Grouping and Structure from Motion

Raquel Urtasun

TTI Chicago

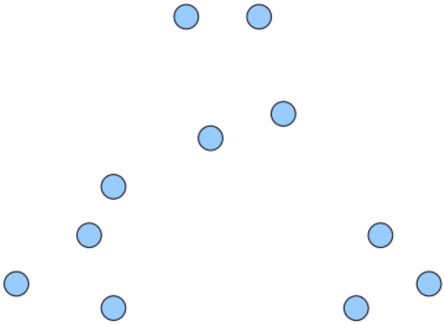
March 12, 2013

Example of grouping techniques

- K-means style clustering, e.g., SLIC superpixels
- Normalized cuts
- Graph-based superpixels
- Mean-shift
- Watershed transform

Simple K-means

- Find three clusters in this data



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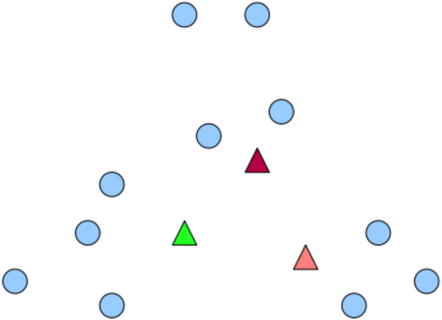

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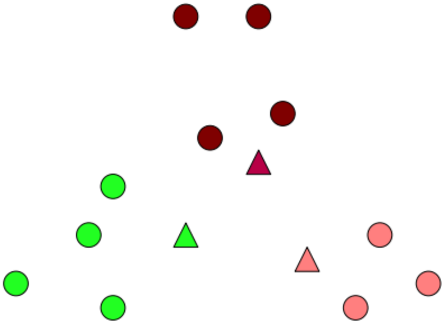

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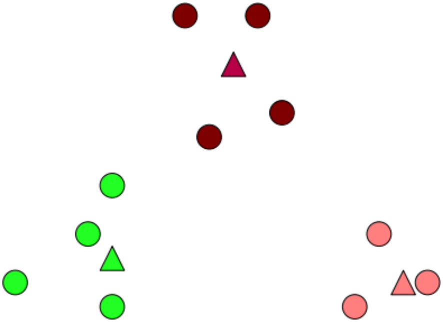

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Results

[R. Achanta and A. Shaji and K. Smith and A. Lucchi and P. Fua and S. Susstrunk, PAMI12]

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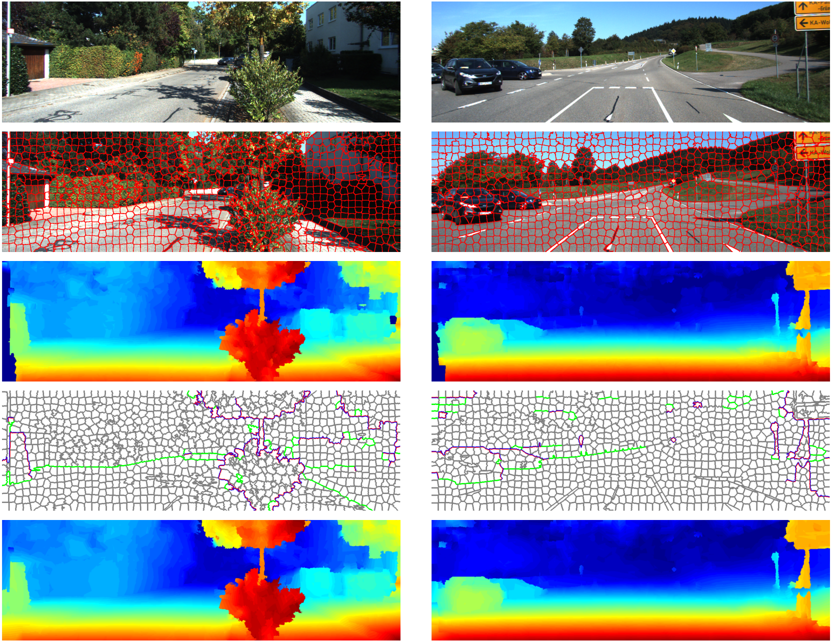
[K. Yamaguchi, D. McAllester and R. Urtasun, CVPR13]



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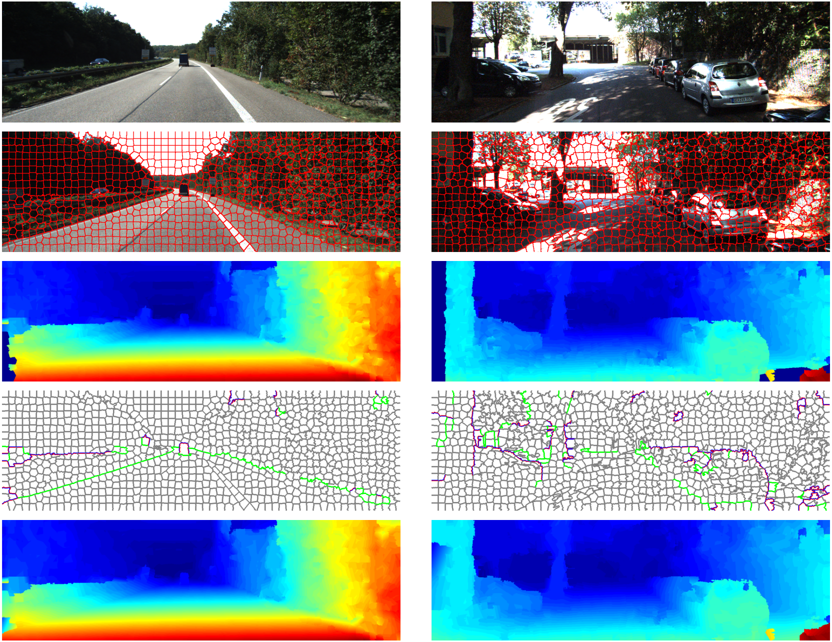

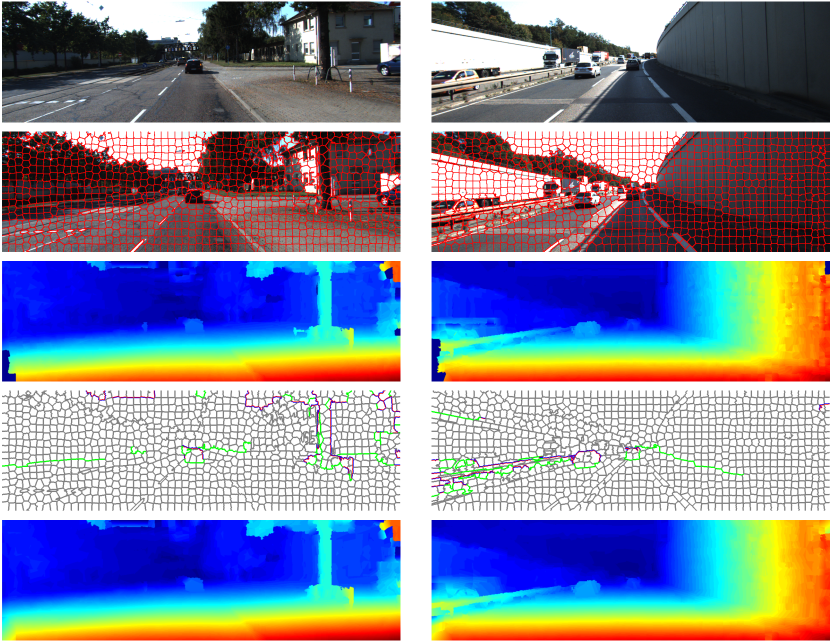

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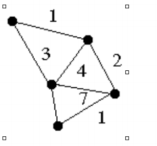
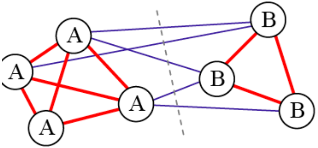
Segmentation as a mincut problem



$$\begin{bmatrix} 0 & 1 & 3 & \infty & \infty \\ 1 & 0 & 4 & \infty & 2 \\ 3 & 4 & 0 & 6 & 7 \\ \infty & \infty & 6 & 0 & 1 \\ \infty & 2 & 7 & 1 & 0 \end{bmatrix}$$

Weight Matrix: W

- Examines the **affinities** (similarities) between nearby pixels and tries to separate groups that are connected with weak affinities.



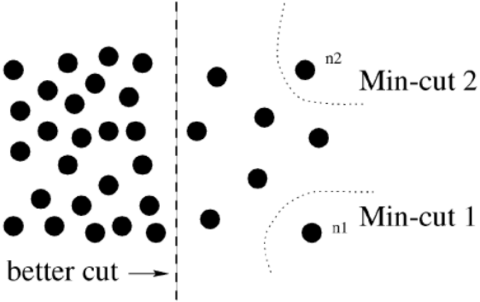
- The cut separate the nodes into two groups

Minimun Cuts

- The cut between two groups A and B is defined as the sum of all the weights being cut

$$cut(A,B) = \sum_{i \in A, j \in B} w_{i,j}$$

- Problem: Results in small cuts that isolates single pixels



- We need to normalize somehow

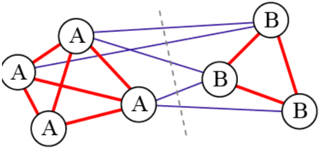
Normalized Cuts

[J. Shi and J. Malik, PAMI00]

- Better measure is the normalized cuts

$$N_{cut}(A,B) = \frac{cut(A,B)}{assoc(A,V)} + \frac{cut(A,B)}{assoc(B,V)}$$

with $assoc(A,A) = \sum_{i \in A, j \in A} w_{ij}$ is the association term within a cluster and $Assoc(A,V) = assoc(A,A) + cut(A,B)$ is the sum of all the weights associated with nodes in A.



	A	B	sum
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B	$cut(B,A)$	$assoc(B,B)$	$assoc(B,V)$
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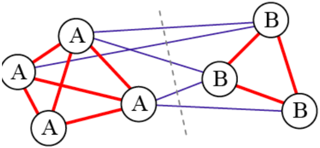

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- Minimizing this **Rayleigh quotient** is equivalent to solving the generalized eigenvalue system

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Algorithm

[J. Shi and J. Malik, PAMI00]

1. Given an image or image sequence, set up a weighted graph $\mathbf{G} = (\mathbf{V}, \mathbf{E})$ and set the weight on the edge connecting two nodes to be a measure of the similarity between the two nodes.
2. Solve $(\mathbf{D} - \mathbf{W})\boldsymbol{x} = \lambda \mathbf{D}\boldsymbol{x}$ for eigenvectors with the smallest eigenvalues.
3. Use the eigenvector with the second smallest eigenvalue to bipartition the graph.
4. Decide if the current partition should be subdivided and recursively repartition the segmented parts if necessary.

Examples

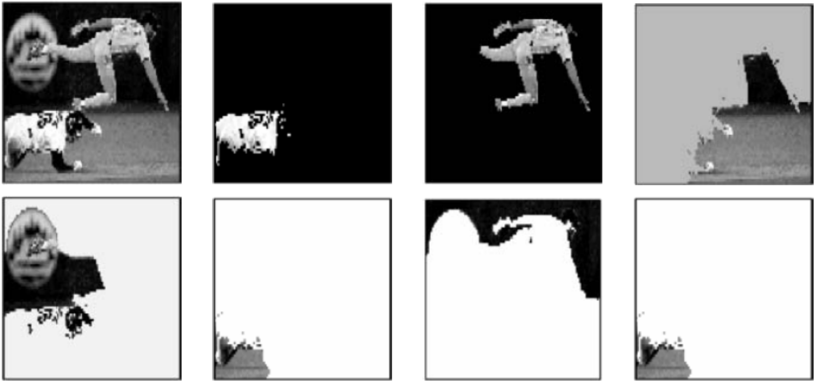

Figure: Shi and Malik N-Cuts

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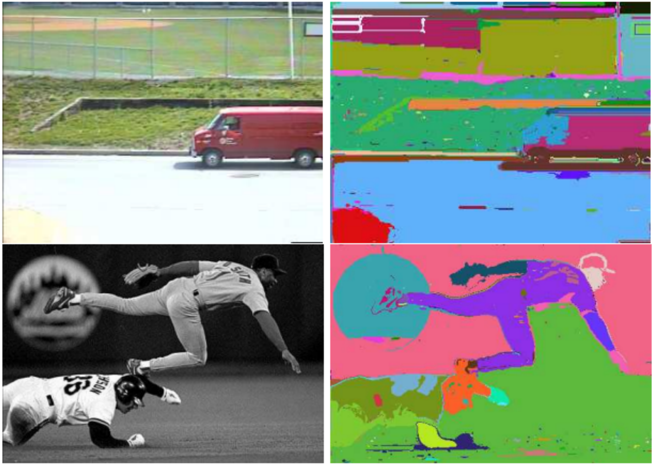
Results

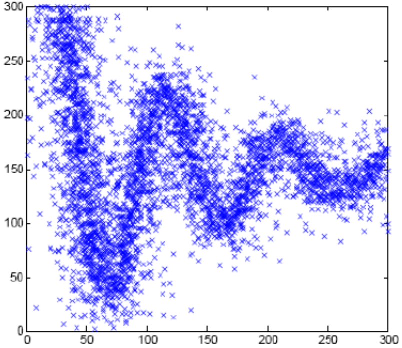
[P. Felzenszwald and D. Huttenlocher, IJCV04]



Example of grouping techniques

- K-means style clustering, e.g., SLIC superpixels
- Normalized cuts
- Graph-based superpixels
- Mean-shift
- Watershed transform

- We have a bunch of points drawn from some distribution
- What's the distribution that generated these points?



[Source: M. Tappen]

Parametric vs Non-Parametric

- We can fit a parametric distribution, e.g., mixture of Gaussians
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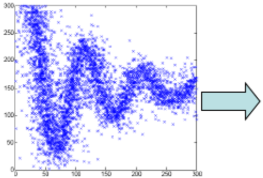

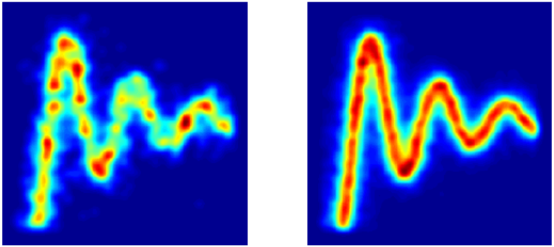

(a) 2000 Samples (b) 20000 Samples

Figure 2-2: Kernel density estimates of the density function shown in Figure 2-1(a). Figure (a) shows the estimate found with a relatively small number of samples. It is uneven and does not approximate the true density well. (b) With more samples, the estimate of the density improves significantly.

[Source: M. Tappen]

KDE

- We approximate the density by

$$\hat{f}(\mathbf{x}) = \frac{1}{n}\sum_{i=1}^{n} K_H(\mathbf{x} - \mathbf{x}_i)$$

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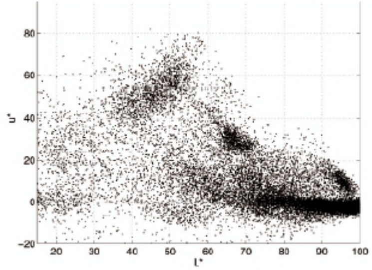

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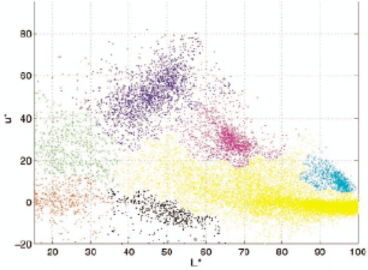

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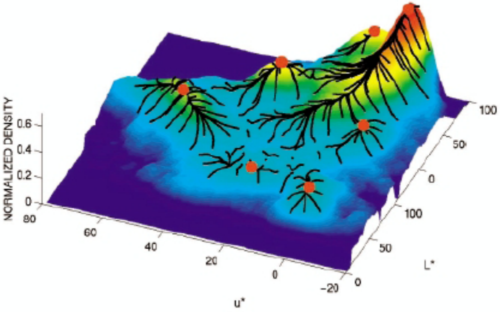
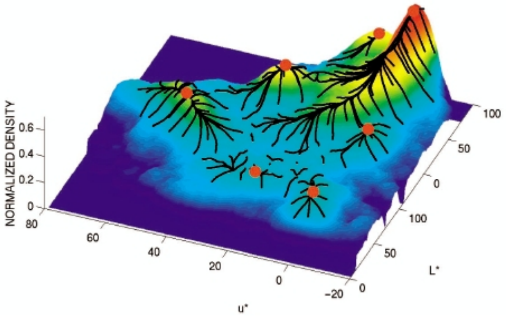


(c)

What is mean-shift

- The density will have peaks (also called modes)
- If we started at point and did gradient-ascent, we would end up at one of the modes
- Cluster based on which mode each point belongs to



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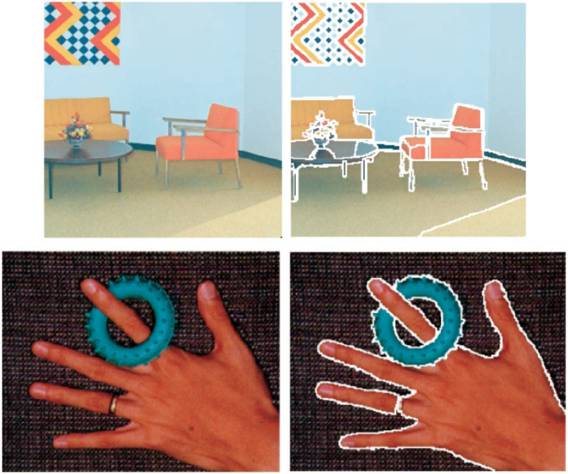

[Source: M. Tappen]

Let's look at Structure from Motion

Structure from motion

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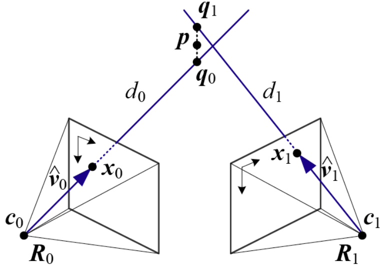
Triangulation

- It is the problem of determining a point's 3D position from a set of corresponding image locations and known camera positions
- This problem is the converse of the pose estimation problem



- Simplest solution: find 3D point $\mathbf{p}$ that lies closest to all the 3D rays corresponding to the 2D feature locations $\{\mathbf{x}_j\}$ observed by cameras $\mathbf{P}_j = \mathbf{K}_j[\mathbf{R}_j|\mathbf{t}_j]$, with $\mathbf{t}_j = -\mathbf{R}_j\mathbf{c}_j$.

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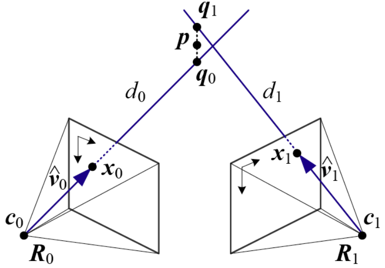

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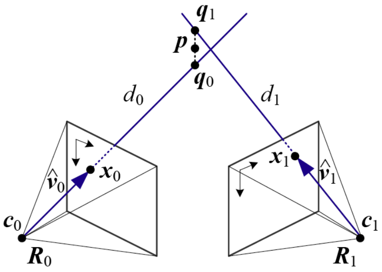

- The rays originate at $\mathbf{c}_j$ in a direction $\hat{\mathbf{v}}_j = \mathcal{N}(\mathbf{R}_j^{-1} K_j^{-1} \mathbf{x}_j)$
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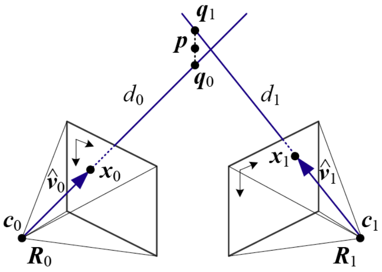

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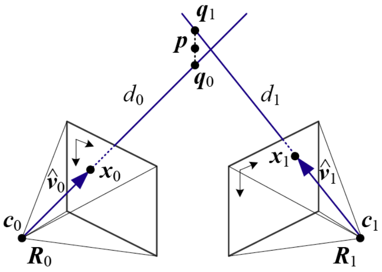

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Alternative formulation

- Can produce significantly better estimates if some cameras are closer to the 3D point than others is to minimize the residual in the measurement equations

$$
\begin{aligned}
x_j &= \frac{p^j_{00}X + p^j_{01}Y + p^j_{02}Z + p^j_{03}W}{p^j_{20}X + p^j_{21}Y + p^j_{22}Z + p^j_{23}W} \\
y_j &= \frac{p^j_{10}X + p^j_{11}Y + p^j_{12}Z + p^j_{13}W}{p^j_{20}X + p^j_{21}Y + p^j_{22}Z + p^j_{23}W}
\end{aligned}
$$

  where (x_j, y_j) are the measured 2D feature locations, and $\{p^j_{00}, \cdots, p^j_{23}\}$ are the known entries in the camera matrix $\mathbf{P}$, and $\mathbf{p} = (X, Y, Z, W)$ in homogeneous coordinates

- Why is this better?

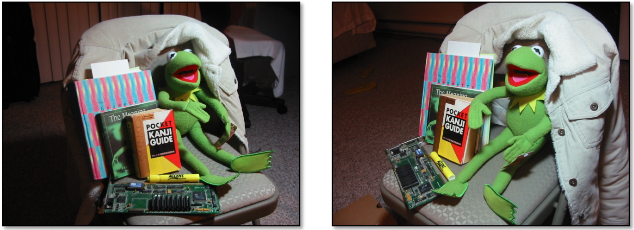
- How do we solve this now?

Two frame structure from motion

Figure: Images from N. Snavely

- Simultaneous recovery of 3D structure and pose from image correspondences
- Why is this not the stereo problem?

Two frame structure from motion



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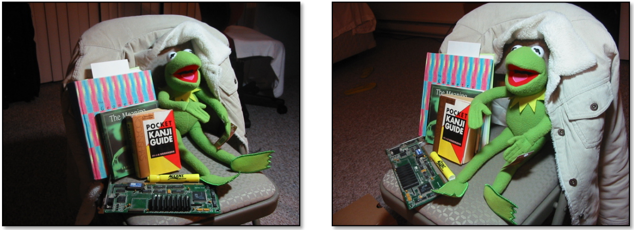

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8 Point Algorithm

- We can write a system of equations

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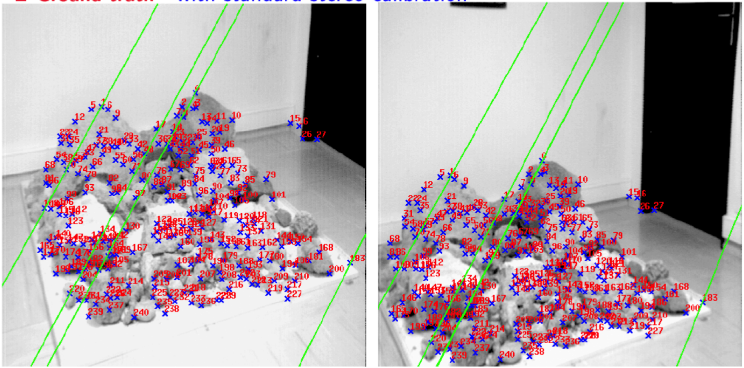
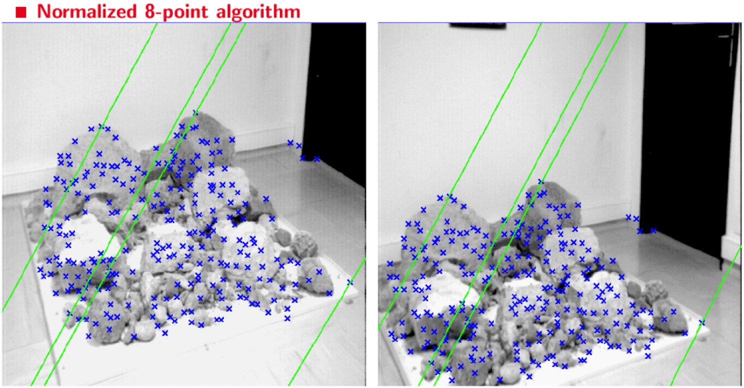
Results

■ **Ground truth** with standard stereo calibration



[Source: N. Snavely]

Results



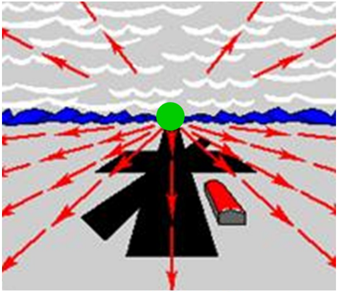
[Source: N. Snavely]

Now what?

- Given the fundamental matrix, we can calibrate the cameras
- Given this calibration we can triangulate
- This is a chicken and egg problem so we can re-iterate this process.

Pure Translational Motion (known rotation)

- If we know the rotation, we can pre-rotate all the points in the second image to match the viewing direction of the first.
- The resulting set of 3D points move towards (or away from) the **focus of expansion (FOE)**
- See exercise about this ...



adapted from Gibson 1978

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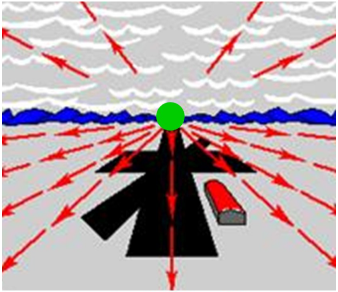

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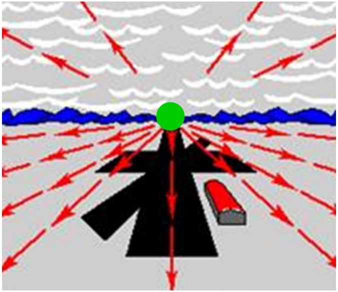

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More views...

- The geometry of three views is described by a 3 x 3 x 3 tensor called the **trifocal tensor**
- The geometry of four views is described by a 3 x 3 x 3 x 3 tensor called the **quadrifocal tensor**
- After this it starts to get complicated ...
- We will not see this in class

Structure from motion

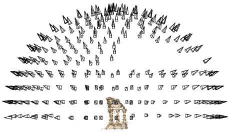
Given many images, how can we

- figure out where they were all taken from?
- build a 3D model of the scene?

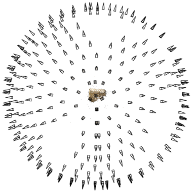
[Source: N. Snavely]

Structure from Motion

- Input: images with points in correspondence $p_{i,j} = (u_{i,j}, v_{i,j})$
- Output:
  - **structure**: 3D location $\mathbf{x}_i$ for each point $\mathbf{p}_i$
  - **motion**: camera parameters $\mathbf{R}_j$, $\mathbf{t}_j$ possibly $\mathbf{K}_j$
- Objective function: minimize reprojection error



Reconstruction (side)



(top)

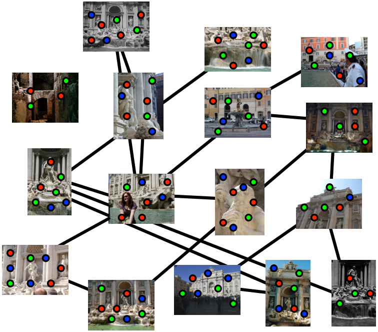
[Source: N. Snavely]

Reconstructions from Video

[Source: N. Snavely]

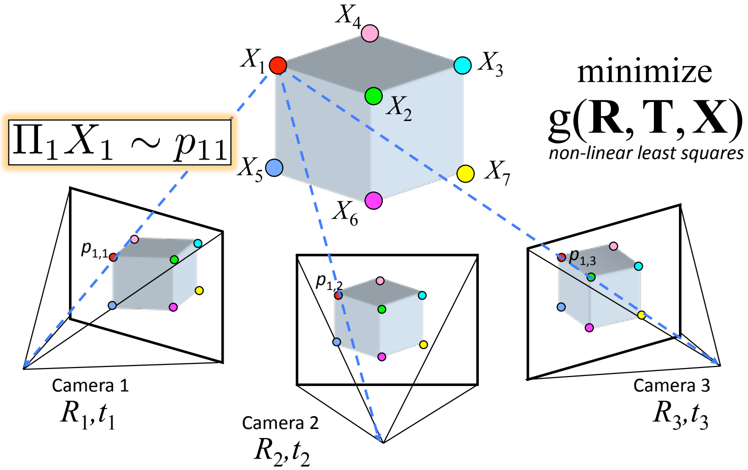
How do we get correspondences?

- Feature detection and matching
- We can construct a graph of matches
- Use RANSAC to estimate fundamental matrices between each pair



[Source: N. Snavely]

Structure from Motion Problem



[Source: N. Snavely]

Problem Size

- What are the variables?
- How many variables per camera?
- How many variables per point?
- E.g., Trevi Fountain collection, 466 input photos, $> 100,000$ 3D points

[Source: N. Snavely]

Bundle Adjustment

- Minimize sum of squared reprojection errors:

$$g(\mathbf{X}, \mathbf{R}, \mathbf{T}) = \sum_{i=1}^{m} \sum_{j=1}^{n} w_{ij} || \mathbf{P}(\mathbf{x}_i, \mathbf{R}_j, \mathbf{t}_j) - \begin{bmatrix} u_{i,j} \\ v_{i,j} \end{bmatrix} ||_2^2$$

  with w_{ij} indicator whether they are visible or not, $\mathbf{P}(\mathbf{x}_i, \mathbf{R}_j, \mathbf{t}_j)$ the predicted image location, and $(u_{i,j}, v_{i,j})$ the observed image location

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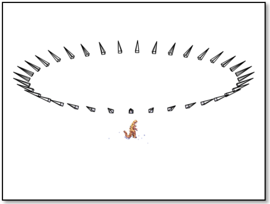
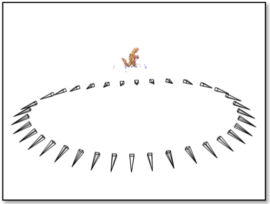
Failure cases

- Necker reversal

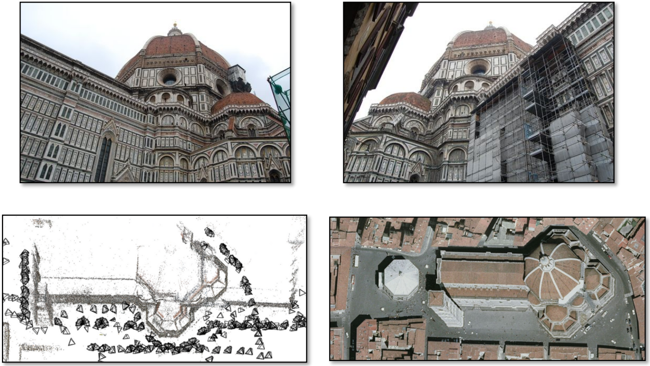




[Source: N. Snavely]

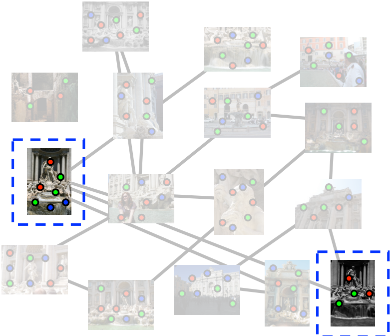
Failure cases

- Repetitive Patterns



[Source: N. Snavely]

Incremental Structure From Motion



[Source: N. Snavely]

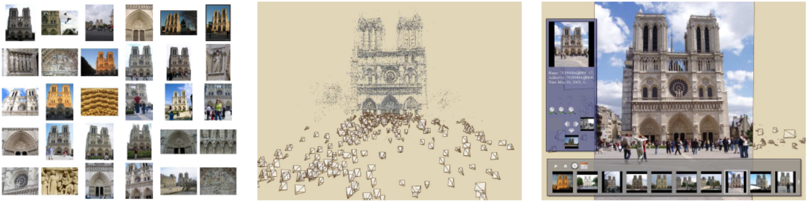
Incremental Structure From Motion

- What's the problem with this approach?

More Results

[Source: N. Snavely]

Applications: 3D Reconstruction from Photo Collections



(a) (b) (c)

Figure 1: Our system takes unstructured collections of photographs such as those from online image searches (a) and reconstructs 3D points and viewpoints (b) to enable novel ways of browsing the photos (c).

[N. Snavely et al. Siggraph 2006]