Turing 1936

Seeking mathematical definition for "algorithm" or "mechanical computation" to solve a problem.

> Finite list of unambiguous instructions that can be executed mechanically so that applying them to *any* input (a string) results in an appropriate output (also a string) in a finite number of steps.

This is what Turing machines are.

Also what electronic computers are!

Why?

Axioms of predicate logic: Formulas (strings over some alphabet) of specific forms expressing true facts.

E.g.,

$$\forall x\,(F \to G) \to (\forall x\, F \to \forall x\, G)$$

$$\neg \exists x\, F \;\leftrightarrow \forall x\, \neg F$$

Inference rules of predicate logic: Given that certain formulas $F_1, F_2, \ldots, F_k$ hold, conclude that some other formula G also holds.

E.g.,

$$\frac{F \qquad F \to G}{G}$$

$$\frac{F(c)}{\exists x\, F(x)}$$

Why?

Hilbert-Ackerman's "Entscheidungsproblem":

Is there an algorithm that takes as input a formula in predicate logic and determines whether that formula can be proved using the axioms and inference rules of predicate logic?

(Can we turn mathematics into a mindless endeavour?)

To settle this question mathematically we must have a clear definition of what counts as an algorithm.

This is why Turing developed his machines.

TM for even-length palindromes

Palindrome = string that reads the same forward and backward

We want a TM that:

- accepts all strings in $\{0,1\}^*$ that are even length palindromes
- does not accept any other strings.

E.g., it should accept: ε and 011110, but
it should not accept: 010100 or 10101

TM for even-length palindromes

1. If the symbol under the head is ⊔, accept;
   else "remember" that symbol, replace it by ⊔ and move R
2. While scanning a symbol ≠ ⊔ move R
3. Move L from the first ⊔ found
4. If the symbol under the head is different from the one "remembered", reject
   else replace it by ⊔ and move L
5. While scanning a symbol ≠ ⊔ move L
6. Move R and go to step 1

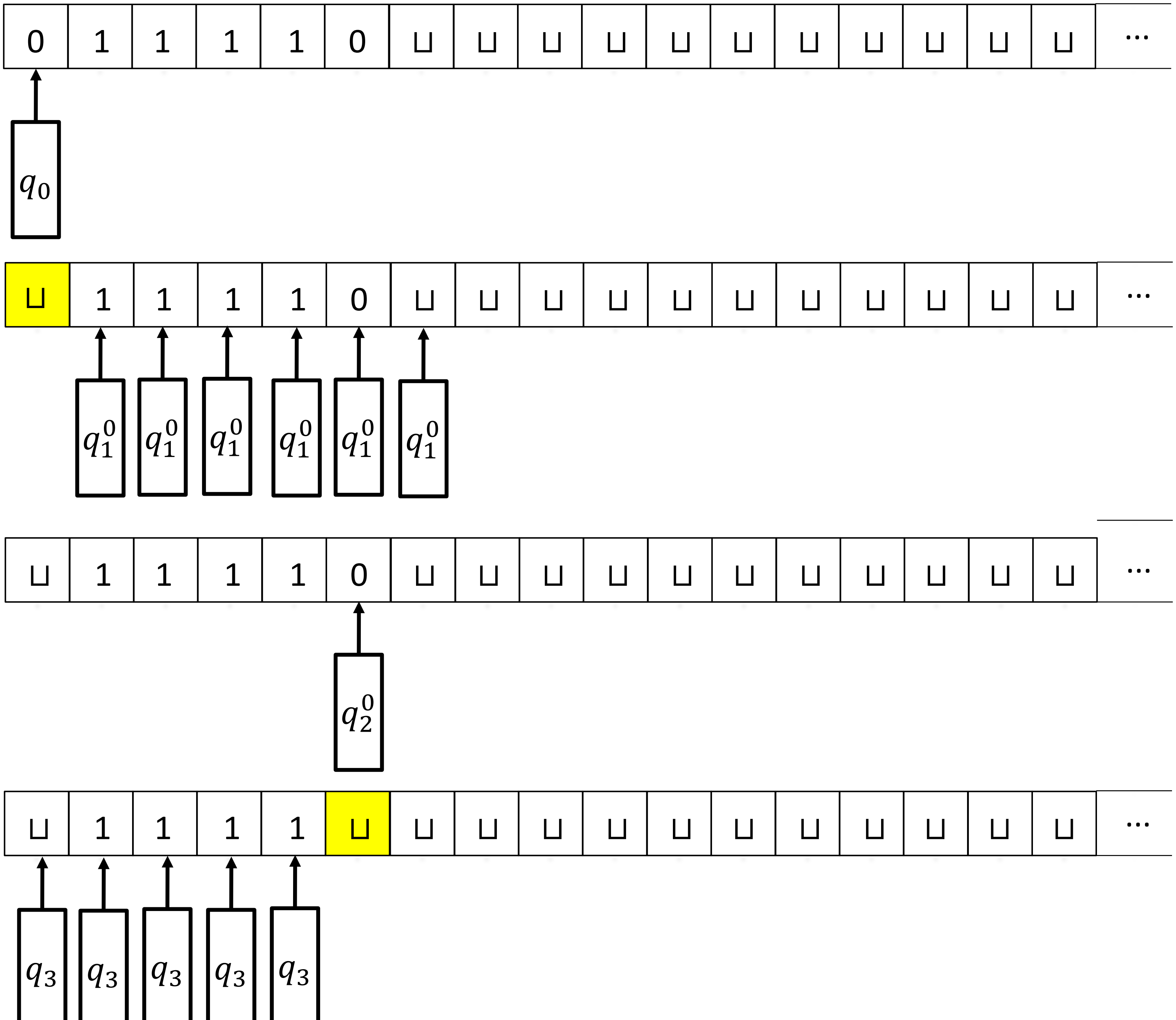
TM for even-length palindromes

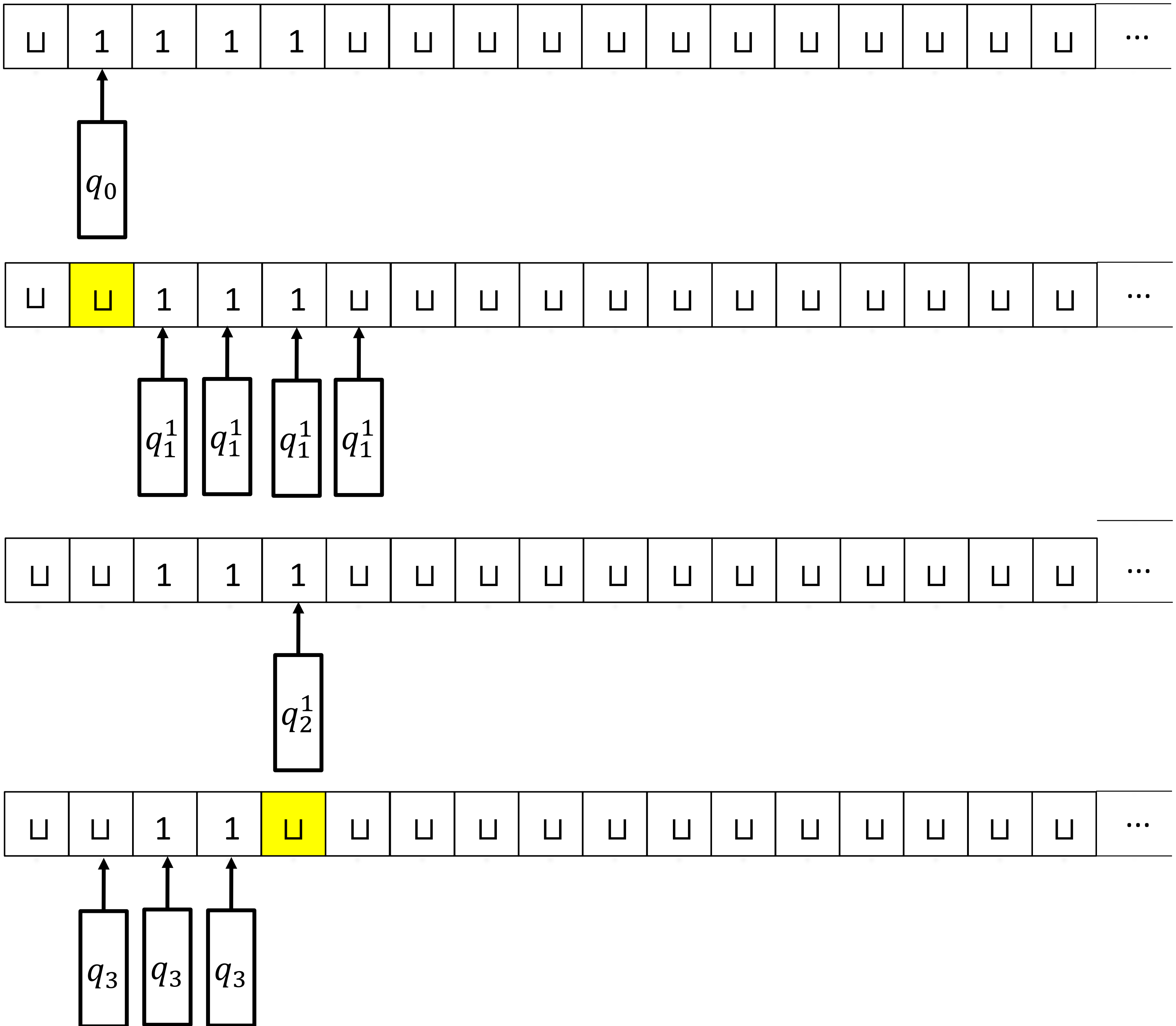
TM instructions ("finite control"):

- new state
- new symbol
- direction of move

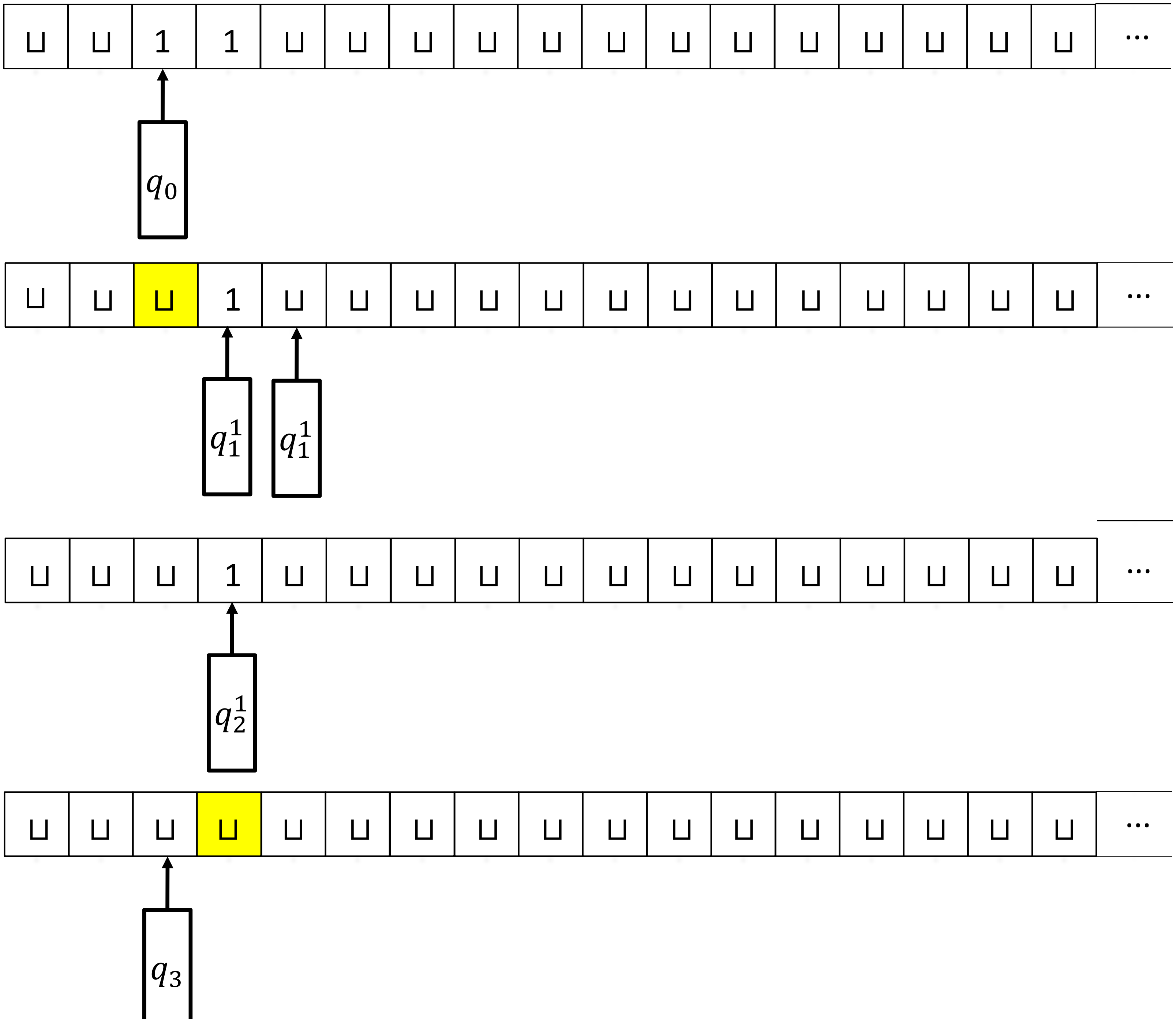
Current symbol / Current state

Current state	0	1	$\sqcup$
q_0: initial state	$(q_1^0, \sqcup, R)$	$(q_1^1, \sqcup, R)$	Accept
q_1^0: scan right, 1st symbol 0	$(q_1^0, 0, R)$	$(q_1^0, 1, R)$	$(q_2^0, \sqcup, L)$
q_1^1: scan right, 1st symbol 1	$(q_1^1, 0, R)$	$(q_1^1, 1, R)$	$(q_2^1, \sqcup, L)$
q_2^0: at right end, 1st symbol 0	$(q_3, \sqcup, L)$	Reject	X
q_2^1: at right end, 1st symbol 1	Reject	$(q_3, \sqcup, L)$	X
q_3: scan left ("rewind")	$(q_3, 0, L)$	$(q_3, 1, L)$	$(q_0, \sqcup, R)$

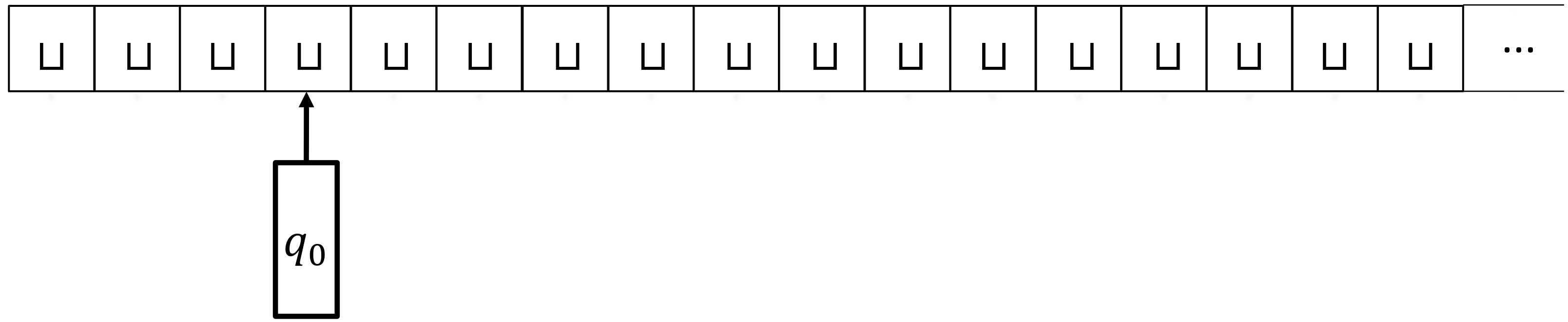

0 1 1 1 1 0 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_0
⊔ 1 1 1 1 0 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_1^0 q_1^0 q_1^0 q_1^0 q_1^0 q_1^0
⊔ 1 1 1 1 0 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_2^0
⊔ 1 1 1 1 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_3 q_3 q_3 q_3 q_3

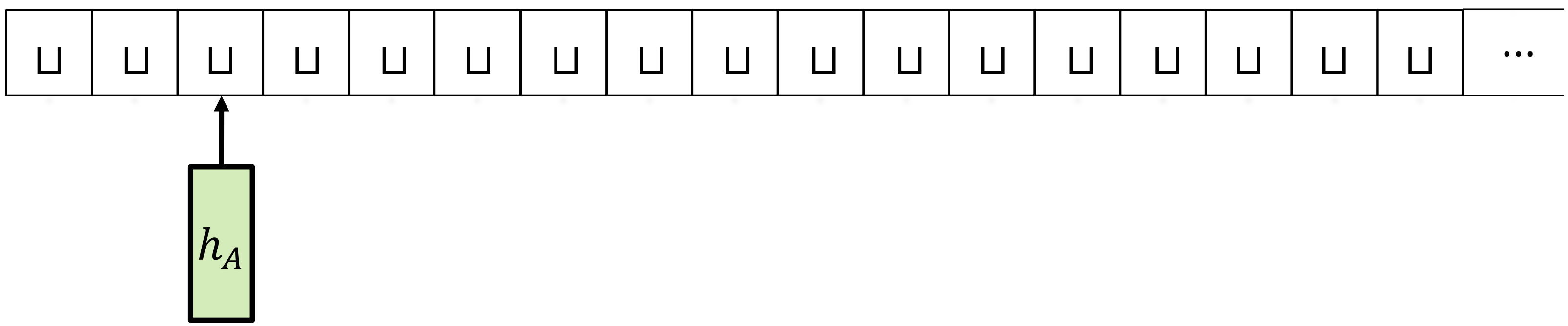


⊔ 1 1 1 1 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_0
⊔ ⊔ 1 1 1 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_1^1 q_1^1 q_1^1 q_1^1
⊔ ⊔ 1 1 1 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_2^1
⊔ ⊔ 1 1 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_3 q_3 q_3



⊔ ⊔ 1 1 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_0
⊔ ⊔ ⊔ 1 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_1^1
q_1^1
⊔ ⊔ ⊔ 1 ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_2^1
⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_3


⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
q_0


⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ ⊔ …
h_A

TM for even-length palindromes

$M = (Q, \Sigma, \Gamma, \delta, q_0, h_A, h_R)$
$Q = \{q_0, q_1^0, q_1^1, q_2^0, q_2^1, q_3, h_A, h_R\}$

- q_0 - initial state
- q_1^0 - scanning right, first symbol was 0
- q_1^1 - scanning right, first symbol was 1
- q_2^0 - reached right end, first symbol was 0
- q_2^1 - reached right end, first symbol was 1
- q_3 - scanning left ("rewinding")
- h_A - accept state
- h_R - reject state

TM for even-length palindromes

The transition function:

- $\delta(q_0, a) = \begin{cases} (h_A, \sqcup, R), \text{if } a = \sqcup \\ (q_1^a, \sqcup, R), \text{if } a \neq \sqcup \end{cases} \quad \forall a \in \{0,1,\sqcup\}$
- $\delta(q_1^b, a) = \begin{cases} (q_1^b, a, R), \text{if } a \neq \sqcup \\ (q_2^b, \sqcup, L), \text{if } a = \sqcup \end{cases} \quad \forall a \in \{0,1,\sqcup\}, b \in \{0,1\}$
- $\delta(q_2^b, a) = \begin{cases} (h_R, \sqcup, L), \text{if } a \neq b \\ (q_3, \sqcup, L), \text{if } a = b \end{cases} \quad \forall a \in \{0,1,\sqcup\}, b \in \{0,1\}$
- $\delta(q_3, a) = \begin{cases} (q_3, a, L), \text{if } a \neq \sqcup \\ (q_0, \sqcup, R), \text{if } a = \sqcup \end{cases} \quad \forall a \in \{0,1,\sqcup\}$

TM for even-length palindromes

The transition function in tabular form:

Current symbol

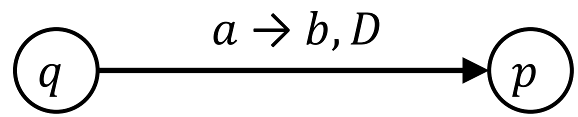
Current state

	0	1	$\sqcup$
q_0	$(q_1^0, \sqcup, R)$	$(q_1^1, \sqcup, R)$	$(h_A, \sqcup, L)$
q_1^0	$(q_1^0, 0, R)$	$(q_1^0, 1, R)$	$(q_2^0, \sqcup, L)$
q_1^1	$(q_1^1, 0, R)$	$(q_1^1, 1, R)$	$(q_2^1, \sqcup, L)$
q_2^0	$(q_3, \sqcup, L)$	$(h_R, \sqcup, L)$	X
q_2^1	$(h_R, \sqcup, L)$	$(q_3, \sqcup, L)$	X
q_3	$(q_3, 0, L)$	$(q_3, 1, L)$	$(q_0, \sqcup, R)$

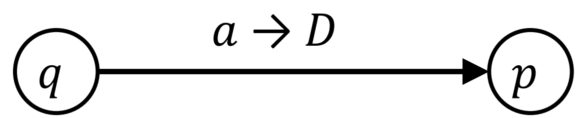
TM for even-length palindromes

The transition function in graphical form:

- $\delta(q, a) = (p, b, D)$, for $b \neq a$



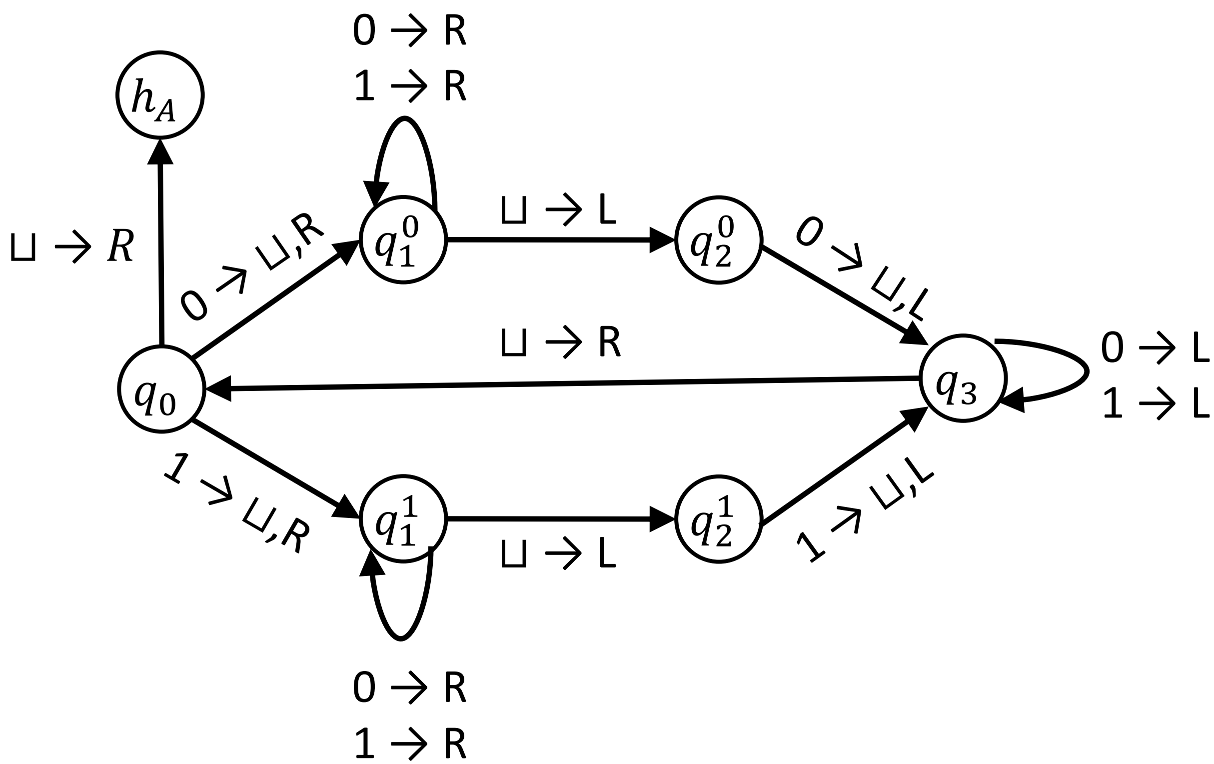
- $\delta(q, a) = (p, b, D)$, for $b = a$



- Missing transitions: implicitly going to h_R (reject)

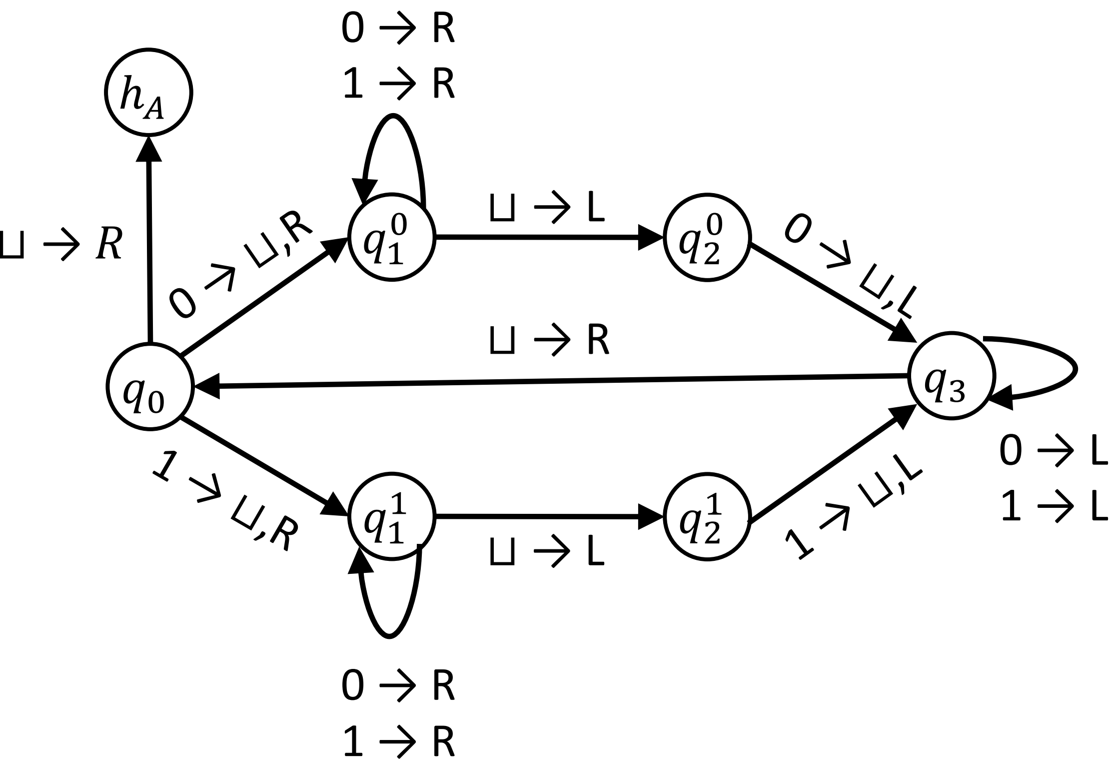
TM for even-length palindromes

Diagram for transition function of this TM:



TM for even-length palindromes

Computation on input 0110:



$q_0\, 0110$

$\vdash \sqcup\, q_1^0 110$

$\vdash \sqcup\, 1q_1^0 10$

$\vdash \sqcup\, 11q_1^0 0$

$\vdash \sqcup\, 110q_1^0$

$\vdash \sqcup\, 11q_2^0 0$

$\vdash \sqcup\, 1q_3 1$

$\vdash \sqcup\, q_3 11$

$\vdash q_3 \sqcup 11$

$\vdash \sqcup\, q_0 11$

$\vdash \sqcup\sqcup\, q_1^1 1$

$\vdash \sqcup\sqcup\, 1q_1^1$

$\vdash \sqcup\sqcup\, q_2^1 1$

$\vdash \sqcup\, q_3$

$\vdash \sqcup\sqcup\, q_0$

$\vdash \sqcup\sqcup\sqcup\, h_A$

Exercise: Trace the computation on inputs 0111 and 010.