mathbb{R}^n$ be a change of co-ordinates that maps M_i to $\square_n$, and maps $F_i^{\pm j}$ to $E^{\pm j}$ for $j = 1, \ldots, n$. Let $\phi_i = \psi_{i+1} \circ \varphi \circ \psi_i^{-1}$. Then the Standardized (n,k)-ICP holds for each ϕ_i. Let $\square_k$ denote the unit k-cube in $\mathbb{R}^n$,

$$\square_k = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : |x_i| \leq 1 \text{ for } 1 \leq i \leq k,\ x_{k+1} = \cdots = x_n = 0\}.$$

This is a k-dimensional submanifold of $\square_n$, and its boundary

$$\partial\square_k = \{(x_1, \ldots, x_n) \in \square_k : x_j = \pm 1 \text{ for some } j \in \{1, \ldots, k\}\},$$

is contained in $\partial_X\square_n$. By Lemma 3 and Definition 6, the homology class $[\partial\square_k] \in H_{k-1}(\partial_X\square_n)$ is 1. By Theorem 6 (Sard's Theorem), we can homotope $\square_k$ relative to its boundary to a k-manifold Γ with $\partial\Gamma = \partial\square_k$ such that Γ intersects $\phi_0^{-1}\phi_1^{-1}\cdots\phi_i^{-1}(\partial_X\square_n)$ transversally for each $i = 0, \ldots, N$. Thus $[\partial\Gamma] = [\partial\square_k] \neq 0$, and

$$\phi_i\phi_{i-1}\cdots\phi_0(\Gamma) \text{ is transverse to } \partial_X\square_n \text{ for all } i. \tag{4}$$

We start the induction by taking $\Gamma_0 = \Gamma$. Then Γ_0 and ϕ_0 satisfy the hypotheses of Lemma 1. Let $\Gamma_1 = \phi_0(\Gamma_0) \cap \square_n$. Then Γ_1 is a non-empty k-manifold with boundary $\partial\Gamma_1 \subset \partial_X\square_n$, and $[\partial\Gamma_1] \neq 0$.

At the i-th step of the induction, we have a non-empty k-manifold $\Gamma_i \subset \square_n$ with $\partial\Gamma_i \subset \partial_X\square_n$ and $[\partial\Gamma_i] \neq 0$ in $H_{k-1}(\partial_X\square_n)$. Moreover $\Gamma_i \subset \phi_{i-1}(\Gamma_{i-1}) \subset \cdots \subset \phi_{i-1}\phi_{i-2}\cdots\phi_0(\Gamma_0)$, which implies that $\phi_i(\Gamma_i) \subset \phi_i\cdots\phi_0(\Gamma_0)$. From (4), we see that $\phi_i(\Gamma_i)$ is transverse to $\partial_X\square_n$, so we can apply Lemma 1. If we set $\Gamma_{i+1} = \phi_i(\Gamma_i) \cap \square_n$, then Γ_{i+1} and ϕ_{i+1} satisfy the hypotheses of Lemma 1, and the induction continues.

By the N-th step of the induction, we have produced non-empty k-manifolds Γ_i, $i = 0, \ldots, N$, with $Int(\Gamma_i) \subset Int(\square_n)$, $\partial\Gamma_i = \Gamma_i \cap \partial_X\square_n$,

$$[\partial\Gamma_i] \neq 0 \text{ for all } i \ ,$$

and

$$\Gamma_{i+1} \subset \phi_i(\Gamma_i) \text{ for all } i \ .$$

Let $\gamma_i = \psi_i^{-1}(\Gamma_i)$ for each $i = 0, \ldots, N$. Then $Int(\gamma_i) \subset Int(M_i)$, $\partial\gamma_i = \gamma_i \cap \partial_X M_i$, and

$$\gamma_{i+1} = \psi_{i+1}^{-1}(\Gamma_{i+1}) \subset \psi_{i+1}^{-1}(\phi_i(\Gamma_i)) = \psi_{i+1}^{-1}(\psi_{i+1}\,\varphi\,\psi_i^{-1})(\Gamma_i) = \varphi(\gamma_i).$$

This completes the proof of Theorem 1.

Corollary 1 (Shadowing Containment Theorem). *Let $\{M_i\}_{i=0}^N$ be a sequence of n-dimensional parallelepipeds enclosing a pseudo-trajectory $\{\mathbf{y}_i\}_{i=0}^N$ such that $\mathbf{y}_i \in M_i, i = 0, \ldots, N$, and suppose that M_i and M_{i+1} satisfy (n,k)-ICP for all $i = 0, \ldots, N-1$. Let ε be the maximum diameter of M_i over all i. Then there exists a sequence of k-dimensional manifolds $\{\gamma_i \subset M_i\}_{i=0}^N$ such that any point $\mathbf{x}_j \in \gamma_j, j = 0, \ldots, N$ admits an exact orbit $\{\mathbf{x}_i\}_{i=0}^N$ which is an ε-shadow of $\{\mathbf{y}_i\}_{i=0}^N$. That is, $\|\mathbf{x}_i - \mathbf{y}_i\| \leq \varepsilon$ for all $i = 0, \ldots, N$.*

Proof. By the Inductive Containment Theorem, there is a sequence of non-empty k-manifolds $\gamma_i \subset M_i$ such that $\gamma_i \subset Int(\varphi(\gamma_{i-1}))$ for all i. Let $\mathbf{x}_N$ be any point in γ_N. As φ is a diffeomorphism, there is a unique point $\mathbf{x}_0 \in \gamma_0$ such that $\varphi^N(\mathbf{x}_0) = \mathbf{x}_N$. For each i we set $\mathbf{x}_i = \varphi^i(\mathbf{x}_0)$. Then $\mathbf{x}_{i+1} = \varphi(\mathbf{x}_i) \in M_{i+1}$ for all i and $\{\mathbf{x}_i\}_{i=0}^N$ is an ε-shadow of $\{\mathbf{y}_i\}_{i=0}^N$. Since this is true for all $\mathbf{x}_N \in \gamma_N$, γ_N admits a k-dimensional family of shadows. ☐

3. **Sard's Theorem and Transversality.** For the purpose of proving the Inductive Containment Theorem in arbitrary dimensions, we would like to determine under what conditions two objects in $\mathbb{R}^n$ intersect in a "nice" way. It turns out that when the objects in question are smooth manifolds, then a good answer is provided by *transversality theory.* In this section we review some of the basic concepts from transversality theory that we need. Guillemin and Pollack (1974) supply a more detailed introduction, together with some applications to geometry.

3.1. **Manifolds.** Let X be a smooth manifold of dimension k. That is, every point $x \in X$ has an open neighborhood V which is homeomorhic to an open set $U \subset \mathbb{R}^k$:

$$\theta : U \xrightarrow{\cong} V \subset X. \tag{5}$$

The triplet (θ, U, V) is sometimes referred to as a *coordinate chart* near x. If (θ_1, U_1, V_1), (θ_2, U_2, V_2) are two coordinate charts (near points x_1 and x_2, say) that *overlap* in the sense that $V_1 \cap V_2 \neq \emptyset$, then we further require that

$$\theta_1^{-1} \circ \theta_2 : \theta_2^{-1}(V_1 \cap V_2) \to \theta_1^{-1}(V_1 \cap V_2)$$
$$\text{and} \quad \theta_2^{-1} \circ \theta_1 : \theta_1^{-1}(V_1 \cap V_2) \to \theta_2^{-1}(V_1 \cap V_2)$$

be smooth, as maps between open sets in $\mathbb{R}^k$.

3.2. **Tangent space.** In the cases of interest to us, the manifold X sits in some higher-dimensional Euclidean space $\mathbb{R}^n$, so the coordinate map (5) is simply a smooth map from $U \subset \mathbb{R}^k$ to $\mathbb{R}^n$, with image equal to V. Thus the derivative of θ at a point $u \in U$ makes sense as a linear map from $\mathbb{R}^k$ to $\mathbb{R}^n$, *i.e.,* it is a real $n \times k$ matrix. We denote this derivative by $d\theta_u$.

Suppose (θ, U, V) is a coordinate chart near $x \in X$, and $\theta(u) = x$. We define the *tangent space to X at x* to be the image of $d\theta_u : \mathbb{R}^k \to \mathbb{R}^n$. (Equivalently, the tangent space is the linear span of the column vectors of $d\theta_u$.) This is a vector subspace of $\mathbb{R}^n$ which we denote by T_xX. Geometrically,

$$x + T_xX = \{x + v \mid v \in T_xX\}$$

consists of all vectors starting at x that are tangent there to X. In other words, it is the best approximation of X near x by a linear subspace. One can easily check that if (θ', U', V') is another coordinate chart near x with $\theta'(u') = x$, then $\text{Image}(d\theta_u) = \text{Image}(d\theta'_u)$ – that is, T_xX is well-defined and independent of our choice of coordinate chart.

3.3. **Smooth maps and differentials.** If X^k and Y^l are smooth manifolds of respective dimensions k and l, then a continuous map $f : X \to Y$ is called *smooth* if, for every coordinate chart (θ, U, V) for X and $(\tilde{\theta}, \tilde{U}, \tilde{V})$ for Y, the restriction

$$\tilde{\theta}^{-1} \circ f \circ \theta|_{(f \circ \theta)^{-1}(\tilde{U})}$$

is smooth when considered as a map defined in an open subset of $\mathbb{R}^k$ with values in $\mathbb{R}^l$. If $X \subset \mathbb{R}^m$ and $Y \subset \mathbb{R}^n$ for some $m > k$, $n > l$, it is equivalent to say that

$f : X \to Y$ is smooth if around any point $x \in X$ there is an open ball B_x and a smooth map $F : B_x \to \mathbb{R}^n$ such that the restriction of F to $X \cap B_x$ equals f:

$$F|_{X\cap B_x} = f.$$

If f is smooth and $f(x) = y$, then the *derivative* of f at x is a linear map

$$df_x : T_xX \to T_yY.$$

To define the derivative, suppose that (θ, U, V) is a coordinate chart near x with $\theta(u) = x$, and $(\tilde{\theta}, \tilde{U}, \tilde{V})$ is a coordinate chart near y with $\tilde{\theta}(\tilde{u}) = y$. Set $h = \tilde{\theta}^{-1} \circ f \circ \theta$ on $(f \circ \theta)^{-1}(\tilde{V})$ so that the diagram

$$\begin{tikzcd} V \arrow[r, "f"] & \tilde{V} \\ U \arrow[u, "\theta"] \arrow[r, "h"'] & \tilde{U} \arrow[u, "\tilde{\theta}"'] \end{tikzcd}$$

commutes. Note in particular that $h(u) = \tilde{u}$. We then set

$$df_x(\nu) = \; d\tilde{\theta}_{\tilde{u}} \circ dh_u \circ (d\theta_u)^{-1}(\nu), \text{ for } \nu \in \mathrm{Image}(d\theta_u)\ .$$

(If we think of the derivatives of $\tilde{\theta}, h$ and θ as matrices, then the latter composition is a product of matrices.) This clearly maps $T_xX = \mathrm{Image}(d\theta_u)$ into $T_yY = \mathrm{Image}(d\tilde{\theta}_{\tilde{u}})$. If $X \subset \mathbb{R}^m$, $Y \subset \mathbb{R}^n$ and f near x equals the restriction of $F : B_x \to \mathbb{R}^n$, then it is not hard to show that df_x equals the restriction of dF_x to $T_xX \subset \mathbb{R}^m$.

If $x \in X$ and $df_x = 0$, that is, $df_x(\nu) = 0$ for all $\nu \in T_xX$, then x is called a *critical point* of f. If $y \in Y$ and there is some $x \in f^{-1}(y)$ such that $df_x = 0$, then y is called a *critical value* of f. If no such x exists for y, then y is called a *regular value* of f.

3.4. **Transversality.** The smooth map $f : X \to Y$ is *transversal* to the submanifold $Z \subset Y$ if

$$\mathrm{Image}(df_x) + T_{f(x)}(Z) = T_{f(x)}Y$$

for every point x in the preimage of Z.

Theorem 2 (Preimage Theorem)**.** *Let $f : X \to Y$ be a smooth map of manifolds. If f is transversal to a submanifold $Z \subset Y$, then $f^{-1}(Z)$ is a submanifold of X and the codimension of $f^{-1}(Z)$ in X equals the codimension of Z in Y.*

Note: The *codimension* of Z in Y is $\mathrm{codim}\, Z = \dim Y - \dim Z$.

Definition 2. Suppose $X \subset Y$. The **inclusion map** $i : X \to Y$ is defined by $i(x) = x$ for all $x \in X$.

When X and Z are both submanifolds of the same manifold Y, then a point $x \in X$ lies in the intersection $X \cap Z$ if and only if x lies in the preimage of Z under the inclusion map $i : X \hookrightarrow Y$. We say that X and Z *intersect transversally* in Y if

$$T_xX + T_xZ = T_xY$$

for every $x \in X \cap Z$. Since the derivative $di_x : T_xX \to T_xY$ is simply the inclusion of T_xX into T_xY, the next result is a direct consequence of the Preimage Theorem.

Theorem 3. *If the submanifolds X and Z intersect transversally in Y, then their intersection $X \cap Z$ is again a submanifold and*

$$\operatorname{codim}(X \cap Z) = \operatorname{codim} X + \operatorname{codim} Z\,.$$

Theorem 4 (Transversality Theorem, (Guillemin and Pollack 1974))**.** *Let $F : X \times S \to Y$ be a smooth map of manifolds. Let $Z \subset Y$ be a smooth submanifold without boundary. If F is transverse to Z then for almost every $s \in S$, $F_s = F(\cdot, s)$ is transverse to Z.*

The theorem follows from an application of Sard's theorem, which we now state. (See (Guillemin and Pollack 1974, Chap 2,§1).)

Theorem 5 (Sard's Theorem)**.** *For any smooth map of a manifold X (with boundary) into a boundaryless manifold Y, almost every point of Y is a regular value of $f : X \to Y$ (and of $\partial f = f|_{\partial X} : \partial X \to Y$).*

The idea behind the proof of Theorem 4 is this. By the Preimage theorem, $W = F^{-1}(Z)$ is a submanifold of $X \times S$. Let $\pi : X \times S \to S$ be the natural projection map, and consider its restriction to $W \subset X \times S$. By Sard's theorem, almost every value of $s \in S$ is a regular value of $\pi : W \to S$. Using the fact that F is transversal to Z, one can show that the regular values of $\pi|_W$ correspond to the values of s for which F_s is transversal to Z. For details of the proof, we refer the reader to (Guillemin and Pollack 1974, Chap 2, §3).

Combining the Transversality theorem with the Preimage theorem, one can prove that any smooth map $X \to Y$ can be deformed by an arbitraily small amount to a map that is transversal to a submanifold $Z \subset Y$. As a special case of this, we have the following.
Notation: Given an open set V, we write $U \subset\subset V$ to signify that there is a compact set K with $U \subset K \subset V$.

Theorem 6. *Let $\square_k$ denote the k-cube*

$$\square_k = \{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| \le 1 \text{ for } 1 \le i \le k\ ,\ x_{k+1} = \cdots = x_n = 0\},$$

with boundary

$$\partial\square_k = \{(x_1, \dots, x_n) \in \square_k : x_i = \pm 1 \text{ for some } i \in \{1, \dots, k\}.\}.$$

Let $Z_1, \dots, Z_L$ be smooth submanifolds of $\mathbb{R}^n$, and suppose each has the property that $Z_l \cap \square_k \subset\subset \operatorname{int} \square_k$. Then we can homotope $\square_k$ relative to its boundary to a k-manifold Γ with $\partial\Gamma = \partial\square_k$, so that Γ intersects Z_l transversally for all $l = 1, \dots, L$.

Proof. There exists a compact set K such that $Z_l \cap \square_k \subset K \subset \operatorname{int} \square_k$ for all l. Take $\varepsilon : \square_k \to \mathbb{R}$ to be a smooth, compactly supported bump function with $\operatorname{spt}(\varepsilon) \subset \operatorname{int} \square_k$. We can assume that $K \subset \{x : \varepsilon(x) \ne 0\}$.

Let S denote the open unit ball in $\mathbb{R}^n$ and let $F : \square_k \times S \to \mathbb{R}^n$ be the smooth map $F(x, s) = x + \varepsilon(x) \cdot s$. For any fixed point x where $\varepsilon(x) \ne 0$, the map $s \mapsto F(x, s)$ is a rescaling followed by translation of the ball S, hence is a submersion. If $Z \subset \mathbb{R}^n$ is a submanifold and $Z \cap \square_k \subset \{x : \varepsilon(x) \ne 0\}$, then it follows that F is transversal to Z. So by the transversality theorem of Guillemin-Pollack, the map $x \mapsto F(x, s)$ is transverse to Z for almost every $s \in S$.

By our hypotheses, $Z_l \cap \square_k \subset \{x : \varepsilon(x) \ne 0\}$ for every l. Thus for each l, there is a subset $\Omega_l \subset S$ of measure zero such that $F_s = F(\cdot, s)$ is transverse to Z_l for any $s \in S \backslash \Omega_l$. The union $\Omega = \Omega_1 \cup \cdots \cup \Omega_L$ again has measure zero, and for any

$s \in S \backslash \Omega$, F_s is transverse to all the Z_l. Now the set $S \cap \Box_k$ also has measure zero and for any $s \in S \backslash \Box_k$,

$$\Gamma_s = \{x + \varepsilon(x) \cdot s \,:\, x \in \Box_k\}$$

is a smooth submanifold of $\mathbb{R}^n$ with boundary (and corners). Moreover, if $s \in S \backslash (\Omega \cup \Box_k)$, then saying that F_s is transverse to Z_l is equivalent to saying that Γ_s and Z_l intersect transversally.

Let us fix one such s and take $\Gamma = \Gamma_s$ to be our desired k-manifold. The homotopy from $\Box_k$ to Γ is given by

$$h_t(x) = x + t\varepsilon(x) \cdot s\,, \qquad t \in [0,1].$$

Clearly $h_0 : \Box_k \to \Box_k$ is the identity map and h_1 maps $\Box_k$ homeomorphically onto Γ. For any $x \in \partial\Box_k$, we have $\varepsilon(x) = 0$ because $\mathrm{spt}(\varepsilon) \subset \mathrm{int}\,\Box_k$. Therefore $h_t(x) = x$ for all $x \in \partial\Box_k$ and all $t \in [0,1]$, which is to say that the homotopy fixes the boundary and, in particular, $\partial\Gamma = \partial\Box_k$. $\Box$

4. **Proof of Lemma 1.**

4.1. **Review of Singular Homology (with integer coefficients).**

For a quick introduction to singular homology (and cohomology), we refer the reader to Appendix A of Milnor and Stasheff (1974). More details can also be found in the graduate text by Greenberg (1981).

The basic objects of singular homology are *equivalence classes* of *singular simplices* in a predetermined topological space. These in turn are modeled on standard simplices in Euclidean space.

Definition 3. Let $K \geq 0$. The **standard K-simplex** is the convex set $\Delta^K \subset \mathbb{R}^{K+1}$, consisting of all $(K+1)$-tuples $(y_0, \dots, y_K)$ with

$$y_i \geq 0 \qquad , \qquad y_0 + y_1 + \cdots + y_K = 1.$$

Any continuous map σ from Δ^K to a topological space X is called a **singular K-simplex** in X.

Let $K \geq 0$ be an integer and let $C_K(X)$ be the free $\mathbb{Z}$-module obtained by taking one generator $[\sigma]$ for each singular K-simplex in X. We call $C_K(X)$ the K-th *singular chain group* of X. For $K < 0$, $C_K(X)$ is defined to be zero.

To define the equivalence relation on $C_K(X)$, we need to introduce the following *boundary operator*.

Definition 4. Let $\sigma : \Delta^K \to X$ be a singular K-simplex in X. The i-th *face* of σ is the singular $(K-1)$-simplex

$$\sigma \circ \lambda_i : \Delta^{K-1} \to X,$$

where the linear embedding $\lambda_i : \Delta^{K-1} \to \Delta^K$ is defined by

$$\lambda_i(y_0, \dots, y_{i-1}, y_{i+1}, \dots, y_K) = (y_0, \dots, y_{i-1}, 0, y_{i+1}, \dots, y_K).$$

The homomorphism

$$\partial : C_K(X) \to C_{K-1}(X)$$

given by

$$\partial[\sigma] = [\sigma \circ \lambda_0] - [\sigma \circ \lambda_1] + - \cdots + (-1)^K [\sigma \circ \lambda_K]$$

is called the **boundary homomorphism**.

It is an exercise in algebra to verify that

$$\partial \circ \partial = 0. \tag{6}$$

Let $\mathcal{Z}_K(X)$ be the kernel of $\partial : C_K(X) \to C_{K-1}(X)$, and let $\mathcal{B}_K(X)$ be the image of $\partial : C_{K+1}(X) \to C_K(X)$. By (6), $\mathcal{B}_K(X) \subset \mathcal{Z}_K(X)$ so the quotient

$$H_K(X) = \mathcal{Z}_K(X)/\mathcal{B}_K(X) \tag{7}$$

makes sense. We call $H_K(X)$ the K-th *singular homology group* of X, and an element of $H_K(X)$ is called a *homology class.*

Suppose $f : X \to Y$ is a continuous map. By composing with f, we get a map

$$f\circ : \{K\text{-simplices in } X\} \to \{K\text{-simplices in } Y\}$$

which maps σ to $f \circ \sigma$. One can show further that there is an "induced" map

$$f_* : H_K(X) \to H_K(Y).$$

See Appendix A of Milnor and Stasheff (1974), or Greenberg (1981) for details.

Definition 5. We call $f_* : H_K(X) \to H_K(Y)$ the **push forward map** of f.

The following is a basic result in homology theory.

Proposition 1. *Let $f_1, f_2 : X \to Y$ be continuous maps and suppose that f_1 is homotopic to f_2. Then the push-forward maps $(f_1)_* : H_K(X) \to H_K(Y)$ and $(f_2)_* : H_K(X) \to H_K(Y)$ are equal. That is, $(f_1)_* = (f_2)_*$ as maps from $H_K(X)$ to $H_K(Y)$.*

Proposition 2. *Let m be a positive integer and let S^m be the m-dimensional sphere. Then*

$$H_i(S^m) \cong \begin{cases} \mathbb{Z} & \text{if } i = 0 \text{ or } i = m \\ 0 & \text{otherwise.} \end{cases}$$

Lemma 2. *$Q \backslash \Box_n$ is homotopic to S^{k-1}.*

Lemma 3. *$\partial_X \Box_n$ is homotopic to S^{k-1}.*

Lemma 4. *If ϕ satisfies the Standardized (n,k)-ICP then $\phi|_{\partial_X \Box_n} : \partial_X \Box_n \to Q \backslash \Box_n$ induces an isomorphism in homology, that is,*

$$\phi_* : H_r(\partial_X \Box_n) \to H_r(Q \backslash \Box_n)$$

is an isomorphism for all r.

Proof of Lemma 2. By definition,

$$\Box_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n : |x_i| \le 1 \text{ for all } i = 1, \dots, n\} \tag{8}$$

$$\Box_k = \{(x_1, \dots, x_k) \in \mathbb{R}^k : |x_i| \le 1 \text{ for all } i = 1, \dots, k\}$$

$$Q = \{(x_1, \dots, x_n) \in \mathbb{R}^n \;:\; |x_i| \le 1 + \varepsilon \text{ for } i = 1, \dots, k, \text{ and } |x_i| \le 1 - \varepsilon \text{ for } i = k+1, \dots, n\}. \tag{9}$$

We identify $\Box_k$ with the cross section

$$\{(x_1, \dots, x_n) \in \Box_n : x_{k+1} = \dots = x_n = 0\}.$$

By (8) and (9),

$$Q \backslash \Box_n = \{(x_1, \dots, x_n) \in \mathbb{R}^n \;:\; |x_i| \le 1 + \varepsilon \text{ for } i = 1, \dots, k \textit{ and } |x_i| \le 1 - \varepsilon \text{ for } i = k+1, \dots, n \textit{ and } |x_j| > 1 \text{ for some } j\}.$$

If $j \geq k+1$, then $|x_j| \leq 1-\varepsilon < 1$; therefore $|x_j| > 1$ is only possible when $1 \leq j \leq k$. Thus

$$Q\backslash\Box_n = \left\{ (x_1,\ldots,x_n) \in \mathbb{R}^n : \begin{array}{l} |x_i| \leq 1+\varepsilon \text{ for } i = 1,\ldots,k \\ |x_i| \leq 1-\varepsilon \text{ for } i = k+1,\ldots,n \\ |x_j| > 1 \text{ for some } j,\ 1 \leq j \leq k \end{array} \right\}. \tag{10}$$

Next we define a retraction of $Q\backslash\Box_n$ onto $\Box_k(1+\varepsilon)\backslash\Box_k$. For each $t \in [0,1]$, set $f_t(x_1,\ldots,x_n) = (x_1,\ldots,x_k,tx_{k+1},\ldots,tx_n)$. If $|x_i| \leq 1-\varepsilon$ then $|tx_i| \leq |x_i| \leq 1-\varepsilon$, so $f_t(Q\backslash\Box_n) \subset Q\backslash\Box_n$ for all t. Note also that f_1 is the identity map and $f_0(x_1,\ldots,x_n) = (x_1,\ldots,x_k,0,\ldots,0)$ lies in the set

$$\begin{aligned}\Box_k(1+\varepsilon)\backslash\Box_k = \{(x_1,\ldots,x_k,0,\ldots,0) \quad &: \quad |x_i| \leq 1+\varepsilon \text{ for } i = 1,\ldots,k \text{ and} \\ &|x_j| > 1 \text{ for some } j = 1,\ldots,k\}.\end{aligned}$$

This proves that f_0 is homotopic to the identity map and is a retraction. It follows that $Q\backslash\Box_n$ is homotopic to $\Box_k(1+\varepsilon)\backslash\Box_k$.

On the other hand, we can show that $\Box_k(1+\varepsilon)\backslash\Box_k$ retracts onto $\partial\Box_k$. For any $x = (x_1,\ldots,x_n)$, let $m(x) = \{|x_i| : 1 \leq i \leq n\}$. If $x \neq 0$ then $m(x) \neq 0$, so $p(x) = \frac{1}{m(x)} \cdot x$ is well-defined. In particular, p is defined on $\Box_k(1+\varepsilon)\backslash\Box_k$ and maps this set onto $\partial\Box_k$. The function $p_t(x) := (\frac{1}{m(x)})^t \cdot x$, for $t \in [0,1]$, gives a homotopy from $p_0 = id$ to $p = p_1$. This proves that p is a retraction.

In conclusion, $p \circ f_0$ maps $Q\backslash\Box_n$ onto $\partial\Box_k$ and is a homotopy equivalence. As $\partial\Box_k$ is homotopy equivalent to S^{k-1}, the lemma is proved. $\Box$

We will need a preferred generator for the proof of Lemma 4 below, so let us specify one now. By Proposition 2 and Lemma 2, $H_{k-1}(Q\backslash\Box_n) \cong \mathbb{Z}$ has two possible generators. First, observe that $\partial\Box_k$ has a natural decomposition as a formal sum of $(k-1)$-simplices, and that $\partial(\partial\Box_k) = 0$. It follows that $\partial\Box_k$ represents an element in $\mathcal{Z}_{k-1}(\partial\Box_k)$. In fact, the homology class represented by $\partial\Box_k$ generates the group $H_{k-1}(\partial\Box_k) \cong \mathbb{Z}$. We denote this class by $[\partial\Box_k]$. Next let $i_1 : \partial\Box_k \hookrightarrow Q\backslash\Box_n$ denote the natural inclusion. Note that i_1 is a homotopy inverse to $p \circ f_0$.

Definition 6. The class $[i_1] = (i_1)_*[\partial\Box_k]$ will be our preferred generator for $H_{k-1}(Q\backslash\Box_n)$.

Proof of Lemma 3. By definition,

$$\partial_X\Box_n = \{(x_1,\ldots,x_n) \in \Box_n : |x_j| = 1 \text{ for some } j = 1,\ldots,k\ \}. \tag{11}$$

For $t \in [0,1]$, define $g_t : \partial_X\Box_n \to \partial_X\Box_n$ by

$$g_t(x_1,\ldots,x_n) = (x_1,\ldots,x_k,tx_{k+1},\ldots,tx_n).$$

Note that if $|x_i| \leq 1$, then $|tx_i| \leq |x_i| \leq 1$, so indeed $g_t(\partial_X\Box_n) \subset \partial_X\Box_n$. As in the proof of Lemma 2, g_1 is the identity map and $g_0(x_1,\ldots,x_n) = (x_1,\ldots,x_k,0,\ldots,0) \in \partial\Box_k$, which proves that $g_0 \simeq g_1$. Thus g_0 is a retraction of $\partial_X\Box_n$ onto $\partial\Box_k$.

On the other hand, $\partial\Box_k \simeq \partial D^k = S^{k-1}$, so we have proved

$$\partial_X\Box_n \simeq \partial\Box_k \simeq S^{k-1},$$

as required.

Remark. Let i_2 denote the natural inclusion of $\partial\Box_k$ into $\partial_X\Box_n$, and note that i_2 is a homotopy inverse to g_0. We will take $[i_2] = (i_2)_*[\partial\Box_k]$ to be the preferred generator of $H_{k-1}(\partial_X\Box_n) \cong \mathbb{Z}$.

Proof of Lemma 4. Let φ be a diffeomorphism satisfying (n,k)-ICP, and ϕ be the associated form of φ in standardized co-ordinates that satisfies the Standardized (n,k)-ICP. Then ϕ maps $\partial_X\Box_n$ into the set $Q\backslash\Box_n$. We claim that *i)* $\phi|_{\partial_X \square_n}$ is homotopic to $h|_{\partial_X \square_n}$, where h is the "hyperbolic map" defined by

$$h(x_1, \ldots, x_n) = ((1+\varepsilon)x_1, \ldots, (1+\varepsilon)x_k, (1-\varepsilon)x_{k+1}, \ldots, (1-\varepsilon)x_n); \qquad (12)$$

ii) $h|_{\partial_X \square_n} : \partial_X \square_n \to Q \backslash \square_n$ induces an isomorphism in homology.

We begin by proving (i). For each $j = 1, \ldots, k$, let

$$Q_{j+} = Q \cap \{x_j > +1\}, \qquad Q_{j-} = Q \cap \{x_j < -1\}.$$

Thus, for example, Q_{j+} consists of all points $(x_1, \ldots, x_n)$ such that $|x_i| \le 1+\varepsilon$ for $1 \le i \le k$, $|x_i| \le 1-\varepsilon$ for $k+1 \le i \le n$, and $x_j > +1$.

Each of these sets is a product of intervals; namely,

$$Q_{j+} = [-1-\varepsilon, 1+\varepsilon]^{k-1} \times (1, 1+\varepsilon] \times [-1+\varepsilon, 1-\varepsilon]^{n-k}$$

$$Q_{j-} = [-1-\varepsilon, 1+\varepsilon]^{k-1} \times [-1-\varepsilon, -1) \times [-1+\varepsilon, 1-\varepsilon]^{n-k}.$$

Therefore $Q_{j\pm}$ is fully contractible.

By the Standardized (n,k)-ICP, ϕ maps the set

$$B_{j\pm} := \partial_X \square_n \cap \{x_j = \pm 1\}$$

into $Q_{j\pm}$. The hyperbolic map h defined by (12) also maps $B_{j\pm}$ into $Q_{j\pm}$. As $Q_{j\pm}$ is contractible, the two maps

$$\phi|_{B_{j\pm}} : B_{j\pm} \to Q_{j\pm} \text{ and } h|_{B_{j\pm}} : B_{j\pm} \to Q_{j\pm}$$

must be homotopic. Let $H_{j\pm} : B_{j\pm} \times I \to Q_{j\pm}$ be a homotopy between them with $H_{j\pm}(\cdot, 0) = \phi$ and $H_{j\pm}(\cdot, 1) = h$.

For any $j' \ne j$, $H_{j\pm}$ maps the "overlap" $(B_{j\pm} \cap B_{j'\pm}) \times I$ into $Q_{j\pm} \cap Q_{j'\pm}$. The same is true for $H_{j'\pm}$. But the intersection of $Q_{j\pm}$ and $Q_{j'\pm}$ is also a product of intervals, hence contractible. This implies that we can choose the homotopy maps $H_{j\pm}$ in such a way that they agree on overlaps; *i.e.,* if $x \in B_{j\pm} \cap B_{j'\pm}$ and $t \in I$, then $H_{j\pm}(x,t) = H_{j'\pm}(x,t)$.

Now $\partial_X \square_n$ equals the union, over all $j = 1, \ldots, k$, of $B_{j+} \cup B_{j-}$. Thus, given any $(x,t) \in \partial_X \square_n \times I$, we can find j such that $x \in B_{j+}$ or B_{j-}. We therefore construct a homotopy $H : \partial\Gamma \times I \to Q \backslash \square_n$ from $\phi = H(\cdot, 0)$ to $h = H(\cdot, 1)$ by patching together the various maps $H_{j\pm}$. To be precise, if $(x,t) \in \partial\Gamma \times I$, then choose $j \in \{1, \ldots, k\}$ such that x lies in B_{j+} or B_{j-}. Suppose for example that $x \in B_{j+}$. Then we define $H(x,t) := H_{j+}(x,t)$. If $x \in B_{j'\pm}$ for some other $j' \ne j$ in the set $\{1, \ldots, k\}$, then $H_{j'\pm}(x,t) = H_{j+}(x,t)$ by the remarks of the preceding paragraph. Thus H is a well-defined map, and (i) is proved.

To prove (ii), we will show that h_* maps $[i_2]$ to $[i_1]$. Thus we need to show that $h_*(i_2)_*[\partial\square_k] = (i_1)_*[\partial\square_k]$. Since i_1 and $p \circ f_0$ are homotopy inverses, it is equivalent to show that

$$(p \circ f_0)_* h_* (i_2)_* [\partial\square_k] = [\partial\square_k]. \qquad (13)$$

Now consider $p \circ f_0 \circ h \circ i_2$, which maps $\partial\square_k$ to itself. It is simple to check that for any $(x_1, \ldots, x_k, 0, \ldots, 0) \in \partial\square_k$,

$$\begin{aligned} & p \circ f_0 \circ h \circ i_2(x_1, \ldots, x_k, 0, \ldots, 0) \\ = \; & p \circ f_0((1+\varepsilon)x_1, \ldots, (1+\varepsilon)x_k, 0, \ldots, 0) \\ = \; & p((1+\varepsilon)x_1, \ldots, (1+\varepsilon)x_k, 0, \ldots, 0) \\ = \; & \frac{1}{1+\varepsilon} \cdot ((1+\varepsilon)x_1, \ldots, (1+\varepsilon)x_k, 0, \ldots, 0). \end{aligned}$$

Thus $p \circ f_0 \circ h \circ i_2|_{\partial \Box_k}$ equals the identity map. But the identity map induces the identity map in homology, so we have proved (13).

Proof of Lemma 1. By transversality, $\Gamma' = \phi(\Gamma) \cap \Box_n$ is a k-manifold with boundary, and its boundary equals $\phi(\Gamma) \cap \partial \Box_n$. To see that $\Gamma' \neq \emptyset$, we use Lemma 4. For suppose that $\phi(\Gamma) \cap \Box_n = \emptyset$. Then $\phi(\Gamma) \subset Q \backslash \Box_n$ and $\partial(\phi(\Gamma)) = \phi(\partial\Gamma)$, which is to say that $\phi(\partial\Gamma)$ is a boundary element in $Q\backslash\Box_n$, *i.e.,* it represents an element in $\mathcal{B}_{k-1}(Q\backslash\Box_n)$. By the definition of singular homology (7), $[\phi(\partial\Gamma)] = 0$ in $H_{k-1}(Q\backslash\Box_n)$. On the other hand, $[\phi(\partial\Gamma)] = \phi_*[\partial\Gamma]$ because ϕ is a diffeomorphism, and $[\partial\Gamma] \neq 0$ by our wrap-around assumption. By Lemma 4, $\phi_* : H_{k-1}(\partial_X \Box_n) \to H_{k-1}(Q\backslash\Box_n)$ is an isomorphism, meaning that $\phi_*[\partial\Gamma] \neq 0$, a contradiction. Thus our assumption on $\phi(\Gamma) \cap \Box_n$ must have been false. We conclude that $\phi(\Gamma) \cap \Box_n$ is non-empty. This proves (i).

By the Standardized (n,k)-ICP, $\phi(\Gamma) \subset Q$ and so $\phi(\Gamma) \cap \partial_C \Box_n = \emptyset$. This proves (ii).

Let ι denote the inclusion $\partial_X \Box_n \hookrightarrow Q\backslash\Box_n$. We have

$$\iota \circ i_2 = i_1, \tag{14}$$

where $i_1 : \partial\Box_k \hookrightarrow Q\backslash\Box_n$ and $i_2 : \partial\Box_k \hookrightarrow \partial_X\Box_n$ denote, as before, the natural inclusion maps. Since $[i_2]$ generates $H_{k-1}(\partial_X\Box_n) \cong \mathbb{Z}$, there are uniquely determined integers $d, d' \in \mathbb{Z}$ such that

$$[\partial\Gamma] = d \cdot [i_2] \qquad \text{and} \qquad [\partial\Gamma'] = d' \cdot [i_2].$$

Let $A = \phi(\Gamma)\backslash Int(\Box_n)$. By our transversality assumption, A is a k-manifold with boundary, and its boundary equals

$$\phi(\partial\Gamma) - \partial\Gamma'. \tag{15}$$

This is to say that ∂A is the disjoint union of $\phi(\partial\Gamma)$ and $\partial\Gamma'$, and that $\partial\Gamma'$ is included with its orientation reversed (hence the minus sign in (15)). By the Theorem of Whitehead, A is triangulable, hence represents a class $[A] \in C_k(Q\backslash\Box_n)$ with the property that $\partial[A] = [\phi(\partial\Gamma)] - [\iota(\partial\Gamma')]$. By the definition of singular homology (7), we therefore have

$$[\phi(\partial\Gamma)] - [\iota(\partial\Gamma')] = 0 \tag{16}$$

in $H_{k-1}(Q\backslash\Box_n)$.

Since ϕ is a diffeomorphism, $[\phi(\partial\Gamma)] = \phi_*[\partial\Gamma]$. From the proof of Lemma 4 we also know that $\phi \simeq h$ and $h_*[i_2] = [i_1]$. Therefore

$$[\phi(\partial\Gamma)] = \phi_*[\partial\Gamma] = h_* d \cdot [i_2] = d \cdot h_*[i_2] = d \cdot [i_1]. \tag{17}$$

On the other hand, by (14),

$$[\iota(\partial\Gamma')] = \iota_*[\partial\Gamma'] = \iota_* d' \cdot [i_2] = \iota_*(i_2)_* d' \cdot [\partial\Box_k] = (i_1)_* d' \cdot [\partial\Box_k] = d' \cdot [i_1]. \tag{18}$$

Combining (16), (17) and (18), we find that $d[i_1] = d'[i_1]$. Since $[i_1]$ generates $H_{k-1}(Q\backslash\Box_n)$, it follows that $d' = d$. Thus $[\partial\Gamma'] = [\partial\Gamma]$, and we are done.

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E-mail address: `wayne@ics.uci.edu`