CSC 411: Lecture 05: Nearest Neighbors

Rich Zemel, Raquel Urtasun and Sanja Fidler

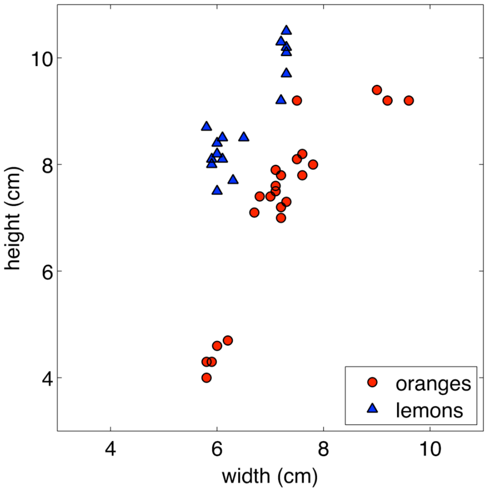
University of Toronto

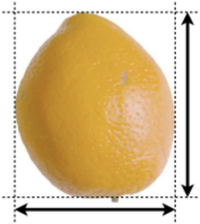
Today

- Non-parametric models
  - distance
  - non-linear decision boundaries

Note: We will mainly use today's method for classification, but it can also be used for regression

Classification: Oranges and Lemons





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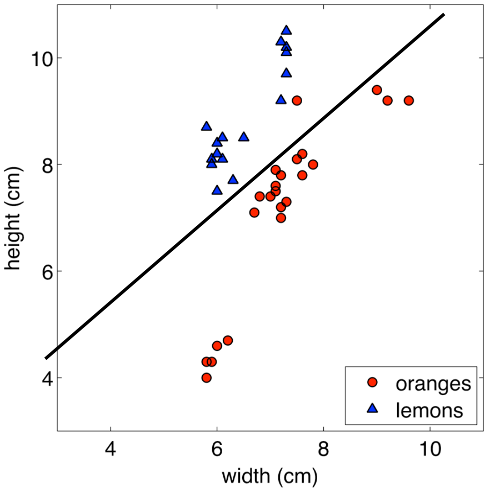
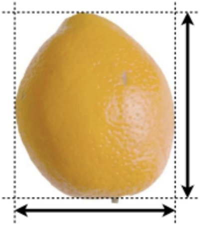


Can construct simple linear decision boundary:

$y = \text{sign}(w_0 + w_1 x_1 + w_2 x_2)$



What is the meaning of "linear" classification

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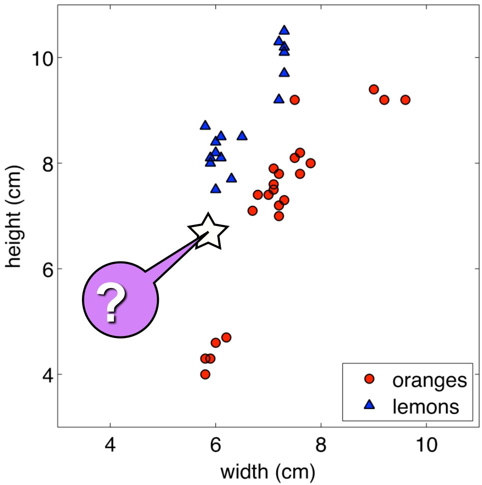
- What functions $f()$ have we seen so far in class?

Classification as Induction



Instance-based Learning

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- Learning amounts to simply storing training data
- Test instances classified using similar training instances
- Embodies often sensible underlying assumptions:
  - Output varies smoothly with input
  - Data occupies sub-space of high-dimensional input space

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- Note: we don't really need to compute the square root. Why?

Nearest Neighbors: Decision Boundaries

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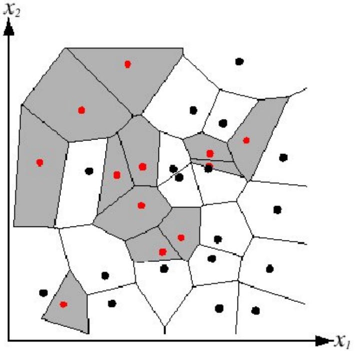
- Nearest neighbor algorithm does not explicitly compute decision boundaries, but these can be inferred
- Decision boundaries: Voronoi diagram visualization
  - show how input space divided into classes
  - each line segment is equidistant between two points of opposite classes



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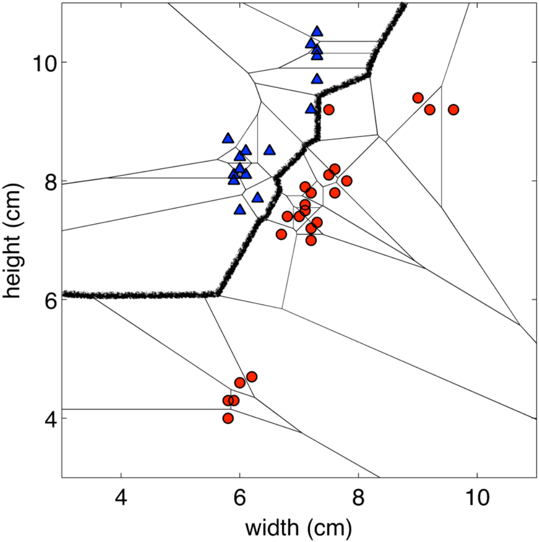

Example: 2D decision boundary

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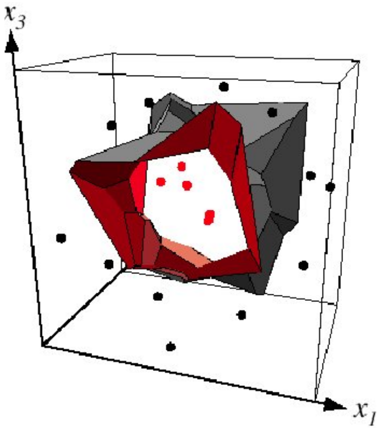
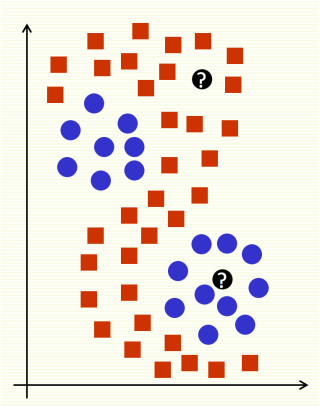


Example: 3D decision boundary

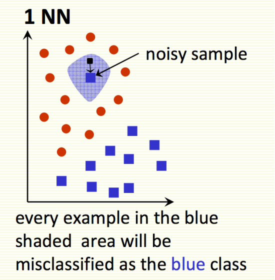
Nearest Neighbors: Multi-modal Data

- Nearest Neighbor approaches can work with multi-modal data



[Slide credit: O. Veksler]

Nearest Neighbors



[Pic by Olga Veksler]

- Nearest neighbors sensitive to mis-labeled data ("class noise"). Solution?

k-Nearest Neighbors

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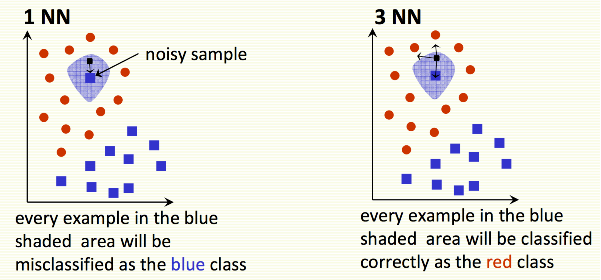

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k-Nearest Neighbors

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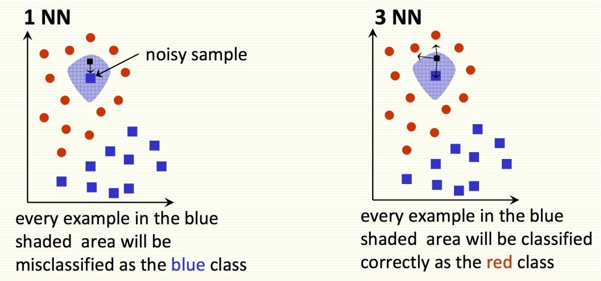

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Algorithm (kNN):

1. Find k examples $\{\mathbf{x}^{(i)}, t^{(i)}\}$ closest to the test instance $\mathbf{x}$
2. Classification output is majority class

$$y = arg \max_{t^{(z)}} \sum_{r=1}^{k} \delta(t^{(z)}, t^{(r)})$$

k-Nearest Neighbors

How do we choose k?

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- Larger k may lead to better performance
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- We can use cross-validation to find k
- Rule of thumb is $k < sqrt(n)$, where n is the number of training examples

[Slide credit: O. Veksler]

k-Nearest Neighbors: Issues & Remedies

- If some attributes (coordinates of $\mathbf{x}$) have larger ranges, they are treated as more important

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    - Simple option: Linearly scale the range of each feature to be, e.g., in range [0,1]
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- Non-metric attributes (symbols)
  - Hamming distance

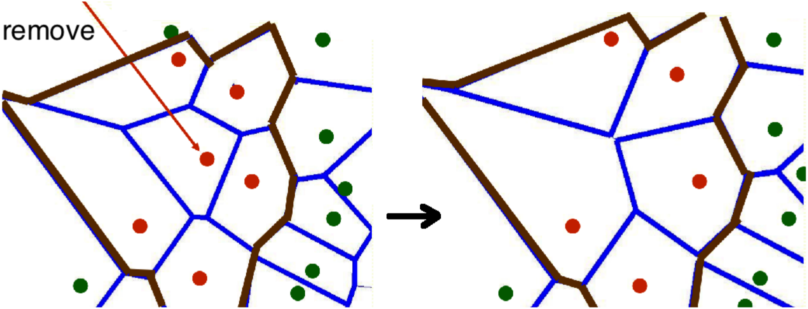
k-Nearest Neighbors: Issues (Complexity) & Remedies

- Expensive at test time: To find one nearest neighbor of a query point **x**, we must compute the distance to all N training examples. Complexity: $O(kdN)$ for kNN
  - Use subset of dimensions
  - Pre-sort training examples into fast data structures (e.g., kd-trees)
  - Compute only an approximate distance (e.g., LSH)
  - Remove redundant data (e.g., condensing)
- Storage Requirements: Must store all training data
  - Remove redundant data (e.g., condensing)
  - Pre-sorting often increases the storage requirements
- High Dimensional Data: "Curse of Dimensionality"
  - Required amount of training data increases exponentially with dimension
  - Computational cost also increases

[Slide credit: David Claus]

k-Nearest Neighbors Remedies: Remove Redundancy

- If all Voronoi neighbors have the same class, a sample is useless, remove it



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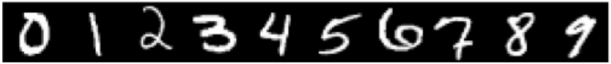
Example: Digit Classification

- Decent performance when lots of data



- Yann LeCunn – MNIST Digit Recognition
  - Handwritten digits
  - 28x28 pixel images: d = 784
  - 60,000 training samples
  - 10,000 test samples
- Nearest neighbour is competitive

	Test Error Rate (%)
Linear classifier (1-layer NN)	12.0
K-nearest-neighbors, Euclidean	5.0
K-nearest-neighbors, Euclidean, deskewed	2.4
K-NN, Tangent Distance, 16x16	1.1
K-NN, shape context matching	0.67
1000 RBF + linear classifier	3.6
SVM deg 4 polynomial	1.1
2-layer NN, 300 hidden units	4.7
2-layer NN, 300 HU, [deskewing]	1.6
LeNet-5, [distortions]	0.8
Boosted LeNet-4, [distortions]	0.7

Fun Example: Where on Earth is this Photo From?

- Problem: Where (e.g., which country or GPS location) was this picture taken?

[Paper: James Hays, Alexei A. Efros. im2gps: estimating geographic information from a single image. CVPR'08. Project page: http://graphics.cs.cmu.edu/projects/im2gps/]

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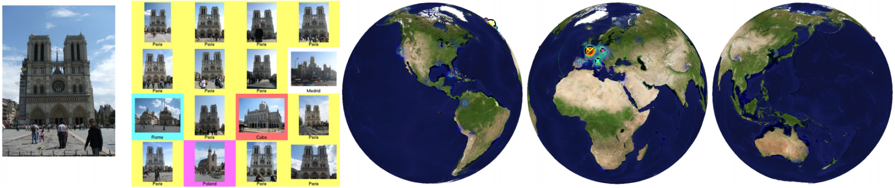
  - Get 6M images from Flickr with GPs info (dense sampling across world)
  - Represent each image with meaningful features
  - Do kNN!



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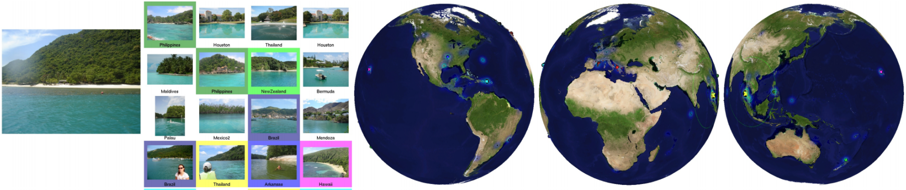

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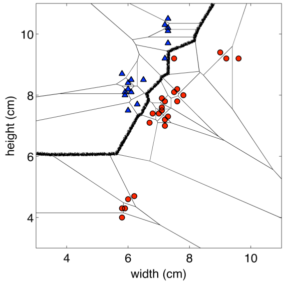

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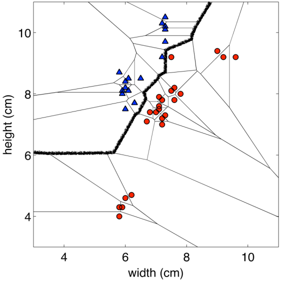

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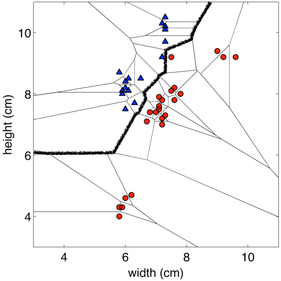

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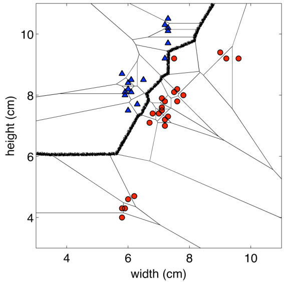

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- Inductive Bias: What kind of decision boundaries do we expect to find?