Computer Science C63 — January 31, 2024
Scarborough Campus — University of Toronto

Homework Assignment #4
Due: February 14, 2024, by 11:59 pm

• ***You must submit your assignment through the Crowdmark system.*** *You will receive by email an invitation through which you can submit your work in the form of separate PDF docucments with your answers to each question of the assignment. To work with a partner, you and your partner must form a group on Crowdmark. Crowdmark does not enforce a limit on the size of groups.* ***The course policy that limits the size of each group to at most two remains in effect:*** *submissions by groups of more than two persons will not be graded.*
• *To minimize bias in grading, Crowdmark does not reveal your name to the grader. Please do not subvert this feature by including your name(s) on the files you submit.*
• *It is your responsibility to ensure that the files you submit are legible. To this end, I encourage you to learn and use the LaTex typesetting system, which is designed to produce high-quality documents that contain mathematical notation. You are not required to produce the files you submit using LaTex; you may produce it any way you wish, as long as the resulting document is legible.*
• *By virtue of submitting this assignment you (and your partner, if you have one) acknowledge that you are aware of the policy on homework collaboration for this course.*[a]
• *For any question, you may use facts previously proved in this course, its prerequisites, or in the assigned sections of the textbook.*
• *Unless we explicitly state otherwise, you should justify your answers. Your paper will be graded based on the correctness a of your answers, and the clarity, precision, and conciseness of your presentation.*

[a] "In each homework assignment you may collaborate with at most one other student who is currently taking CSCC63. If you collaborate with another student on an assignment, you and your partner must submit only one copy of your solution, with both of your names. The solution will be graded in the usual way and both partners will receive the same mark. Collaboration involving more than two students is not allowed. **For help with your homework you may consult only the instructor, TAs, your homework partner (if you have one), your textbook, and your class notes. You may not consult any other source.**"

In your answers for this assignment you may appeal to Church's thesis when arguing that a language is decidable or recognizable, or that a function is computable. That is, you may replace an argument that a Turing machine for a certain task exists by a description, in high-level pseudocode, of an algorithm that achieves this task.

Question 1. (10 marks) Consider the languages

$$\textsc{Unique} = \{\langle M\rangle \colon |\mathcal{L}(M)| = 1\} \quad \text{and} \quad \textsc{Halt} = \langle M, x\rangle \colon M \text{ halts on } x\}.$$

(a) Prove that $\textsc{Unique}$ is unrecognizable.

(b) Show that $\textsc{Unique} \not\leq_m \textsc{Halt}$.

(c) Show that $\textsc{Unique} \leq_T \textsc{Halt}$. (**Hint:** Observe that $\textsc{Unique}$ is the set of codes of Turing machines that (i) accept some string in some number of steps; and (ii) do not accept any other string in any number of steps.)

Comment: We have seen that a mapping reduction is a special case of a Turing reduction (see the notes on reductions). Thus, for any languages L, L', if $L \leq_m L'$ then $L \leq_T L'$. Parts (b) and (c) of this question show that the converse is not true: There exist languages L, L' such that $L \leq_T L'$ but $L \not\leq_m L'$. This can be proved more simply by showing that (i) $\overline{\text{HALT}} \leq_T \text{HALT}$ but (ii) $\overline{\text{HALT}} \not\leq_m \text{HALT}$. (It is a good idea to do this as a warm-up exercise, but do not include it in your answer.) The advantage of the more complicated route suggested in this question is that the proof of part (c) is more interesting than the proof that $\overline{\text{HALT}} \leq_T \text{HALT}$, in that it requires more versatile use of a Turing reduction.

Question 2. (25 marks) For each of the following sets ***and its complement***, classify it as (a) decidable, (b) undecidable but recognizable, or (c) unrecognizable. Justify your answer. In all sets defined below M stands for a Turing machine recognizer, and in the definition of E, M_1, and M_2 stand for Turing machine function computers.

For each set and its complement be sure to write your answer starting on a new sheet of paper, as you will be asked to submit these answers as five separate files.

$A = \{\langle M, x\rangle\colon M$ on input x never moves left$\}$

$B = \{\langle M\rangle\colon$ for some input y, M accepts both y and y^R (the reverse of y)$\}$

$C = \{\langle M\rangle\colon$ for every input y, if M accepts y then M accepts $y^R)\}$

$D = \{\langle M, k\rangle\colon M$ on empty tape makes at most k left moves$\}$

$E = \{\langle M_1, M_2\rangle\colon M_1$ and M_2 compute functions $F_1, F_2\colon \mathbb{N} \to \mathbb{N}$ such that F_1, F_2 are total and $\forall n \in \mathbb{N},\ F_2(n) = F_1(n) + 17\}$