Computer Science C63 — March 13, 2024
Scarborough Campus — University of Toronto

Homework Assignment #6
Due: March 20, 2024, by 11:59 pm

• ***You must submit your assignment through the Crowdmark system.*** *You will receive by email an invitation through which you can submit your work in the form of separate PDF docucments with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups.* ***The course policy that limits the size of each group to at most two remains in effect:*** *submissions by groups of more than two persons will not be graded.*
• *To minimize bias in grading, Crowdmark does not reveal your name to the grader. Please do not subvert this feature by including your name(s) on the files you submit.*
• *It is your responsibility to ensure that the files you submit are legible. To this end, I encourage you to learn and use the LaTex typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the files you submit using LaTex; you may produce it any way you wish, as long as the resulting document is legible.*
• *By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.*[a]
• *For any question, you may use facts previously proved in this course, its prerequisites, or in the assigned sections of the textbook.*
• *Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on the correctness a of your answers, and the clarity, precision, and conciseness of your presentation.*

[a] "In each homework assignment you may collaborate with at most one other student who is currently taking CSCC63. If you collaborate with another student on an assignment, you and your partner must submit only one copy of your solution, with both of your names. The solution will be graded in the usual way and both partners will receive the same mark. Collaboration involving more than two students is not allowed. **For help with your homework you may consult only the instructor, TAs, your homework partner (if you have one), your textbook, and your class notes. You may not consult any other source.**"

UTSC has many, possibly overlapping, student clubs: Computer Science student club, Marxist club, gardening club, theatre club, cross-country running club, and so on. The dean wants to appoint a ***council of representatives*** from ***all*** these clubs to provide advice to the administration. The following questions concern different possible design goals for such a council.

Question 1. (25 marks) There are some pairs of individuals who, if both present in the same room, create a toxic atmosphere that would subvert the work of the council; we will call such a pair ***toxic***. The presence of only one of two individuals in a toxic pair on the council does not create problems. The dean knows the set of toxic pairs and wants to appoint a council that includes at least one representative from each club but does not include any toxic pair. Note that, since the clubs can have overlapping membership, a student who belongs to multiple clubs could represent all of them on the council.

More precisely, the dean wants to solve the HARMONIOUS COUNCIL decision problem:

Instance: $\langle C_1, C_2, \ldots, C_m, T\rangle$, where each C_i, $i \in [1..m]$, is a non-empty set of students (the members of student club i), and T is a set of pairs $\{s, s'\}$ of students (the toxic pairs).

Question: Is there some $R \subseteq C_1 \cup C_2 \cup \cdots \cup C_m$ (R is the council of representatives) so that $R \cap C_i \neq \varnothing$ for all $i \in [1..m]$, and for all $\{s, s'\} \in T$, at least one of s, s' is not in R? That is, is there a council that contains at least one member from each club but does not contain both elements of any toxic pair?

Prove that this problem is **NP**-complete. Give two different arguments to show that the problem is in **NP**, one based on the definition of **NP** in terms of non-deterministic Turing machines and one based on the certificate-verifier characterization of **NP**. (Do not dwell on why something can be done in polynomial time if it is obvious that it can.)

Question 2. (20 marks) The dean is disappointed that choosing a HARMONIOUS COUNCIL is (probably) computationally very hard, since the problem is NP-complete. So, he drops the idea of avoiding toxic pairs, and instead notes that it is undesirable for the council to have multiple members from the same club, because this gives an unfair advantage to such clubs. The dean now wants to explore a second method for picking the members of the council: Choose them so that each club is represented on the council by exactly one member. (It is still possible for one student to represent multiple clubs.) More precisely, the dean wants to solve the FAIR COUNCIL problem:

Instance: $\langle C_1, C_2, \ldots, C_m \rangle$, where each C_i, $i \in [1..m]$, is a non-empty set of students (the members of student club i).

Question: Is there some set $R \subseteq C_1 \cup C_2 \cup \cdots \cup C$ so that $|R \cap C_i| = 1$ for each $i \in [1..m]$? That is, is there a council that has exactly one representative from each club?

Prove that this problem is also **NP**-complete. Now it suffices to give one argument to show that the problem is in **NP**.

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Additional problem to think about but not to submit with your answer: The dean is disappointed that choosing a HARMONIOUS COUNCIL or a FAIR COUNCIL is (probably) computationally very hard, since both problems are NP-complete. But given his experience with committees he observes that if the council is very large, it becomes unwieldy. So he also wants to explore the SMALL COUNCIL decision problem, defined as follows:

Instance: $\langle C_1, C_2, \ldots, C_m, \ell \rangle$, where each C_i, $i \in [1..m]$, is a non-empty set of students (the members of student club i), and $\ell \in \mathbb{Z}^+$.

Question: Is there some set $R \subseteq C_1 \cup C_2 \cup \cdots \cup C_m$ so that $|R| = \ell$ and $R \cap C_i \neq \varnothing$ for each $i \in [1..m]$? That is, is there a council that has only ℓ members but still has at least one representative from each club?

Prove that this problem is also **NP**-complete.