Computer Vision: Calibration and Reconstruction

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TTI Chicago

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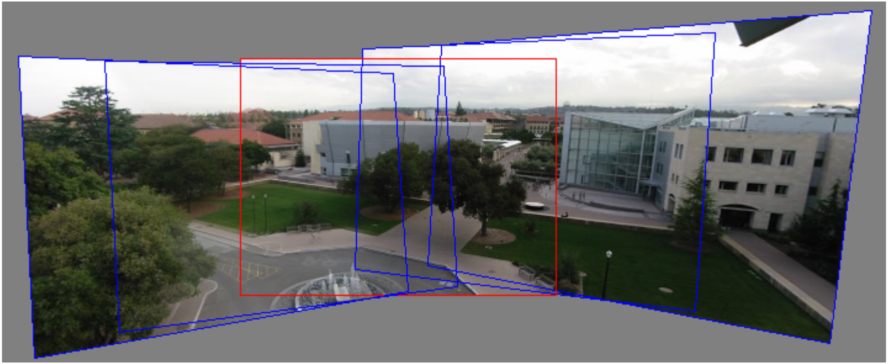
What did we see in class last week?

Panoramas



Today's Readings

- Chapter 6 and 11 of Szeliski's book

Let's look at camera calibration

Camera Calibration

Find the quantities internal to the camera that affect the imaging process as well as the position of the camera with respect to the world

- Rotation and translation

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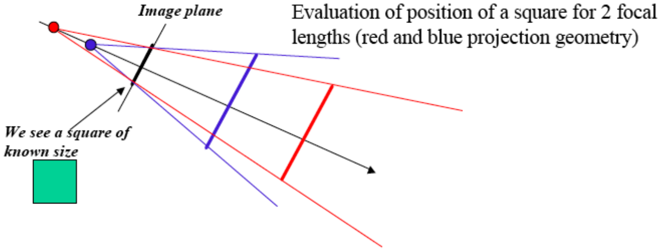
Why do we need calibration?

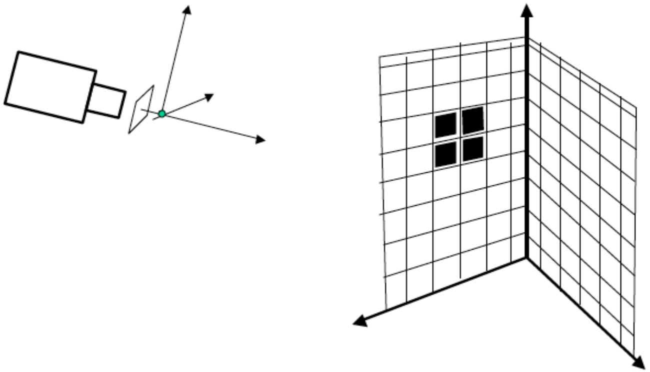
- Have good reconstruction
- Interact with the 3D world



[Source: Ramani]

Camera and Calibration Target

Most methods assume that we have a known 3D target in the scene



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Most common used Procedure

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Calibration

- Estimate matrix $\mathbf{P}$ using 2D-3D correspondences
- Estimate $\mathbf{K}$ and $(\mathbf{R}, \mathbf{t})$ from $\mathbf{P}$

$$\mathbf{P} = \mathbf{K} \cdot \mathbf{R} \cdot [\mathbf{I}_{3\times 3} \mid \mathbf{t}]$$

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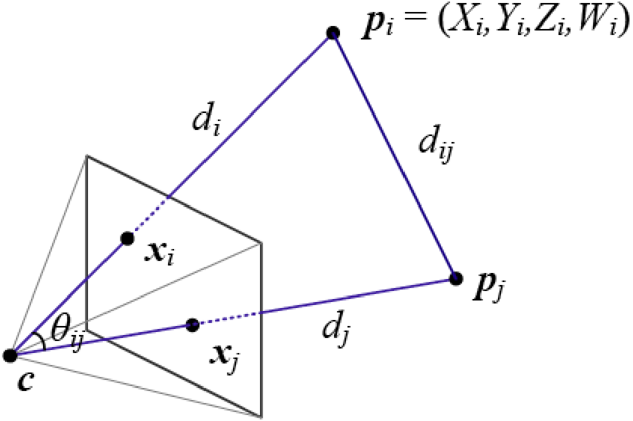
Inherent Constraints

- The visual angle between any pair of 2D points must be the same as the angle between their corresponding 3D points.



Direct Linear Transform

- Simplest to form a set of linear equations (analog to the 2D case)

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\begin{aligned}
x_i &= \frac{p_{00}X_i + p_{01}Y_i + p_{02}Z_i + p_{03}}{p_{20}X_i + p_{21}Y_i + p_{22}Y_i + p_{23}} \\
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  with (x_i, y_i), the measured 2D points, and (X_i, Y_i, Z_i) are known 3D locations

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Finding Camera Translation

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Finding Camera Orientation and Internal Parameters

- Left 3×3 submatrix $\mathbf{M}$ of $\mathbf{P}$ is of form

$$\mathbf{M} = \mathbf{KR}$$

  where $\mathbf{K}$ is an upper triangular matrix, and $\mathbf{R}$ is an orthogonal matrix

- Any non-singular square matrix $\mathbf{M}$ can be decomposed into the product of an upper-triangular matrix $\mathbf{K}$ and an orthogonal matrix $\mathbf{R}$ using the RQ factorization

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RQ Factorization

- Define the matrices

$$\mathbf{R}_x = \begin{bmatrix} 1 & 0 & 0 \\ 0 & c & -s \\ 0 & s & c \end{bmatrix}, \mathbf{R}_y = \begin{bmatrix} c' & 0 & s' \\ 0 & 1 & 0 \\ -s' & 0 & c' \end{bmatrix}, \mathbf{R}_z = \begin{bmatrix} c'' & -s'' & 0 \\ s'' & c'' & 0 \\ 0 & 0 & 1 \end{bmatrix}$$

- We can compute

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RQ Factorization

- Why does this algorithm work?

$$\mathbf{M}\mathbf{R}_x\mathbf{R}_y\mathbf{R}_z = \mathbf{K}$$

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Improved computation of $\mathbf{P}$

- We have equations involving homgeneous coordinates

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- Thus $\mathbf{x}_i$ and $\mathbf{X}_i$ just have to be proportional

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$$\mathbf{P}\mathbf{X}_i = \begin{bmatrix} \mathbf{p}_1^T \mathbf{X}_i \\ \mathbf{p}_2^T \mathbf{X}_i \\ \mathbf{p}_3^T \mathbf{X}_i \end{bmatrix}$$

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$$\mathbf{x}_i \times \mathbf{P}\mathbf{X}_i = \begin{bmatrix} v_i \mathbf{p}_3^T \mathbf{X}_i - w_i \mathbf{p}_2^T \mathbf{X}_i \\ w_i \mathbf{p}_3^T \mathbf{X}_i - u_i \mathbf{p}_2^T \mathbf{X}_i \\ u_i \mathbf{p}_3^T \mathbf{X}_i - v_i \mathbf{p}_2^T \mathbf{X}_i \end{bmatrix}$$

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- We can thus write

$$\begin{bmatrix} 0_4 & -w_i \mathbf{X}_i^T & v_i \mathbf{X}_i^T \\ w_i \mathbf{X}_i^T & 0_4 & -u_i \mathbf{X}_i^T \\ -v_i \mathbf{X}_i^T & u_i \mathbf{X}_i^T & 0_4 \end{bmatrix} \begin{bmatrix} \mathbf{p}_1 \\ \mathbf{p}_2 \\ \mathbf{p}_3 \end{bmatrix} = 0$$

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- 2 independent equations in 11 unknowns (ignoring scale)

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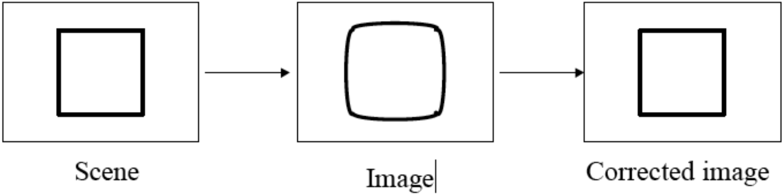
Radial Distorsion

- We have assumed that lines are imaged as lines
- Significant error for cheap optics and for short focal lengths

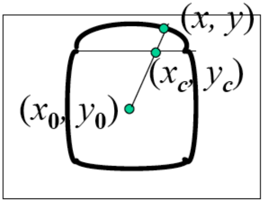


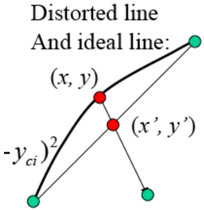
Radial Distorsion Correction

- We can write the correction as

$$\begin{aligned} x_c - x_0 &= L(r)(x - x_0) \\ y_c - y_0 &= L(r)(y - y_0) \end{aligned}$$

with $L(r) = 1 + \kappa_1 r^2 + \kappa_2 r^4$ and $r^2 = (x - x_0)^2 + (y - y_0)^2$





- We thus minimize the following function

$$f(\kappa_1, \kappa_2) = \sum_i (x_i' - x_{ci})^2 + (y_i' - y_{ci})^2$$

using lines known to be straight, with (x', y') the radial projection of (x, y) on straight line

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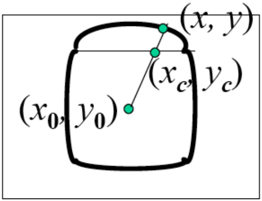

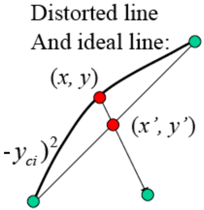

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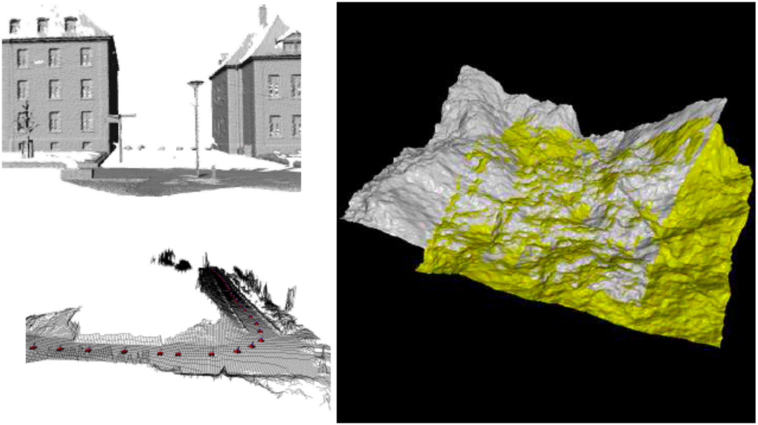
Let's look at 3D alignment

Motivation



[Source: W. Burgard]

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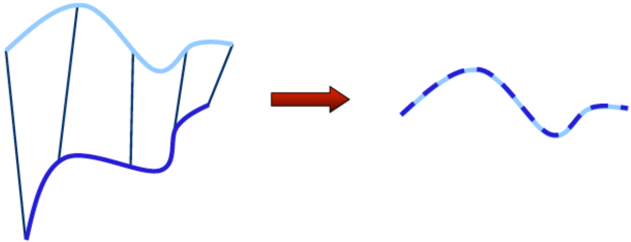
Key Idea

- If the correct correspondences are known, the correct relative rotation/translation can be calculated in closed form



[Source: W. Burgard]

Center of Mass

- The centroids of the two point clouds c and c' can be used to estimate the translation

$$\mu = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}_i \qquad \mu' = \frac{1}{N}\sum_{i=1}^{N}\mathbf{x}'_i$$

- Subtract the corresponding center of mass from every point in the two sets,

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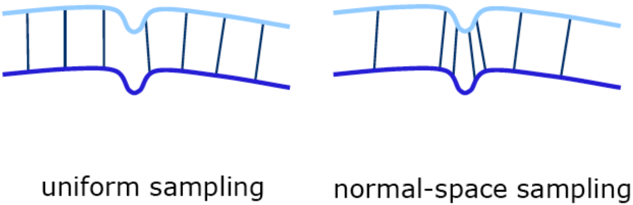
Normal-based sampling

- Ensure that samples have normals distributed as uniformly as possible

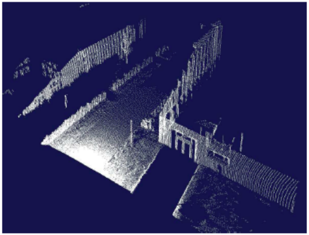


- better for mostly-smooth areas with sparse features

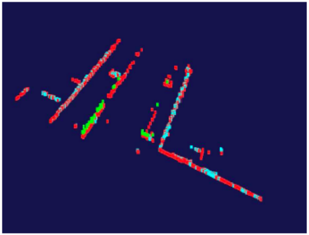
[Source: W. Burgard]

Feature-based sampling

- try to find important points
- decrease the number of correspondences
- higher efficiency and higher accuracy
- requires preprocessing



3D Scan (~200.000 Points)



Extracted Features (~5.000 Points)

[Source: W. Burgard]

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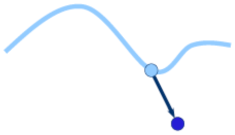

- **Normal shooting**: slightly better than closest point for smooth structures, worse for noisy or complex structures

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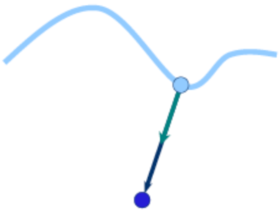

- **Point to plane**: lets flat regions slide along each other

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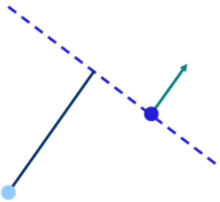

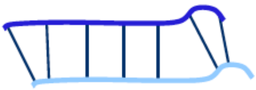

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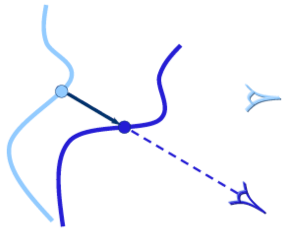

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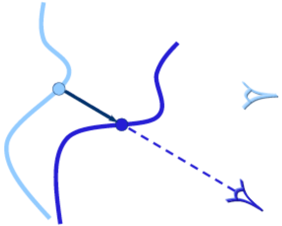

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Where do these 3D points come from?

Let's look into stereo reconstruction

Stereo

Public Library, Stereoscopic Looking Room, Chicago, by Phillips, 1923

[Source: N. Snavely]

Stereo

- Stereo matching is the process of taking two or more images and **estimating a 3D model** of the scene by **finding matching pixels** in the images and converting their 2D positions into 3D depths
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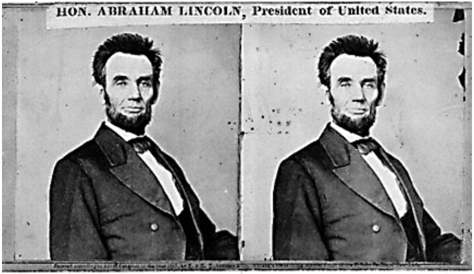

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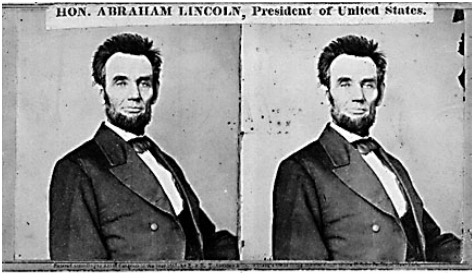

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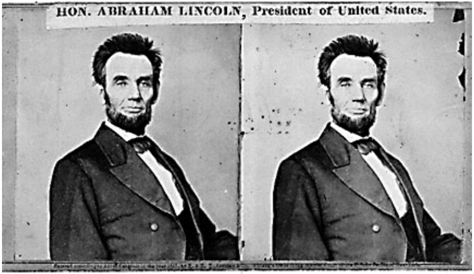

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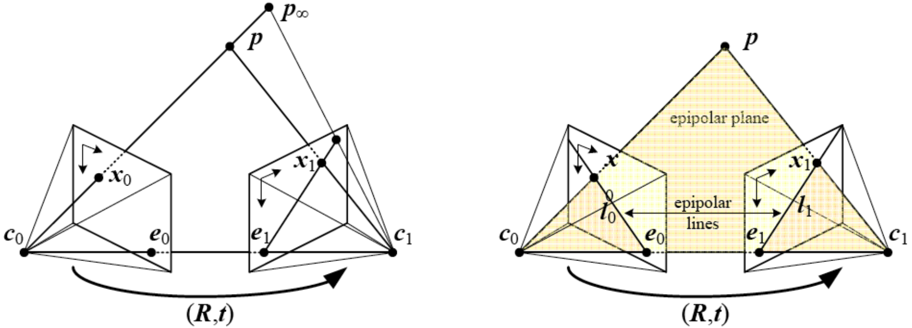
[Source: N. Snavely]

Epipolar Geometry



- Pixel in one image $\mathbf{x}_0$ projects to an **epipolar line** segment in the other image
- The segment is bounded at one end by the projection of the original viewing ray at infinity $\mathbf{p}_\infty$ and at the other end by the projection of the original camera center $\mathbf{c}_0$ into the second camera, which is known as the **epipole** $\mathbf{e}_1$.

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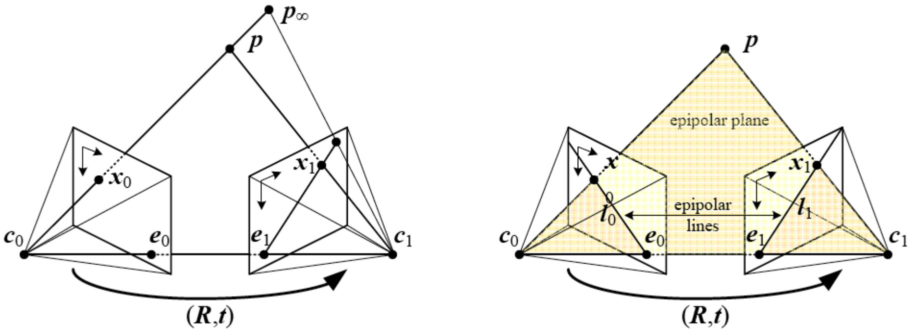

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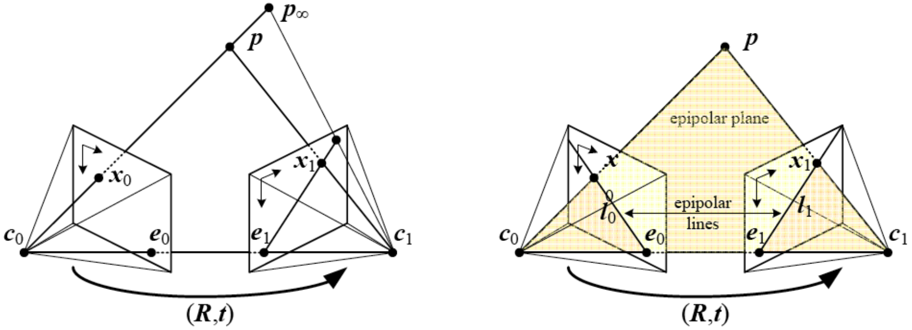
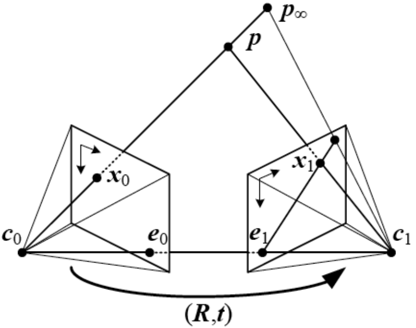
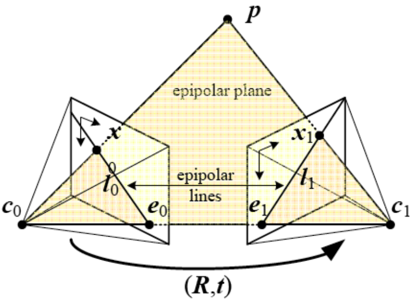


- If we project the epipolar line in the second image back into the first, we get another line (segment), this time bounded by the other corresponding **epipole** e_0
- Extending both line segments to infinity, we get a pair of corresponding epipolar lines, which are the intersection of the two image planes with the **epipolar plane** that passes through both camera centers $\mathbf{c}_0$ and $\mathbf{c}_1$ as well as the point of interest $\mathbf{p}$

Epipolar Geometry





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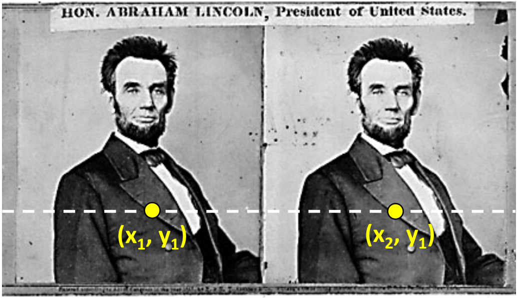
Epipolar Geometry

epipolar
lines



Two images captured by a purely horizontal translating camera
(*rectified* stereo pair)

- The disparity for pixel (x_1, y_1) is $(x_2 - x_1)$ if the images are rectified
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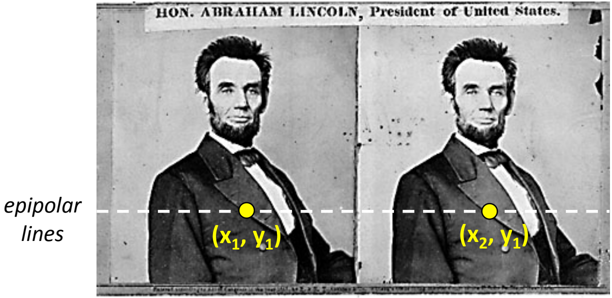
Epipolar Geometry



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[Source: N. Snavely]

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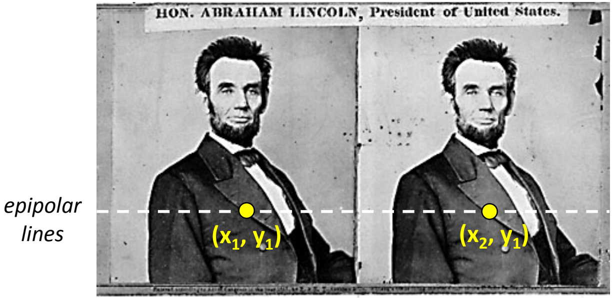

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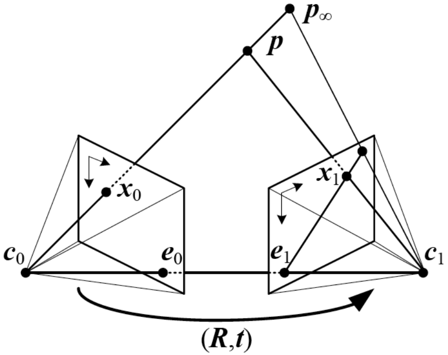
Fundamental Matrix



- Projective geometry depends only on the cameras internal parameters and relative pose of cameras
- Fundamental matrix **F** encapsulates this geometry

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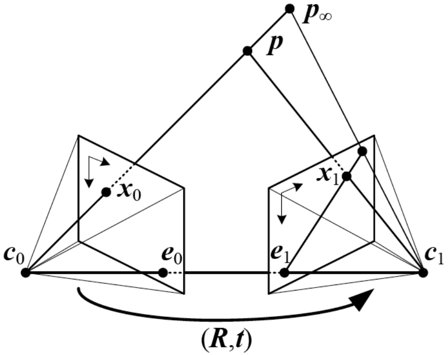

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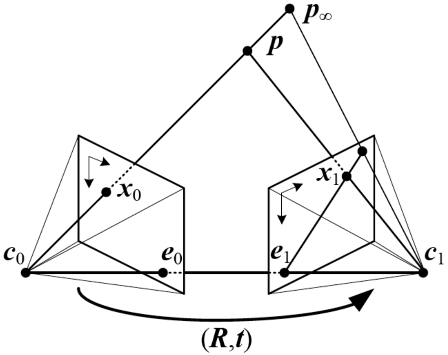

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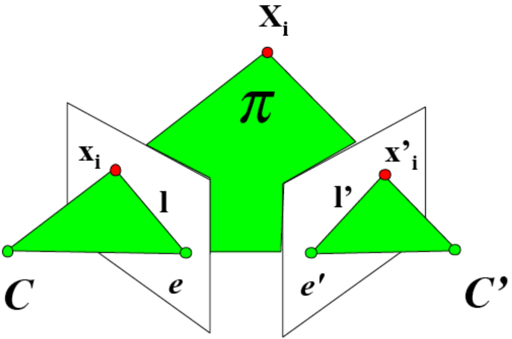
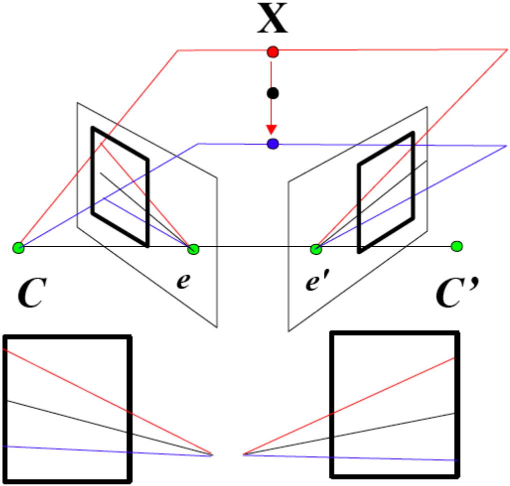
Epipolar Plane



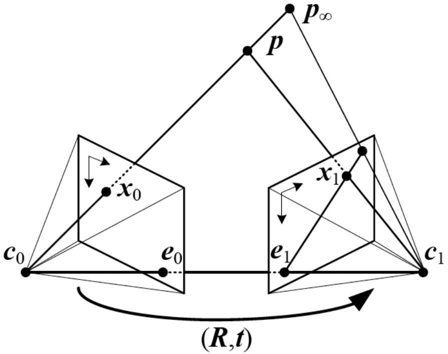
[Source: Ramani]

Pensils of Epipolar Lines



[Source: Ramani]

Computation of Fundamental Matrix



- **F** can be computed from correspondences between image points alone
- No knowledge of camera internal parameters required

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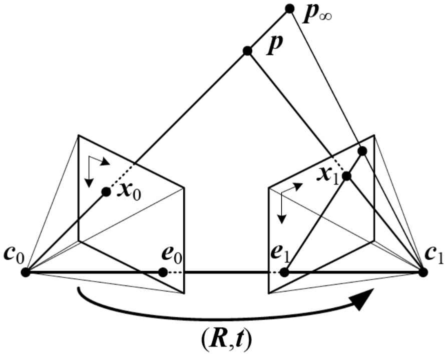

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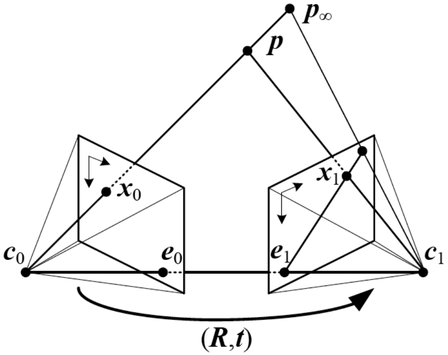

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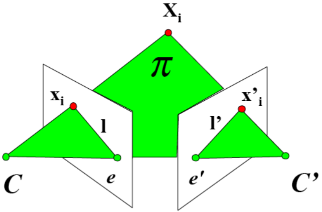
Fundamental Matrix and Projective Geometry



- Take $\mathbf{x}$ in camera $\mathbf{P}$ and find scene point $\mathbf{X}$ on ray of $\mathbf{x}$ in camera $\mathbf{P}$
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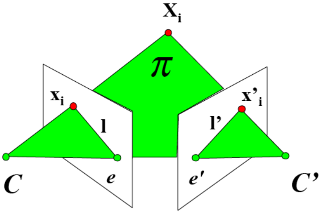

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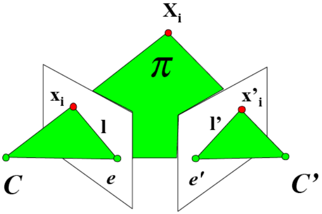

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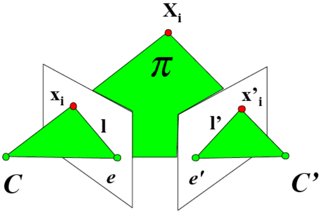

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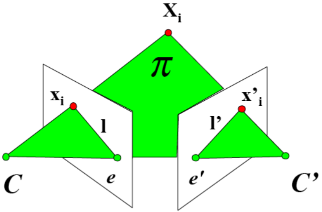

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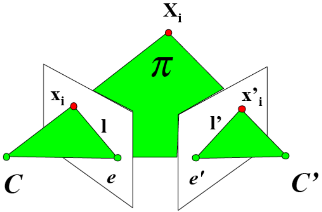
Fundamental Matrix and Projective Geometry



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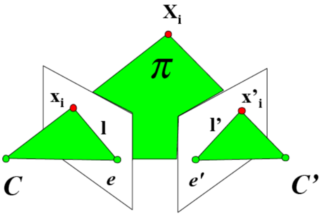
Finding the Fundamental Matrix from known Projections



- Take $\mathbf{x}$ in camera $\mathbf{P}$ and find one scene point on ray from $\mathbf{C}$ to $\mathbf{x}$
- Point $\mathbf{X} = \mathbf{P}^{+}\mathbf{x}$ satisfies $\mathbf{x} = \mathbf{PX}$ with $\mathbf{P}^{+} = \mathbf{P}^{T}(\mathbf{PP}^{T})^{-1}$ so $\mathbf{PX} = \mathbf{PP}^{T}(\mathbf{PP}^{T})^{-1}\mathbf{x} = \mathbf{x}$

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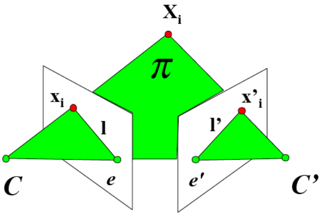

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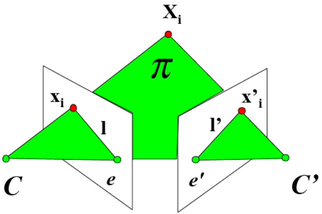

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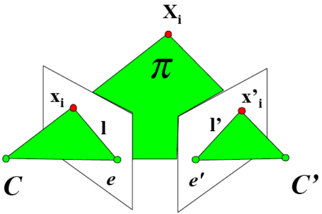

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- Image of this point in camera $\mathbf{P}'$ is $\mathbf{x}' = \mathbf{P}'\mathbf{X} = \mathbf{P}'\mathbf{P}^+\mathbf{x}$
- Image of $\mathbf{C}$ in camera $\mathbf{P}'$ is epipole $\mathbf{e}' = \mathbf{P}'\mathbf{C}$
- Epipolar line of $\mathbf{x}$ in $\mathbf{P}'$ is

$$\mathbf{l}' = (\mathbf{e}') \times (\mathbf{P}'\mathbf{P}^+\mathbf{x})$$

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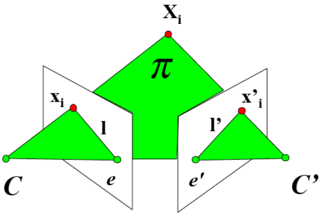

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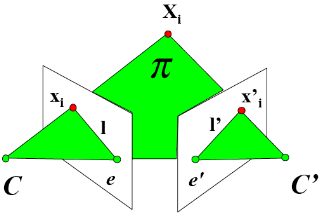

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Properties of the fundamental matrix

- Matrix 3×3 since $\mathbf{x'}^T \mathbf{F} \mathbf{x} = 0$
- Let $\mathbf{F}$ be the fundamental matrix of camera pair $(\mathbf{P}, \mathbf{P}')$, the fundamental matrix of camera pair $(\mathbf{P}', \mathbf{P})$ is $\mathbf{F}' = F^T$

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- Epipolar line of $\mathbf{x}$ is $\mathbf{l'} = \mathbf{F}\mathbf{x}$
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- Epipolar line of $\mathbf{x}'$ is $\mathbf{l} = \mathbf{F}^T \mathbf{x}'$
- Epipole $\mathbf{e}'$ is left null space of $\mathbf{F}$, and $\mathbf{e}$ is right null space.

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- Epipole $\mathbf{e'}$ is left null space of $\mathbf{F}$, and $\mathbf{e}$ is right null space.
- All epipolar lines $\mathbf{l'}$ contains epipole e', so $\mathbf{e'}^T \mathbf{l'} = 0$, i.e. $\mathbf{e'}^T \mathbf{F}\mathbf{x} = 0$ for all $\mathbf{x}$, therefore $\mathbf{e'}^T \mathbf{F} = 0$

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- Epipolar line of $\mathbf{x}'$ is $\mathbf{l} = \mathbf{F}^T \mathbf{x}'$
- Epipole $\mathbf{e}'$ is left null space of $\mathbf{F}$, and $\mathbf{e}$ is right null space.
- All epipolar lines $\mathbf{l}'$ contains epipole e', so $\mathbf{e}'^T \mathbf{l}' = 0$, i.e. $\mathbf{e}'^T \mathbf{F} \mathbf{x} = 0$ for all $\mathbf{x}$, therefore $\mathbf{e}'^T \mathbf{F} = 0$
- $\mathbf{F}$ is of rank 2 because $\mathbf{F} = \mathbf{e}' \times (\mathbf{P}' \mathbf{P}^+)$ and $\mathbf{e}' \times$ is of rank 2

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