CSC321 Lecture 5
Learning in a Single Neuron

Roger Grosse and Nitish Srivastava

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But this raises some questions:

- What if the thing we're trying to predict isn't real-valued or binary-valued?
- What if we don't know the right features Φ?

This week, we cover a much more general learning framework which gets around both of these issues.

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For most of this course, we will instead write down an objective function and optimize it using a technique called gradient descent.

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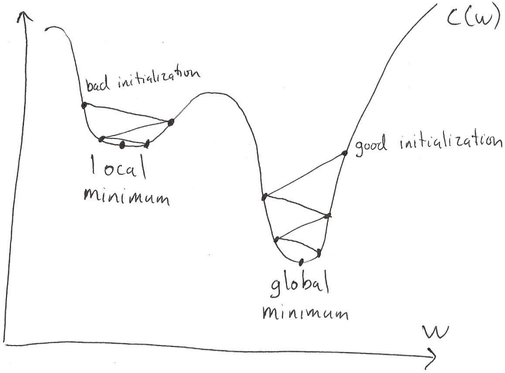
Synonyms : Loss Function/Error Function/Cost Function/Objective Function.

Learning $\equiv$ minimizing unhappiness.

Optimization

Visualizing gradient descent in one dimension: $w \leftarrow w - \epsilon \frac{\mathrm{d}C}{\mathrm{d}w}$



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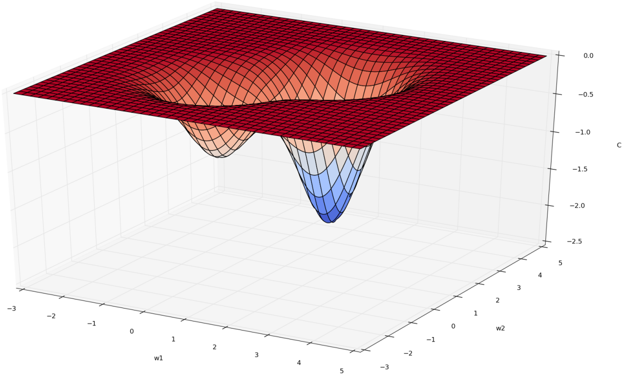


Optimization

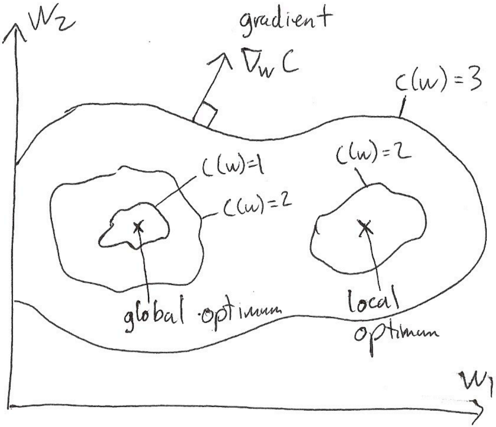
Visualizing it in two dimensions is a bit tricker.

- Level sets (or contours): sets of points on which $C(\mathbf{w})$ is constant
- Gradient: the vector of partial derivatives

$$\nabla_{\mathbf{w}} C = \left(\frac{\partial C}{\partial w_1}, \frac{\partial C}{\partial w_2} \right)$$

  - points in the direction of maximum increase
  - orthogonal to the level set
- The gradient descent updates are opposite the gradient direction.

Optimization



Question 1: Geometry of optimization

Suppose we have a linear regression problem with two training cases and no bias term:

- $\mathbf{x}^{(1)} = (1, 0)$, $t^{(1)} = 0.5$
- $\mathbf{x}^{(2)} = (0, 3)$, $t^{(2)} = 0$

Recall that the objective function is

$$C(\mathbf{w}) = \frac{1}{2}(\mathbf{w}^T\mathbf{x}^{(1)} - t^{(1)})^2 + \frac{1}{2}(\mathbf{w}^T\mathbf{x}^{(2)} - t^{(2)})^2.$$

1. In weight space, sketch the level set of this objective function corresponding to $C(\mathbf{w}) = 1/2$. **Draw both axes with the same scale.**
   - Hint: Write the equation $C(\mathbf{w}) = 1/2$ explicitly in the form

     $$\frac{(w_1 - c)^2}{a^2} + \frac{(w_2 - d)^2}{b^2} = 1.$$

     What geometric object is this, and what do (a, b, c, d) represent?
2. The point $\mathbf{w} = (1.25, 0.22)$ is on the level set for $C(\mathbf{w}) = 1/2$. Sketch the gradient $\nabla_{\mathbf{w}} C$ at this point.
3. Now sketch the gradient descent update if we use a learning rate of $1/2$.

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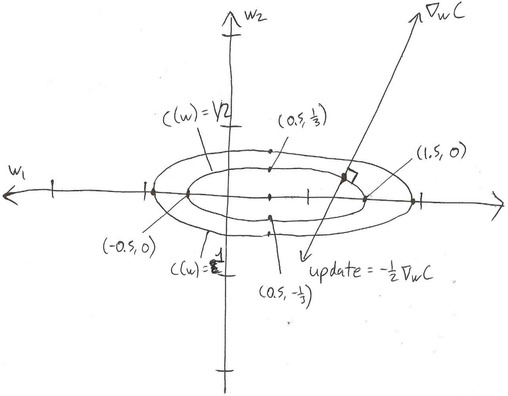

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Note: we chose the numbers for this problem so that the ellipse would be axis-aligned. In general, the level sets for linear regression will be ellipses, but they won't be axis-aligned.

- But all ellipses are rotations of axis-aligned ellipses, so the axis-aligned case gives us all the intuition we need.
- You may have learned how to draw non-axis-aligned ellipses in a linear algebra class. If not, don't worry about it.

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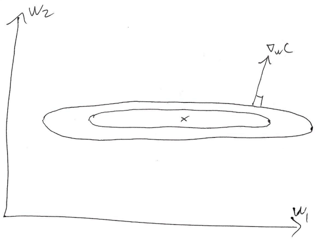

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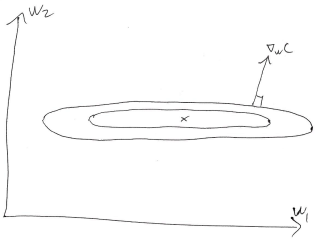

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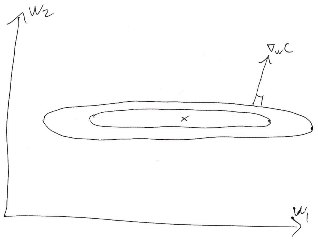

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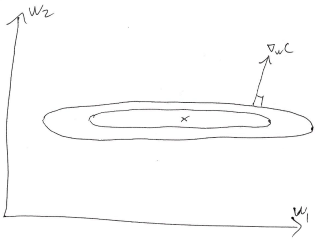

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Note: a more representative picture would show it MUCH more elongated!

Question 2: Computing the gradient

Now let's consider a non-linear neuron whose activation is computed as follows:

$$z = \mathbf{w}^T \mathbf{x} = \sum_j w_j x_j \qquad y = \log(1 + z^2)$$

We will use the cost function

$$C(\mathbf{w}) = |y - t|.$$

Show how to compute the partial derivative $\partial C / \partial w_1$ using the Chain Rule as follows:

1. Compute the derivative $\mathrm{d}C/\mathrm{d}y$. (You may assume $y \neq t$.)
2. Express the derivative $\mathrm{d}C/\mathrm{d}z$ in terms of $\mathrm{d}C/\mathrm{d}y$.
3. Express the partial derivative $\partial C / \partial w_1$ in terms of $\mathrm{d}C/\mathrm{d}z$

Question 2: Computing the gradient

Solution:

$$\frac{\mathrm{d}C}{\mathrm{d}y} = \begin{cases} 1 & \text{if } y > t \\ -1 & \text{if } y < t \end{cases}$$

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$$= \frac{2z}{1+z^2}\frac{\mathrm{d}C}{\mathrm{d}y}$$

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Note: If this were a calculus class, you'd do the substitutions to get $\partial C / \partial w_1$ explicitly. We won't do that here, since at this point you've already derived everything you need to implement the computation in Python.

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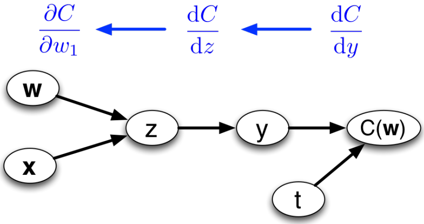
Observe that we compute derivatives going backwards through the computation graph:



This is true in general, not just for neural nets.

This is how we get the term "backpropagation."