On Injective Colourings of Chordal Graphs

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Abstract. We show that one can compute the injective chromatic number of a chordal graph G at least as efficiently as one can compute the chromatic number of $(G-B)^2$, where B are the bridges of G. In particular, it follows that for strongly chordal graphs and so-called power chordal graphs the injective chromatic number can be determined in polynomial time. Moreover, for chordal graphs in general, we show that the decision problem with a fixed number of colours is solvable in polynomial time. On the other hand, we show that computing the injective chromatic number of a chordal graph is NP-hard; and unless $NP = ZPP$, it is hard to approximate within a factor of $n^{1/3-\epsilon}$, for any $\epsilon > 0$. For split graphs, this is best possible, since we show that the injective chromatic number of a split graph is $\sqrt[3]{n}$-approximable. (In the process, we correct a result of Agnarsson et al. on inapproximability of the chromatic number of the square of a split graph.)

1 Introduction

In this paper, a graph is always assumed to be undirected, loopless and simple. An *injective colouring* of a graph G is a colouring c of the vertices of G that assigns different colours to any pair of vertices that have a common neighbour. (That is, for any vertex v, if we restrict c to the (open) neighbourhood of v, this mapping will be injective; whence the name.) Note that injective colouring is not necessarily a proper colouring, i.e., it is possible for two adjacent vertices to receive the same colour. The injective chromatic number of G, denoted $\chi_i(G)$, is the smallest integer k such that G can be injectively coloured with k colours.

Injective colourings are closely related to (but not identical with) the notions of locally injective colourings [9] and $L(h,k)$-labellings [2, 3, 11]. In particular, $L(0,1)$-labellings unlike injective colourings assign distinct colours only to non-adjacent vertices with a common neighbour.

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Injective colourings were introduced by Hahn, Kratochvíl, Širáň and Sotteau in [12]. They attribute the origin of the concept to complexity theory on Random Access Machines. They prove several interesting bounds on $\chi_i(G)$, and also show that, for $k \geq 3$, it is NP-complete to decide whether the injective chromatic number of a graph is at most k. Here we look at the complexity of this problem when the input graphs G are restricted to be chordal. A graph is *chordal* if it does not contain any induced cycle of length four or more [10]. Several difficult combinatorial problems that are NP-complete in general (including graph colouring [10], and many variants [5, 7, 13]) admit a polynomial time solution in chordal graphs.

In Section 4, we show that determining $\chi_i(G)$ is still difficult when restricted to chordal graphs. In fact, it is not only NP-hard, but unless $NP = ZPP$, the injective chromatic number of a chordal graph cannot be efficiently approximated within a factor of $n^{1/3-\epsilon}$, for any $\epsilon > 0$. (Here ZPP is the class of languages decidable by a randomized algorithm that makes no errors and whose expected running time is polynomial.) For split graphs, this is best possible since we show an $\sqrt[3]{n}$-approximation algorithm for the injective chromatic number of a split graph.

On the positive side, we show in Section 5 that for any fixed number k, one can in linear time determine whether a chordal graph can be injectively coloured using no more than k colours. Moreover, we describe large subclasses of chordal graphs that allow computing the injective chromatic number efficiently. We show that for a chordal graph G, one can efficiently compute the injective chromatic number of G from the chromatic number of the square of $G - \mathcal{B}(G)$, that is, the graph G with its bridges $\mathcal{B}(G)$ removed. It follows that for strongly chordal graphs and power chordal graphs (the graphs whose powers are all chordal) the problems is polynomial time solvable.

2 Preliminaries

We follow the terminology of [4, 20]. For a subset S of the vertices (edges) of G, we denote $G[S]$ the subgraph of G induced on the vertices (edges) of S, and $G - S$ the subgraph of G that is obtained by removing from G the vertices (edges) of S. In the case that S consists only of a single element x, we write $G - x$ instead of $G - \{x\}$.

For a connected graph G, a vertex u is a *cutpoint* of G if the graph $G - u$ is disconnected. An edge $e = uv$ is a *bridge* of G if the graph $G - e$ is disconnected. A subset S of vertices of G is a *separator* of G if

$G - S$ is disconnected. As usual, a *clique* of G is a complete subgraph of G, and an *independent set* of G is a subgraph of G having no edges. For any graph G, we denote by $\chi(G)$ and $\alpha(G)$, the chromatic number of G, and the size of a maximum independent set in G, respectively. We denote by G^k the k-th power of G, i.e., the graph obtained from G by making adjacent any two vertices in distance at most k in G. We denote by n, respectively m, the number of vertices, respectively edges of G. For a vertex u in G, we denote by $N(u)$ the set of vertices of G adjacent to u (the *neighbourhood* of u); and for a subset S of vertices of G, we denote by $N(S)$ the set of vertices of $G - S$ adjacent to at least one vertex of S. We let $\deg(u) = |N(u)|$ be the degree of u, and let $\Delta(G)$ be the maximum degree among the vertices of G.

A *split graph* is a graph which can be partitioned into a clique and an independent set with no other restriction on the edges between the two. Any split graph is also chordal. A *tree-decomposition* (T, X) of a connected graph G is a pair (T, X) where T is a tree and X is a mapping from $V(T)$ to the subsets of $V(G)$, such that *(i)* for any edge $ab \in E(G)$, there exists $u \in V(T)$ with $a, b \in X(u)$, and *(ii)* for any vertex $a \in V(G)$, the vertices $u \in V(T)$ with $a \in X(u)$ induce a connected subgraph in T. A *clique-tree* of a chordal graph G is a tree-decomposition (T, X) of G where $\{X(u) \mid u \in V(T)\}$ is precisely the set of all maximal cliques of G.

3 Basic properties

We have the following simple observation.

Observation 1. *For any graph G, $\chi_i(G) \geq \Delta(G)$ and $\chi(G^2) \geq \Delta(G)+1$.*

For trees this is also an upper bound.

Proposition 2. *For any tree T, $\chi_i(T) = \Delta(T)$ and $\chi(T^2) = \Delta(T) + 1$.*

Proof. Let u be a leaf in T and v the parent of u. Then clearly, $\chi(T^2) = \max\{\deg(v)+1, \chi((T-u)^2)\}$ and $\chi_i(T) = \max\{\deg(v), \chi_i(T-u))\}$. The claim follows by induction on $|V(T)|$. $\square$

Now we look at the general case. Let $G^{(2)}$ be the *common neighbour graph* of a graph G, that is, the graph on the vertices of G in which two vertices are adjacent if they have a common neighbour in G. It is easy to see that the injective chromatic number of G is exactly the chromatic number of $G^{(2)}$. In general, as we shall see later, properties of the graph $G^{(2)}$ can be very different from those of G. For instance, even if G is efficiently colourable, e.g. if G is perfect, it may be difficult to colour $G^{(2)}$. Note that any edge of $G^{(2)}$ must be also an edge of G^2 (but not conversely). This yields the following inequality.

Proposition 3. *For any graph G, we have $\chi_i(G) \leq \chi(G^2)$.*

In fact, this inequality can be strengthened. Let $\mathcal{F}(G)$ be the set of edges of G that do not lie in any triangle. Note that an edge of G is also an edge of $G^{(2)}$ if and only if it belongs to a triangle of G. This proves the following proposition.

Proposition 4. *For any graph G, we have $\chi_i(G) = \chi(G^2 - \mathcal{F}(G))$.*

Now we turn to chordal graphs. The following is easy to check.

Observation 5. *Any edge in a bridgeless chordal graph lies in a triangle.*

Let $\mathcal{B}(G)$ be the set of bridges of G. Since a bridge of a graph can never be in a triangle, we have the following fact.

Proposition 6. *For any chordal G, we have $\chi_i(G) = \chi(G^2 - \mathcal{B}(G))$.*

Now since $\mathcal{B}(G - \mathcal{B}(G)) = \emptyset$, we have the following corollary.

Corollary 7. *For any chordal G, $\chi_i(G - \mathcal{B}(G)) = \chi((G - \mathcal{B}(G))^2)$.*

It turns out that there is a close connection between $\chi_i(G - \mathcal{B}(G))$ and both $\chi_i(G)$ and $\chi(G^2)$.

Proposition 8. *For any G, $\chi(G^2) = \max\{\Delta(G) + 1, \chi((G - \mathcal{B}(G))^2)\}$*

Proof. Let $k = \max\{\Delta(G) + 1, \chi((G - \mathcal{B}(G))^2)\}$. It follows from Observation 1 and Corollary 7 that $\chi(G^2) \geq k$. Now fix a set of k colours ($k \geq \chi((G - \mathcal{B}(G))^2)$), and consider a colouring of $(G - \mathcal{B}(G))^2$ using these k colours. We now add the bridges of G one by one, modifying the colouring accordingly. Let uv be a bridge of G and let X and Y be the connected components which become connected by the addition of uv. Suppose that $u \in X$ and $v \in Y$. We can permute the colours of X and Y independently so that u and v obtain the same colour i. Since we have $k \geq \Delta(G) + 1$ colours, there must be a colour $j \neq i$ not used in the neighbourhood of v in Y. By the same argument for u, we may assume that j is not used in the neighbourhood of u in X. Finally, we exchange in X the colours i and j. It is easy to see that after adding all bridges of G one by one, we obtain a proper colouring of G^2. $\square$

A similar argument proves the next proposition.

Proposition 9. *For any split graph G, $\chi_i(G) = \max\{\Delta(G), \chi_i(G - \mathcal{B}(G))\}$*

Finally, combining Corollary 7, and Propositions 8 and 3, we obtain the following tight lower bound on the injective chromatic number of a chordal graph.

Proposition 10. *For any chordal graph G, we have*

$$\chi(G^2) - 1 \leq \max\{\Delta(G), \chi_i(G - \mathcal{B}(G))\} \leq \chi_i(G) \leq \chi(G^2)$$

4 Hardness and approximation results

In this section, we focus on hardness results for the injective chromatic number problem. We begin by observing that it is NP-hard to compute the injective chromatic number of a split graph. This also follows from a similar proof in [12]; we include our construction here, since we shall extend it to prove an accompanying inapproximability result in Theorem 13.

Theorem 11. *It is NP-complete for a given split (and hence chordal) graph G and an integer k, to decide whether the injective chromatic number of G is at most k.*

Proof. First, we observe that the problem is clearly in NP. We show it is also NP-hard. Consider an instance of the graph colouring problem, namely a graph G and an integer l. We may assume that G is connected and contains no bridges. Let H_G be the graph constructed from G by first subdividing each edge of G and then connecting all the new vertices. That is, $V(H_G) = V(G) \cup \{x_{uv} \mid uv \in E(G)\}$ and $E(H_G) = \{ux_{uv}, vx_{uv} \mid uv \in E(G)\} \cup \{x_{st}x_{uv} \mid uv, st \in E(G)\}$. The graph H_G can clearly be constructed in polynomial time. It is not difficult to see that H_G is a split graph, hence it is also chordal. Moreover, one can check that the subgraph of H_G^2 induced on the vertices of G is precisely the graph G. Since G is bridgeless, H_G is also bridgeless, hence using Proposition 6 we have the following.

$$\chi_i(H_G) = \chi(H_G^2) = \chi(G) + m$$

Therefore $\chi_i(H_G)$ is at most $k = l + m$ if and only if $\chi(G)$ is at most l. That concludes the proof. $\square$

By Proposition 10, for any chordal graph G, the injective chromatic number of G is either $\chi(G^2)$ or $\chi(G^2) - 1$. Interestingly, merely distinguishing between these two cases is already NP-complete.

Theorem 12. *It is NP-complete to decide, for a given split (and hence chordal) graph G, whether $\chi_i(G) = \chi(G^2) - 1$.* $\square$

Now we extend the proof of Theorem 11 to show that under a certain complexity assumption, it is not tractable to approximate the injective chromatic number of a split (chordal) graph within a factor of $n^{1/3-\epsilon}$ for all $\epsilon > 0$.

Theorem 13. *Unless $NP = ZPP$, for any $\epsilon > 0$, it is not possible to efficiently approximate $\chi(G^2)$ and $\chi_i(G)$ within a factor of $n^{1/3-\epsilon}$, for any split (and hence chordal) graph G.*

Proof. In [8], it was shown that for any fixed $\epsilon > 0$, unless $NP = ZPP$, the problem of deciding whether $\chi(G) \leq n^\epsilon$ or $\alpha(G) < n^\epsilon$ for a given graph G is not solvable in polynomial time. Consider an instance of this problem, namely a graph G. Again, as in the proof of Theorem 11, we may assume that G is connected and bridgeless. Let $H_{k,G}$ be the split graph constructed from k copies of H_G (the graph used in the proof Theorem 11) by identifying, for each $uv \in E(G)$, all copies of x_{uv}. That is, if $v_1, v_2, \ldots, v_n$ are the vertices of G, we have in $H_{k,G}$ vertices $V(H_{k,G}) = \bigcup_{i=1}^{k} \{v_1^i, v_2^i, \ldots, v_n^i\} \cup \{x_{uv} \mid uv \in E(G)\}$, and edges $E(H_{k,G}) = \bigcup_{i=1}^{k} \{u^i x_{uv},\ v^i x_{uv} \mid uv \in E(G)\} \cup \{x_{uv} x_{st} \mid uv, st \in E(G)\}$.

Now since G is bridgeless, $H_{k,G}$ is also bridgeless. Consider an independent set I of $H_{k,G}^2$. It is not difficult to check that either I trivially contains only a single vertex x_{uv}, or for each pair of vertices $u^i, v^j \in I$, the vertices u and v are not adjacent in G. Hence it follows that from any colouring of $H_{k,G}^2$ one can construct a fractional k-fold colouring of G (i.e., a collection of independent sets covering each vertex of G at least k times) by projecting each non-trivial colour class of $H_{k,G}^2$ to G, i.e., mapping each u^i to u. Using this observation we obtain the following inequalities.

$$\frac{k \cdot n}{\alpha(G)} + m \leq k \cdot \chi_f(G) + m \leq \chi(H_{k,G}^2) = \chi_i(H_{k,G}) \leq k \cdot \chi(G) + m$$

Therefore if $\chi(G) \leq n^\epsilon$ then $\chi(H_{k,G}^2) \leq k \cdot n^\epsilon + m$, and if $\alpha(G) < n^\epsilon$ then $\chi(H_{k,G}^2) > k \cdot n^{1-\epsilon} + m$. Now we fix $k = m$, and denote by N the number of vertices in $H_{m,G}$. For $n \geq 2^{1/\epsilon}$ we obtain the following.

$$\frac{m \cdot n^{1-\epsilon} + m}{m \cdot n^\epsilon + m} \geq \frac{1}{2} n^{1-2\epsilon} \geq n^{1-3\epsilon} \geq (m \cdot n + m)^{\frac{1}{3}(1-3\epsilon)} = N^{\frac{1}{3}-\epsilon}$$

Hence if we can efficiently $(N^{\frac{1}{3}-\epsilon})$-approximate the colouring of $H^2_{m,G}$ then we can decide whether $\chi(G) \leq n^\epsilon$ or $\alpha(G) < n^\epsilon$. That concludes the proof. $\square$

Note that a seemingly stronger result appeared in [1]. Namely, the authors claim that the chromatic number of the square of a split graph is not $(n^{1/2-\epsilon})$-approximable for all $\epsilon > 0$. However this result is not correct. In fact, we show below that there exists a polynomial time algorithm $\sqrt[3]{n}$-approximating the chromatic number of the square of a split graph G, and also $\sqrt[3]{n}$-approximating the injective chromatic number of G. Note that this is also a strengthening of best known $\sqrt{n}$-approximation algorithm for the chromatic number of the square in general graphs (cf. [1]). We need the following lemma.

Lemma 14. *For chordal graphs, the injective chromatic number is α-approximable if and only if the chromatic number of the square is α-approximable.* $\square$

Theorem 15. *There exists a polynomial time algorithm that given a split graph G approximates $\chi(G^2)$ and $\chi_i(G)$ within a factor of $\sqrt[3]{n}$.*

Proof. Let G be a connected split graph with a clique X and an independent set Y. Denote by H the subgraph of G^2 induced on Y. Let $p = |X|$, $N = |V(H)|$, and $M = |E(H)|$. Clearly, $\chi(G^2) = p + \chi(H)$. Consider an optimal colouring of H with colour classes $V_1, V_2, \ldots, V_{\chi(H)}$. Let E_{ij} be the edges of H between V_i and V_j. Clearly, for each edge $uv \in E_{ij}$ there must exist a vertex x_{uv} in X adjacent to both u and v. Moreover, for any two edges $uv, st \in E_{ij}$ we have $x_{uv} \neq x_{st}$, since otherwise we obtain a triangle in $H[V_i \cup V_j]$ which is bipartite. Hence $p \geq |E_{ij}|$ and considering all pairs of colours in H we conclude that $p \geq M/\binom{\chi(H)}{2} \geq 2M/\chi^2(H)$.

A simple edge count shows that any graph with t edges can be coloured with no more than $1/2 + \sqrt{2t + 1/4}$ colours. Such a colouring can be found by a simple greedy algorithm [4]. We can apply this algorithm to H, and use additional p colours to colour the vertices of X. This way we obtain a colouring c of G^2 using at most $p + 1 + \sqrt{2M}$ colours. Using the lower bound from the previous paragraph one can prove the following inequalities (assuming $M \geq 6$ or $p \geq 17$).

$$\frac{p + 1 + \sqrt{2M}}{\chi(G^2)} \leq \frac{p + 1 + \sqrt{2M}}{p + \sqrt{\frac{2M}{p}}} \leq (2M)^{1/6} \leq N^{1/3} \leq n^{1/3}$$

Hence, the colouring c is an $\sqrt[3]{n}$-approximation of $\chi(G^2)$, and by Lemma 14 we can obtain a $\sqrt[3]{n}$-approximation of $\chi_i(G)$. $\square$

5 Exact algorithmic results

Now we focus on algorithms for injective colouring of chordal graphs. Although, computing the injective chromatic number of a chordal graph is hard, the associated decision problem with a fixed number of colours has a polynomial time solution, i.e., the problem is fixed parameter tractable. We need the following lemma.

Lemma 16. *For any chordal G, the treewidth of G^2 is at most $\frac{1}{4}\Delta(G)^2 + \Delta(G)$.* $\square$

Theorem 17. *Given a chordal graph G and a fixed integer k, one can decide in time $O(n \cdot k \cdot k^{(k/2+1)^2})$ whether $\chi_i(G) \leq k$ and also whether $\chi(G^2) \leq k$.*

Proof. It is easy to see that if $\chi_i(G) \leq k$ or if $\chi(G^2) \leq k$, then $\Delta(G)$ must be at most k. Thus if $\Delta(G) > k$, we can reject G immediately. Using Lemma 16, we can construct in time $O(nk^2)$ a tree decomposition (T, X) of G^2 whose width is at most $k^2/4 + k$. Now, using standard dynamic programming techniques on the tree T (cf. [4, 7]), we can decide in time $O(n \cdot k \cdot k^{(k/2+1)^2})$ whether $\chi(G^2) \leq k$ and whether $\chi_i(G) \leq k$. $\square$

Now we show that for certain subclasses of chordal graphs, the injective chromatic number can be computed in polynomial time (in contrast to Theorem 11). First, we summarise the results; the details are presented in subsequent sections.

We call a graph G a *power chordal* graph if all powers of G are chordal. Recall that in Propositions 8 and 9, we showed how, from the chromatic number of the square of the graph $G - \mathcal{B}(G)$, one can compute $\chi(G^2)$ for any graph G, respectively $\chi_i(G)$ for a split graph G. The following theorem describes a similar property for the injective chromatic number in chordal graphs. The proof will follow from Corollary 25 and Theorem 28 which we prove in sections 5.2 and 5.4 respectively.

Theorem 18. *There exists an $O(n + m)$ time algorithm that computes $\chi_i(G)$ given a chordal graph G and $\chi_i(G - \mathcal{B}(G))$. Using this algorithm one can also construct an optimal injective colouring of G from an optimal injective colouring of $G - \mathcal{B}(G)$ in time $O(n + m)$.*

A class $\mathcal{C}$ of graphs is called *induced-hereditary*, if $\mathcal{C}$ is closed under taking induced subgraphs. For an induced-hereditary subclasses of chordal graphs we have the following property.

Proposition 19. *Let $\mathcal{C}$ be an induced-hereditary subclass of chordal graphs. Then the following statements are equivalent.*

(i) One can efficiently compute $\chi(G^2)$ for any $G \in \mathcal{C}$.
(ii) One can efficiently compute $\chi_i(G - \mathcal{B}(G))$ for any $G \in \mathcal{C}$.
(iii) One can efficiently compute $\chi_i(G)$ for any $G \in \mathcal{C}$.

This follows from Theorem 18, Proposition 8, and the fact that each connected component of $G - \mathcal{B}(G)$ must be in $\mathcal{C}$. In some cases, e.g., in the class of power chordal graph, this is true even if $\mathcal{C}$ is not induced-hereditary. The following corollary will follow from Theorem 18 and Corollary 27 which we prove in section 5.3.

Corollary 20. *The injective chromatic number of a power chordal graph can be computed in polynomial time.*

Thus the injective chromatic number of a strongly chordal graph can also be computed in polynomial time.

Finally, observe that due to Theorem 12 one cannot expect the property from Proposition 19 to hold for any subclass of chordal graphs.

5.1 Injective structure

In order to prove Theorem 18, we investigate the structural properties of graphs G that allow efficient computation of $\chi_i(G)$. In this section, G refers to an arbitrary connected graph (not necessarily chordal).

A *clique separator* of G is a separator of G which induces a clique in G. A tree decomposition (T, X) of G is a *decomposition by clique separators*, if for any $uv \in E(T)$, the set $X(u) \cap X(v)$ induces a clique in G. This type of decomposition of graphs was introduced and studied by Tarjan [19]. The decomposition turns out to be particularly useful for the graph colouring problem; namely, one can efficiently construct an optimal colouring of G from optimal colourings of $G[X(u)]$ for all $u \in V(T)$. We define and study a similar concept for the injective colouring problem. Recall that $G^{(2)}$ denotes the common neighbour graph of G defined in section 3.

We say that a subset S of vertices of G is *injectively closed*, if for any two vertices $x, y \in S$ having a common neighbour in G, there exists a common neighbour of x and y that belongs to S. A subset S of vertices of G is called an *injective clique*, if S induces a clique in $G^{(2)}$. Note that an injective clique is not necessarily injectively closed in G. A subset of vertices S of G is called an *injective separator* of G, if S is injectively closed in G, S is a separator of $G^{(2)}$, and $G^{(2)}$ is connected. Note that $G^{(2)}$ can be disconnected even if G is connected, e.g., if G is bipartite. An *injective decomposition* of G is a tree decomposition (T, X) of G such that for any $uv \in E(T)$, the set $X(u) \cap X(v)$ is an injective separator of G. An injective separator S is an *injective clique separator*, if S is also an injective clique. An *injective clique decomposition* is an injective decomposition (T, X) such that for any $uv \in E(T)$, the set $X(u) \cap X(v)$ is an injective clique. Note that any injective clique decomposition of G is a decomposition of $G^{(2)}$ and G^2 by clique separators.

We have the following properties.

Lemma 21. *Let (T, X) be an injective decomposition of a graph G. Then for each $u \in V(T)$, the set $X(u)$ is injectively closed.* □

Theorem 22. *Let (T, X) be an injective clique decomposition of a graph G. Then*

$$\chi_i(G) = \chi(G^{(2)}) = \max_{u \in V(T)} \chi\Big(G^{(2)}[X(u)]\Big) = \max_{u \in V(T)} \chi_i\Big(G[X(u)]\Big).$$

Proof. The first equality is by definition. We obtain the second equality from the fact that (T, X) is a decomposition of $G^{(2)}$ by clique separators. The last equality follows easily, since by Lemma 21, we have $G^{(2)}[X(u)] = G[X(u)]^{(2)}$, and by definition $\chi(G[X(u)]^{(2)}) = \chi_i(G[X(u)])$. □

5.2 Computing $\chi_i(G)$ in chordal graphs

In this section, we focus on injective clique decompositions of chordal graphs. The following is easy to check.

Observation 23. *Let H be a bridgeless graph having a dominating vertex. Then H is an injective clique.* □

We say that a graph G is *decomposable*, if G contains an an injective clique separator S; we say that S *decomposes* G. A graph G is *indecomposable*, if it is not *decomposable*. A graph G is called *perfectly tree-dominated*, if G contains an induced tree T, such that any vertex and any connected component of $G - V(T)$ has exactly one neighbour in T. For such T, we say that T *perfectly dominates* G, or that G is *perfectly dominated* by T.

The following statement relates indecomposable chordal graphs and perfectly tree-dominated graphs.

Proposition 24. *Any perfectly tree-dominated graph is indecomposable. Any indecomposable chordal graph is either perfectly tree-dominated or bridgeless.* □

The property above has an important corollary.

Corollary 25. *For any chordal graph G, there exists an injective clique decomposition (T, X) of G, such that for any $u \in V(T)$, the set $X(u)$ induces either a bridgeless graph or a perfectly tree-dominated graph. This decomposition can be constructed in time $O(n + m)$.*

Proof. First, we find the bridges $\mathcal{B}(G)$ of G. Then, we construct a tree decomposition (T, X) of G such that for $u \in V(T)$, the set $X(u)$ is either a connected component of $G - \mathcal{B}(G)$, or a connected component T of $G[\mathcal{B}(G)]$ augmented with the neighbours of T in G. It follows from the proof of Proposition 24 that (T, X) is a injective clique decomposition. □

5.3 Bridgeless chordal graphs

In this section, we describe some classes of chordal graphs G that allow efficiently computing $\chi(G^2)$.

We focus on chordal graphs whose square is also a chordal graph. Clearly, for any such graph G, one can efficiently colour the square of G. Chordal graphs whose powers are also chordal were already studied in the past. In particular, it was shown by Duchet [16] that for any k, if G^k is chordal, then also G^{k+2} is chordal. Therefore, if a chordal graph G has a chordal square, then any power of G must be chordal, that is, G is power chordal. Interestingly, many known subclasses of chordal graphs, e.g. trees, interval graphs, and strongly chordal graphs, were shown to be power chordal [1]. Moreover, Laskar and Shier [16] found the following subgraph characterisation of power chordal graphs. A k-sun is a graph formed by a cycle $v_0, v_1, \ldots, v_{k-1}$ with edges $v_i v_{i+1}$ (and possibly other edges), and an independent set $w_0, w_1, \ldots w_{k-1}$, where w_i is adjacent only to v_i and v_{i+1} (all indices are taken modulo k). A k-sun of a graph G is *suspended* in G, if there exists a vertex z in G adjacent to w_i and w_j where $j \neq i$ and $j \neq i \pm 1$ modulo k.

Theorem 26. **[16]** *A graph G is power chordal if and only if any k-sun of G, $k \geq 4$, is suspended.*

Based on this characterisation, it is easy to check the following.

Corollary 27. *If G is power chordal, the graph $G - \mathcal{B}(G)$ is also power chordal.* $\square$

Note that by Theorem 26, strongly chordal graphs are trivially power chordal, since no strongly chordal graph can contain an induced k-sun, $k \geq 3$ [6]. Also notice, that the class of power chordal graphs is not induced-hereditary (closed under taking induced subgraphs), since a graph that contains a k-sun can be power chordal, but the k-sun itself (taken as an induced subgraph) is not.

5.4 Perfectly tree-dominated graphs

In this section, we show how to efficiently compute the injective chromatic number of a perfectly tree-dominated graph.

Let G be a perfectly tree-dominated graph. If G is a tree, then by Proposition 2, we have $\chi_i(G) = \Delta(G)$, and a greedy injective colouring of G will be optimal. Otherwise, let T be a minimal tree perfectly dominating G. We define a tree decomposition (T_G, X) of G as follows. We set $T_G = T$, and for $u \in V(T)$, we set $X(u) = N(u) \cup \{u\}$. Clearly, $X(u) \cap X(v) = \{u, v\}$ is injectively closed, and the set $X(u) \cap X(v)$ is a separator of $G^{(2)}$. Hence (T, X) is an injective decomposition of G. Note that for any $u \in V(T)$, the graph $G[X(u)]$ admits only $\deg(u)$ different injective colourings, up to renaming colours. It follows, that using dynamic programming on the rooted tree T, one can determine $\chi_i(G)$ and an optimal injective colouring of G, by computing, for each $u \in V(T)$ and each colouring of $G[X(u)]$, the minimum number of colours needed to injectively colour the subgraph of G induced on the union of $X(u)$ and the sets $X(v)$ for all descendants v of u. Using an additional simple argument it can be shown that the algorithm we just described can be performed in time $O(n + m)$. Hence we have the following theorem.

Theorem 28. *The injective chromatic number $\chi_i(G)$ and an optimal injective colouring of a perfectly tree-dominated graph G can be computed in time $O(n + m)$.*

The above algorithm turns out to be an instance of a more general approach to graph colouring problems [18].

Note added in proof

We have just learned of a related result of Král'[15] showing that $\chi(G^2) = O(\Delta(G)^{3/2})$ for any chordal G. This is easily seen to allow strengthening Theorem 15 from split to chordal graphs.

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