Computer Science C63 | March 20, 2024
Scarborough Campus | University of Toronto

Homework Assignment #7
Due: April 3, 2024, by 11:59 pm

• ***You must submit your assignment through the Crowdmark system.*** *You will receive by email an invitation through which you can submit your work in the form of separate PDF docucments with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups.* ***The course policy that limits the size of each group to at most two remains in effect:*** *submissions by groups of more than two persons will not be graded.*
• *To minimize bias in grading, Crowdmark does not reveal your name to the grader. Please do not subvert this feature by including your name(s) on the files you submit.*
• *It is your responsibility to ensure that the files you submit are legible. To this end, I encourage you to learn and use the LaTex typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the files you submit using LaTex; you may produce it any way you wish, as long as the resulting document is legible.*
• *By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.*[a]
• *For any question, you may use facts previously proved in this course, its prerequisites, or in the assigned sections of the textbook.*
• *Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on the correctness a of your answers, and the clarity, precision, and conciseness of your presentation.*

[a] "In each homework assignment you may collaborate with at most one other student who is currently taking CSCC63. If you collaborate with another student on an assignment, you and your partner must submit only one copy of your solution, with both of your names. The solution will be graded in the usual way and both partners will receive the same mark. Collaboration involving more than two students is not allowed. **For help with your homework you may consult only the instructor, TAs, your homework partner (if you have one), your textbook, and your class notes. You may not consult any other source.**"

Question 1. (20 marks) A courier must deliver packages to various locations in a city. To make her job less boring, she wants to make the deliveries by following a tour (a route that starts and ends in the same place) so that she never traverses the same street segment or even the same corner twice. That is, she wants to solve the COURIER ROUTE problem, defined as follows:

Instance: $\langle G, D\rangle$, where $G = (V, E)$ is a directed graph and $D \subseteq E$. The graph represents a map of the city streets; the edges are street segments and the nodes are intersections of street segments (or ends of no-exit street segments). The edges in the set D are the street segments that contain the addresses where the courier must deliver packages.
Question: Does G have a simple cycle that contains (at least) all the edges in D? (A cycle in a directed graph is ***simple*** if it does not repeat any node, except the one where it starts and ends).

Prove that the COURIER ROUTE problem is **NP**-complete.

Question 2. (35 marks)
a. (15 marks) Just before the exam period, each of students $1, 2, \ldots, n$ submits to the campus librarian a list B_i of the books that student i wishes to borrow from the library to prepare for the exams. The library has only one copy of each book. Student i is ***happy*** if and only if they borrow all the books on their list B_i. A set of students $I \subseteq \{1, 2, \ldots, n\}$ is a ***happy set***, if every student in the set can borrow all the books on their list: that is, no two students in the set have the same book on their wish lists. The

librarian decides which students get to borrow the books on their wish lists. So, she wishes to solve the LIBRARIAN'S DILEMMA decision problem, stated below:

Instance: $\langle B, (B_1, \ldots, B_n), k\rangle$, where B is the set of books that the library owns; for each $1 \le i \le n$, $B_i \subseteq B$ is the set of books that student i wishes to borrow; and $k \in \mathbb{Z}^+$.
Question: Is there a happy set of (at least) k students? That is, is there a set $I \subseteq \{1, 2, \ldots, n\}$ such that $|I| = k$ and for all $i, j \in I$, if $i \neq j$ then $B_i \cap B_j = \varnothing$?

Prove that the LIBRARIAN'S DILEMMA is **NP**-complete.

b. (5 marks) The old librarian retires, and the new librarian is corrupt. In addition to asking the students to submit the set of books they wish to borrow, he also asks them to name the price of their happiness: how much are they willing to pay him for letting them borrow all the books on their set. Thus, the CORRUPT LIBRARIAN'S DILEMMA decision problem now becomes:

Instance: $\langle B, \big((B_1, b_1) \ldots, (B_n, b_n)\big), k\rangle$, where B is the set of books that the library owns; for each $1 \le i \le n$, $B_i \subseteq B$ is the set of books that student i wishes to borrow and b_i is the bribe (in dollars) they are willing to pay the librarian to let them borrow all the books in B_i; and $k \in \mathbb{Z}^+$.
Question: Is it possible to make (at least) k dollars from the bribes of happy students? That is, is there a happy set of students $I \subseteq \{1, 2, \ldots, n\}$ such that $\sum_{i \in I} b_i \ge k$?

Prove that the CORRUPT LIBRARIAN'S DILEMMA problem is **NP**-complete.

c. (15 marks) The problem that the new corrupt librarian really wants to solve is not the above decision problem, but an optimization problem: To whom should he lend books in order to maximize the sum of bribes he receives? (Naturally only happy students pay their bribes.) More precisely, the CORRUPT LIBRARIAN'S OPTIMIZATION problem is:

Instance: $\langle B, \big((B_1, b_1) \ldots, (B_n, b_n)\big)\rangle$, where B is the set of books that the library owns; and for each $1 \le i \le n$, $B_i \subseteq B$ is the set of books that student i wishes to borrow, and b_i is the bribe (in dollars) they are willing to pay the librarian to let them borrow all the books in B_i. (Note the absence of k.)
Output: A happy set of students $I \subseteq \{1, 2, \ldots, n\}$ such that for any happy set of students $J \subseteq \{1, 2, \ldots, n\}$, $\sum_{i \in I} b_i \ge \sum_{j \in J} b_j$.

Give a Cook reduction (i.e., a polynomial-time Turing reduction) of the CORRUPT LIBRARIAN'S OPTIMIZATION problem to the CORRUPT LIBRARIAN'S DILEMMA decision problem. In other words, describe a polynomial-time algorithm for the CORRUPT LIBRARIAN'S OPTIMIZATION problem that uses a black box CLD-ORACLE that solves CORRUPT LIBRARIAN'S DILEMMA, at the cost of one unit of time per use of the black box. Describe clearly your reduction in high-level pseudocode that involves calls to the black box, justify why it is a polynomial-time reduction, and why it is correct.

THAT'S IT WITH HOMEWORK, FOLKS!