Lagrange Duality & PCA
CS 411 Tutorial

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Lagrange Duality

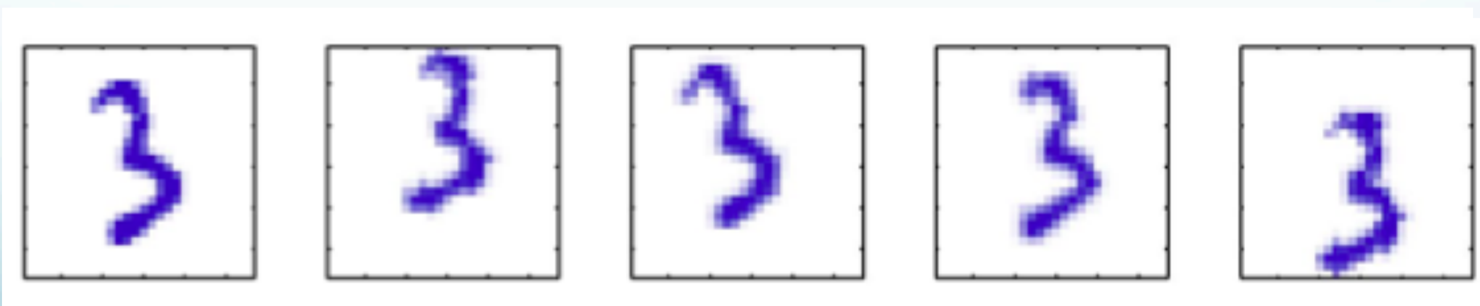
PCA
Tutorial

Dimensionality Reduction

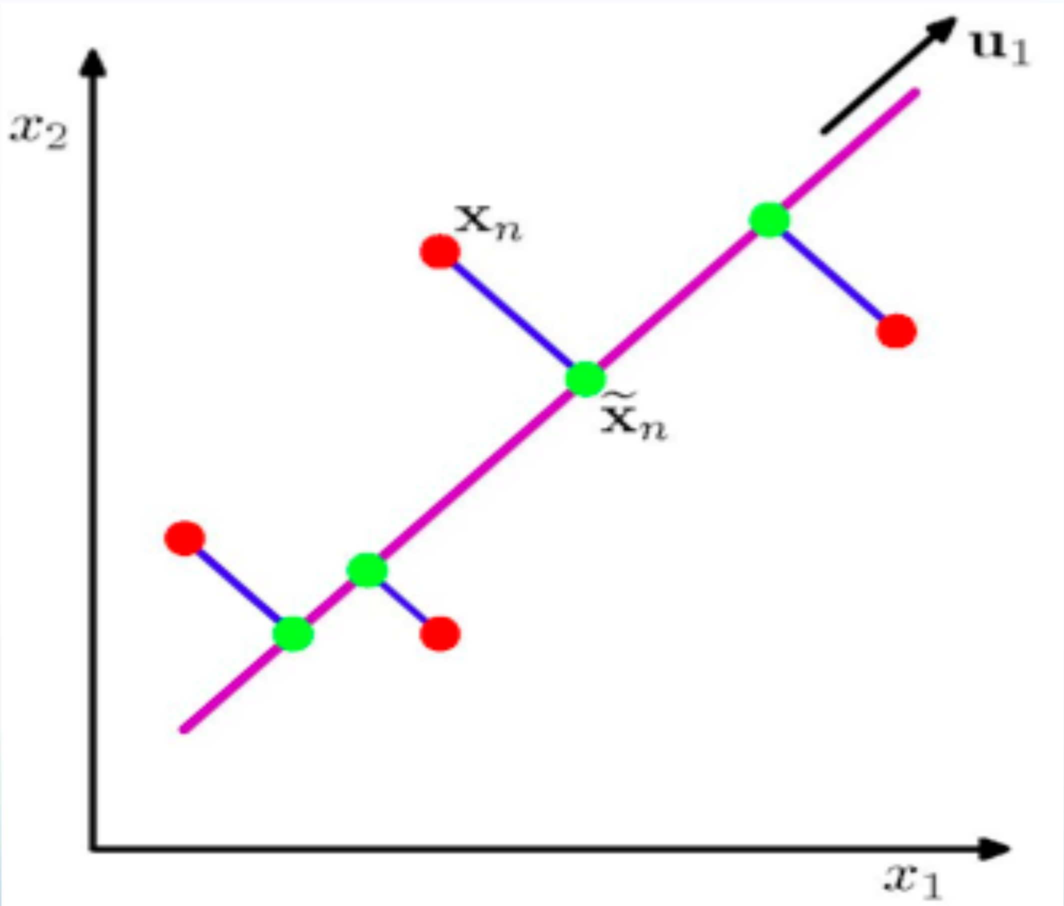
- We have some data $X \in \mathbb{R}^{N \times D}$
- D may be huge, etc.
- We would like to find a new representation $Z \in \mathbb{R}^{N \times K}$ where K << D.
  - For computational reasons.
  - To better understand (e.g., visualize) the data.
  - For compression.
  - ...
- We will restrict ourselves to linear transformations for the time being.

Example

- In this dataset, there are only 3 degrees of freedom: horizontal and vertical translations, and rotations.
- Yet each image contains 784 pixels, so X will be 784 elements wide.

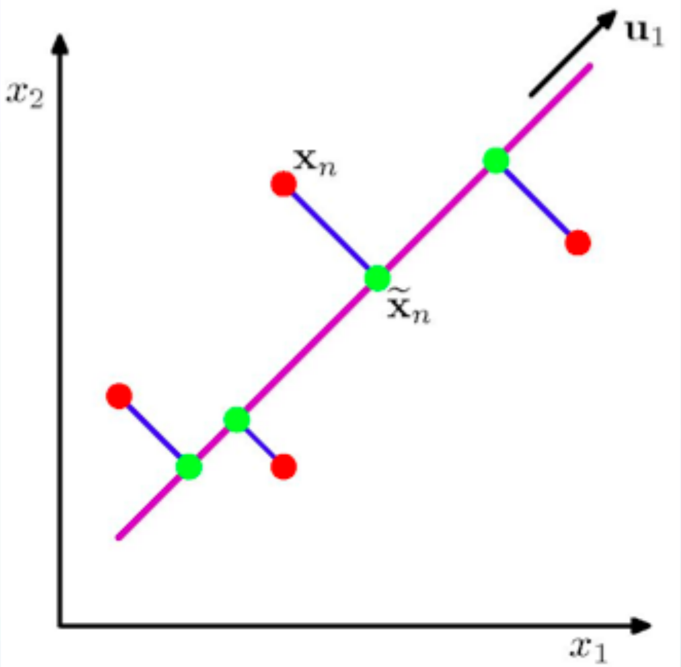


Abstract Visualization



What is a Good Transformation?

- Goal is to find good directions u that preserves “important” aspects of the data.
- In a linear setting: $z = x^T u$
- This will turn out to be the top-K eigenvalues of the data covariance.
- Two ways to view this:
  1. Find directions of *maximum variation*
  2. Find projections that *minimize reconstruction error*



Principal Component Analysis (Maximum Variance)

$$\begin{aligned} &\text{maximize} \frac{1}{2N} \sum_{n=1}^{N} (u_1^T x_n - u_1^T \bar{x}_n)^2 \\ &= u_1^T S u_1 \end{aligned}$$

i.e., variance of the projected data

where the sample mean and covariance are given by:

$$\begin{aligned} \bar{x} &= \frac{1}{N} \sum_{n=1}^{N} x_n \\ S &= \frac{1}{N} \sum_{n=1}^{N} (x_n - \bar{x})(x_n - \bar{x})^T \end{aligned}$$

Finding u1

- We want to maximize $u_1^T S u_1$

  subject to $||u_1|| = 1$
  (since we are finding a direction)

- Use Lagrange multiplier α_1 to express this as

$$u_1^T S u_1 + \alpha_1(1 - u_1^T u_1)$$

Finding u1

- Take derivative and set to 0

$$\begin{aligned} Su_1 - \alpha_1 u_1 &= 0 \\ Su_1 &= \alpha_1 u_1 \end{aligned}$$

- So u_1 is an eigenvector of S with eigenvalue α_1
- In fact it must be the eigenvector with maximum eigenvalue, since this minimizes the objective.

Finding u2

$$\begin{aligned} \text{maximize} \quad & u_2^T S u_2 \\ \text{subject to} \quad & ||u_2|| = 1 \\ & u_2^T u_1 = 0 \end{aligned}$$

Lagrange form: $u_2^T S u_2 + \alpha_2(1 - u_2^T u_2) - \beta u_2^T u_1$

Finding β: $\frac{\partial}{\partial u_2} = S u_2 - \alpha_2 u_2 - \beta u_1 = 0$

$$\begin{aligned} &\implies u_1^T S u_2 - \alpha_2 u_1^T u_2 - \beta u_1^T u_1 = 0 \\ &\implies \alpha_1 u_1^T u_2 - \alpha_2 u_1^T u_2 - \beta u_1^T u_1 = 0 \\ &\implies \alpha_1 \cdot 0 - \alpha_2 \cdot 0 - \beta \cdot 1 = 0 \\ &\implies \beta = 0 \end{aligned}$$

Finding u2

$$
\begin{aligned}
\text{maximize} \quad & u_2^T S u_2 \\
\text{subject to} \quad & ||u_2|| = 1 \\
& u_2^T u_1 = 0
\end{aligned}
$$

Lagrange form: $$u_2^T S u_2 + \alpha_2(1 - u_2^T u_2) - \cancelto{0}{\beta u_2^T u_1}$$

Finding α_2: $$\frac{\partial}{\partial u_2} = S u_2 - \alpha_2 u_2 = 0$$

$$\implies S u_2 = \alpha_2 u_2$$

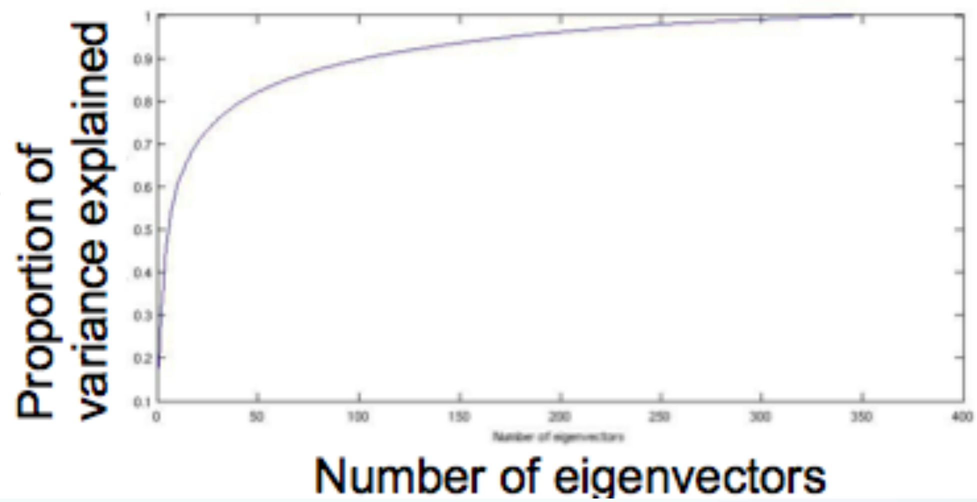
So α_2 must be the second largest eigevalue of S.

PCA in General

- We can compute the entire PCA solution by just computing the eigenvectors with the top-k eigenvalues.
- These can be found using the singular value decomposition of S.

- How do we choose the number of components?

$$\frac{\sum_{i=1}^{M} \alpha_i}{\sum_{i=1}^{N} \alpha_i}$$



- Look at the spectrum of covariance, pick K to capture most of the variation.
- More principled: Bayesian treatment (beyond this course).

Demo

- Eigenfaces

Principal Component Analysis (Minimum Reconstruction Error)

- We can also think of PCA as minimizing the *reconstruction error* of the compressed data.

$$\text{minimize} = \frac{1}{2N} \sum_{n=1}^{N} ||x_n - \hat{x}_n||^2$$

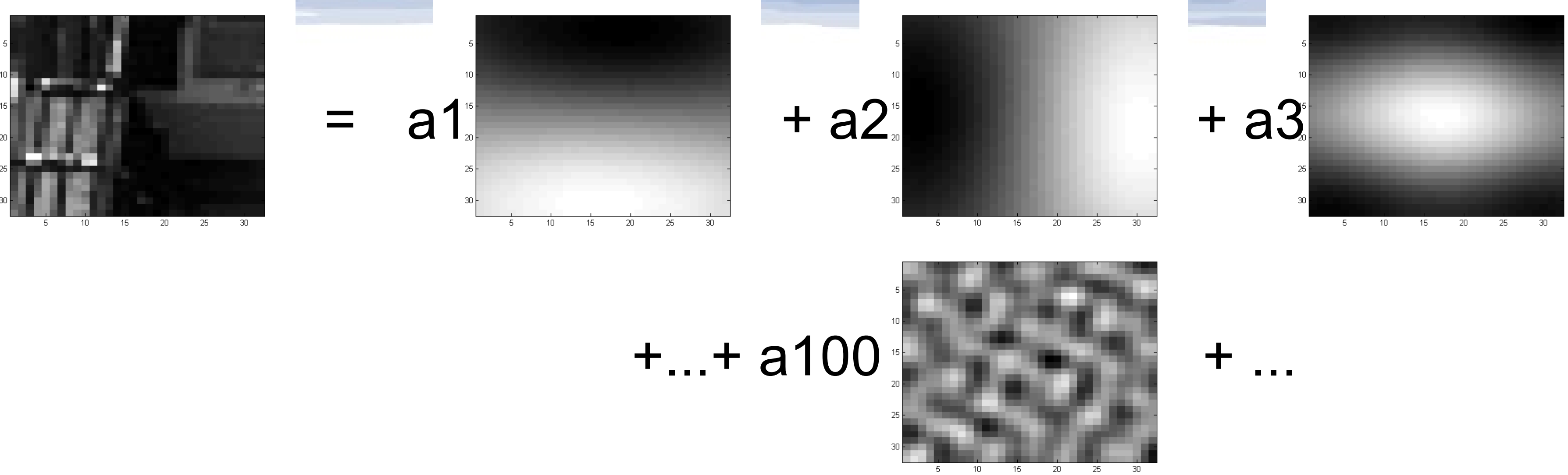
- We will omit the details for now, but the key is that we define some K-dimensional basis such that:

$$\hat{x} = Wx + const$$

- The solution will turn out to be the same as the minimum variance formulation.

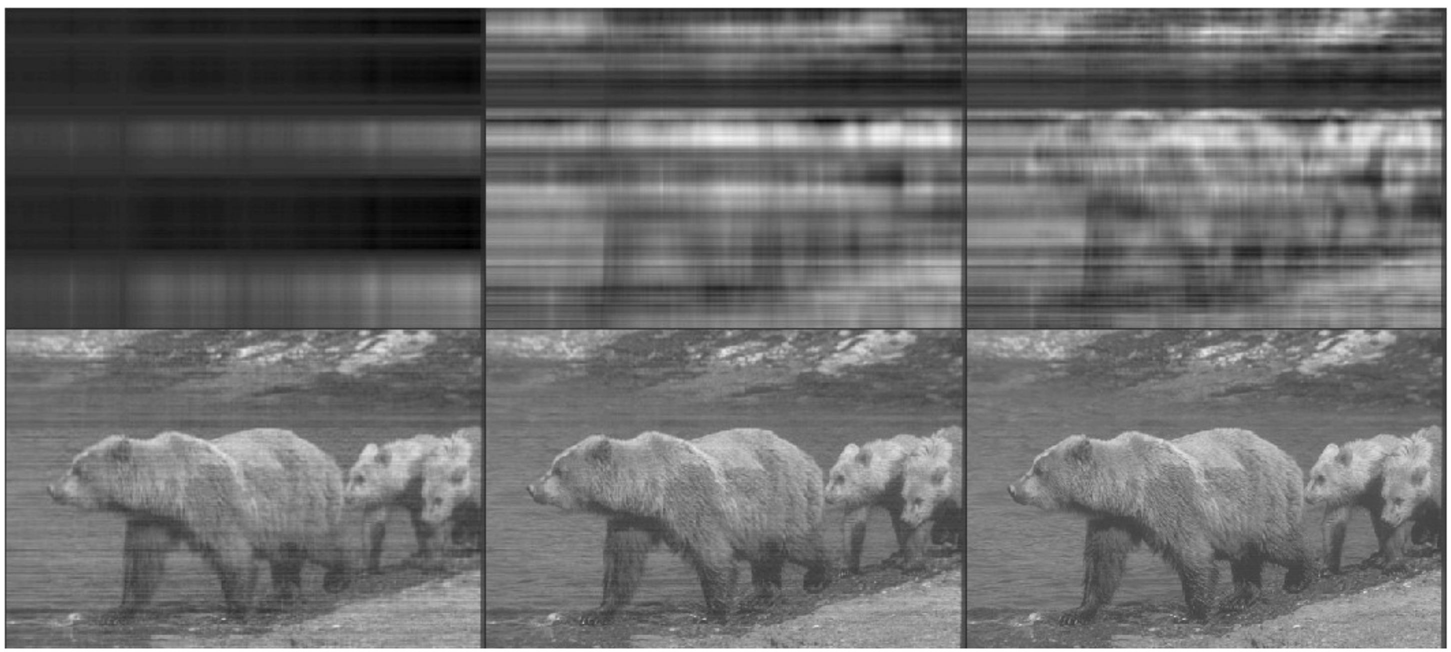
Reconstruction

- PCA learns to represent vectors in terms of sums of basis vectors.
- For images, e.g.,



PCA for Compression

D=1 D=5 D=10



D=50 D=100 D=200

321x481 image, D is the number of basis vectors used

D in this slide is the same as K in the previous slides

Summary

Thank You ;-)