CSC321 Lecture 4
The Perceptron Algorithm

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Recap: Perceptron Model

Inputs : $\mathbf{x}$.

Parameters : $\mathbf{w}$.

$$y = \begin{cases} 1 & \text{if } \mathbf{w}^T\mathbf{x} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

An example of a binary linear classifier.

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- Binary : Two possible classification decisions (0 or 1).
- Linear: $\mathbf{w}^T\mathbf{x}$.

Recap: Perceptron Learning Algorithm

$\mathbf{w} \leftarrow \mathbf{0}$

Repeat until all data points are classified correctly:

Choose a data point $\mathbf{x}$ with target t

Compute

$$y = \begin{cases} 1 & \text{if } \mathbf{w}^T\mathbf{x} \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

If $y \neq t$, then update

$$\mathbf{w} \leftarrow \mathbf{w} + (t - y)\mathbf{x}$$

Theoretical guarantee: if the data are linearly separable, it will make only a finite number of mistakes, then find a $\mathbf{w}$ which correctly classifies all training cases.

Note: after giving this lecture, we realized we've been inconsistent about what happens when an input lies on the decision boundary $\mathbf{w}^T\mathbf{x} = 0$. This isn't a case we want to emphasize in this course. We won't ask any exam or homework questions where inputs lie on the decision boundary. Sorry for the confusion.

Question 1: Perceptron example

Suppose we have the following data points, and no bias term:

- $\mathbf{x} = (1, -2), t = 1$
- $\mathbf{x} = (0, -1), t = 0$

The initial weight vector is $(0, -2)$.

- Draw the feasible regions in weight space.
  - Draw the axes in weight space w_1, w_2.
  - Draw each data point as a line that separates "good" and "bad" regions.
  - Shade the feasible region.
- Carry out the perceptron algorithm until you get a feasible solution.
  - It's easiest to do it on the plot you made. Here is the algorithm - Choose a data point $\mathbf{x}$ with target t
    Compute

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Question 2: Feature space

We're given a problem with a single input and no bias parameter:

- $x = -1, t = 1$
- $x = 1, t = 0$
- $x = 3, t = 1$

Sketch the data in input space. Is this dataset linearly separable?

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Design 2 basis functions (features) ϕ_1 and ϕ_2 such that

- $(\phi_1(-1), \phi_2(-1)), t = 1$
- $(\phi_1(1), \phi_2(1)), t = 0$
- $(\phi_1(3), \phi_2(3)), t = 1$

becomes linearly separable.

$$y = \begin{cases} 1 & \text{if } \mathbf{w}^T \Phi(x) \geq 0 \\ 0 & \text{otherwise} \end{cases}$$

Sketch the feature space - axes will be $\phi_1(x)$ and $\phi_2(x)$.

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Sketch the constraints in weight space.

Question 3: Linear regression in weight space

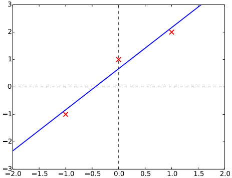
Recall that linear regression fits the model -

$$y = wx + b.$$

Suppose we're given the following training examples:

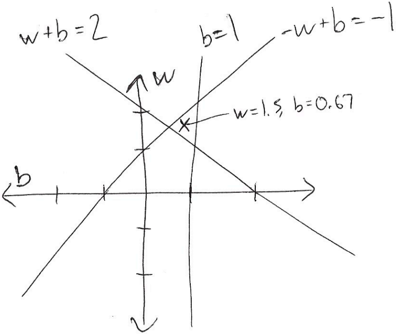
$$x = -1, t = -1 \qquad x = 0, t = 1 \qquad x = 1, t = 2$$

The optimal solution is (approximately) $w = 1.5, b = 0.67$.



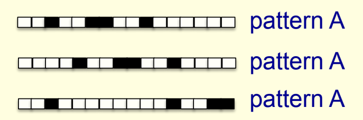
For each example, sketch the sets of points in weight space which predict each target exactly, and plot the optimal solution. (The axes are b and w.) What do you notice?

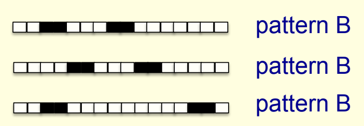
Question 3: Linear regression in weight space



What linear classifiers can't represent.

Recall that Geoff said perceptrons can't distinguish between two different binary patterns with wrap-around if they have the same number of nonzero entries.





Here's another way of looking at it.

- Show that if a linear classifier classifies all the inputs $\mathbf{x}^{(1)}, \ldots, \mathbf{x}^{(N)}$ the same, then it also classifies their average the same.
- What is the average input for patterns A and B?

Your questions from the quiz

- How to initialize weights and biases?
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- How to initialize weights and biases?
  - by default, initialize to 0; but this depends on the situation
- Perceptrons with something other than a binary threshold?
  - We will cover neural net models which make other types of predictions (and these are sometimes called perceptrons)

Thinking about high-dimensional spaces

Geoff says that to think about 14-D space, you should "think about 3-D space and say 14 really loudly." But some intuitions don't carry over:

- "Most" sets of D points in D dimensions are linearly separable.
- "Most" points (inside a hypercube, say) are about the same distance from each other.
- "Most" vectors are approximately orthogonal to each other.