MixtureModel

March 24, 2017

1 Tutorial: Mixture Model

Agenda:

1. Multivariate Gaussian
2. Maximum Likelihood estimation of the mean parameter
3. Bayesian estimation of the mean parameter
4. Expectation Maximization for multivariate Gaussian mixture

References:

- http://www.eecs.yorku.ca/course_archive/2012-13/F/4404-5327/lectures/03%20Multivariate%20Norma (Slides 6, 9, 41-42, 44-52)
- http://disi.unitn.it/~passerini/teaching/2010-2011/MachineLearning/slides/06_07_bayesian_learning/t
- http://stats.stackexchange.com/questions/28744/multivariate-normal-posterior
- http://cs.nyu.edu/~dsontag/courses/ml12/slides/lecture21.pdf
- http://statweb.stanford.edu/~tibs/stat315a/LECTURES/em.pdf (a little advanced)

```
In [2]: import matplotlib
        import numpy as np
        import matplotlib.pyplot as plt
        %matplotlib inline
```

1.1 1. Multivariate Gaussian

$$p(x;\mu,\Sigma) = \frac{1}{(2\pi)^{D/2}|\Sigma|^{1/2}} \exp\left(-\frac{1}{2}(x-\mu)^T\Sigma^{-1}(x-\mu)\right)$$

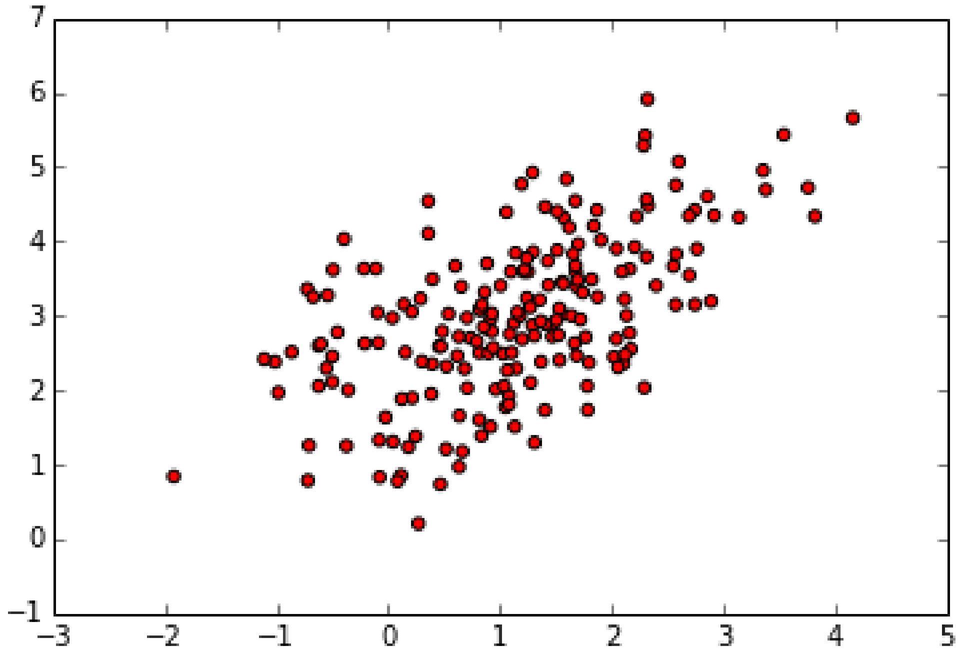
Compared to univariate Gaussian:

$$p(x;\mu,\sigma^2) = \frac{1}{\sqrt{2\pi}\sigma} \exp\left(-\frac{1}{2\sigma^2}(x-\mu)^2\right)$$

```
In [15]: mean1 = [1.0, 3.0]
         cov1 = [[1.0, 0.5],
                 [0.5, 1.0]]
         data1 = np.random.multivariate_normal(mean1, cov1, size=200)
```

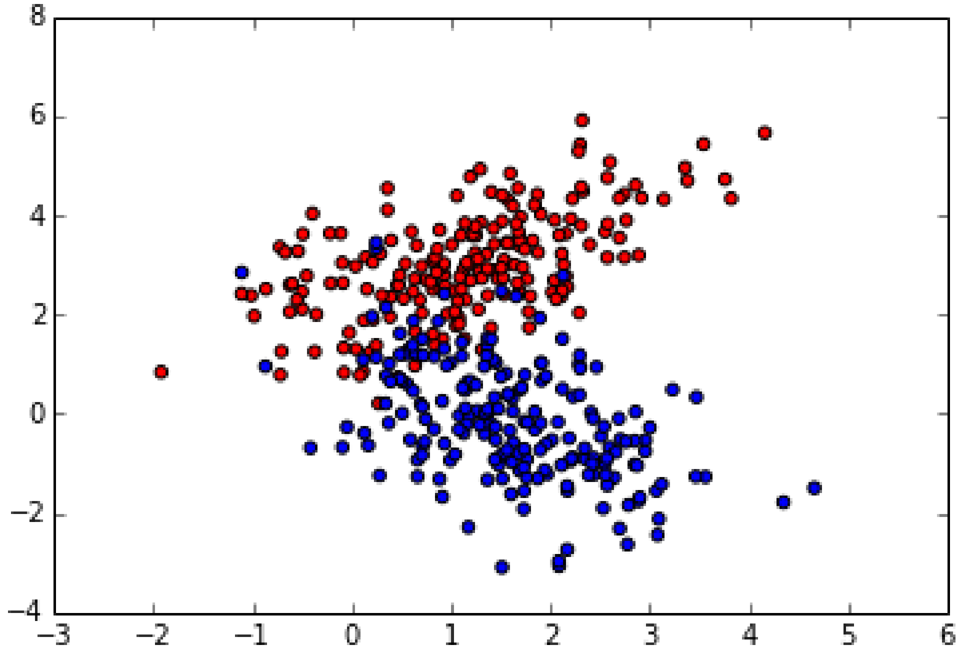
```
In [16]: plt.scatter(data1[:,0], data1[:,1],c='r')
```

Out[16]: <matplotlib.collections.PathCollection at 0x105b0a7d0>




```
In [17]: mean2 = [1.5, 0.0]
         cov2 = [[ 1.0, -0.5],
                 [-0.5,  1.5]]
         data2 = np.random.multivariate_normal(mean2, cov2, size=200)

In [18]: plt.scatter(data1[:,0], data1[:,1], c='r')
         plt.scatter(data2[:,0], data2[:,1], c='b')
```

Out[18]: <matplotlib.collections.PathCollection at 0x105c3c310>



1.2 2. Maximum Likelihood for Mean Parameter

$$\log p(x_1, ..., x_N|\mu, \Sigma) = -\frac{ND}{2}log(2\pi) - \frac{N}{2}\log|\Sigma| - \frac{1}{2}\sum_n (x_n - \mu)^T \Sigma^{-1}(x_n - \mu)$$

Take derivative with respect to μ to get:

$$\frac{\partial}{\partial \mu}\log p = \sum_n \Sigma^{-1}(x_n - \mu)$$

So

$$\mu_{ML} = \frac{1}{N}\sum_n x_n$$

Just like in the univariate case.

Switch to: http://www.eecs.yorku.ca/course_archive/2012-13/F/4404-5327/lectures/03%20Multivariate%20Normal%20Distribution.pdf

1.3 3. Bayesian Estimatiof Mean Parameter

From: http://stats.stackexchange.com/questions/28744/multivariate-normal-posterior

By Bayes's rule the posterior distribution looks like:

$$p(\mu|\{\mathbf{x}_i\}) \propto p(\mu)\prod_{i=1}^{N} p(\mathbf{x}_i|\mu)$$

So:

$$\ln p(\mu|\{\mathbf{x}_i\}) = -\frac{1}{2}\sum_{i=1}^{N}(\mathbf{x}_i-\mu)'\mathbf{\Sigma}^{-1}(\mathbf{x}_i-\mu) - \frac{1}{2}(\mu-\mu_0)'\mathbf{\Sigma}_0^{-1}(\mu-\mu_0) + const$$

$$= -\frac{1}{2}N\mu'\mathbf{\Sigma}^{-1}\mu + \sum_{i=1}^{N}\mu'\mathbf{\Sigma}^{-1}\mathbf{x}_i - \frac{1}{2}\mu'\mathbf{\Sigma}_0^{-1}\mu + \mu'\mathbf{\Sigma}_0^{-1}\mu_0 + const$$

$$= -\frac{1}{2}\mu'(N\mathbf{\Sigma}^{-1}+\mathbf{\Sigma}_0^{-1})\mu + \mu'(\mathbf{\Sigma}_0^{-1}\mu_0 + \mathbf{\Sigma}^{-1}\sum_{i=1}^{N}\mathbf{x}_i) + const$$

$$= -\frac{1}{2}(\mu-(N\mathbf{\Sigma}^{-1}+\mathbf{\Sigma}_0^{-1})^{-1}(\mathbf{\Sigma}_0^{-1}\mu_0+\mathbf{\Sigma}^{-1}\sum_{i=1}^{N}\mathbf{x}_i))'(N\mathbf{\Sigma}^{-1}+\mathbf{\Sigma}_0^{-1})(\mu-(N\mathbf{\Sigma}^{-1}+\mathbf{\Sigma}_0^{-1})^{-1}(\mathbf{\Sigma}_0^{-1}\mu_0+\mathbf{\Sigma}^{-1}\sum_{i=1}^{N}\mathbf{x}_i))+const$$

Which is the log density of a Gaussian:

$$\mu|\{\mathbf{x}_i\} \sim N((N\mathbf{\Sigma}^{-1}+\mathbf{\Sigma}_0^{-1})^{-1}(\mathbf{\Sigma}_0^{-1}\mu_0+\mathbf{\Sigma}^{-1}\sum_{i=1}^{N}\mathbf{x}_i),(N\mathbf{\Sigma}^{-1}+\mathbf{\Sigma}_0^{-1})^{-1})$$

Using the Woodbury identity on our expression for the covariance matrix:

$$(N\mathbf{\Sigma}^{-1}+\mathbf{\Sigma}_0^{-1})^{-1} = \mathbf{\Sigma}(\frac{1}{N}\mathbf{\Sigma}+\mathbf{\Sigma}_0)^{-1}\frac{1}{N}\mathbf{\Sigma}_0$$

Which provides the covariance matrix in the desired form. Using this expression (and its symmetry) further in the expression for the mean we have:

$$\mathbf{\Sigma}(\frac{1}{N}\mathbf{\Sigma}+\mathbf{\Sigma}_0)^{-1}\frac{1}{N}\mathbf{\Sigma}_0\mathbf{\Sigma}_0^{-1}\mu_0 + \frac{1}{N}\mathbf{\Sigma}_0(\frac{1}{N}\mathbf{\Sigma}+\mathbf{\Sigma}_0)^{-1}\mathbf{\Sigma}\mathbf{\Sigma}^{-1}\sum_{i=1}^{N}\mathbf{x}_i$$

$$= \mathbf{\Sigma}(\frac{1}{N}\mathbf{\Sigma}+\mathbf{\Sigma}_0)^{-1}\frac{1}{N}\mu_0 + \mathbf{\Sigma}_0(\frac{1}{N}\mathbf{\Sigma}+\mathbf{\Sigma}_0)^{-1}\sum_{i=1}^{N}(\frac{1}{N}\mathbf{x}_i)$$

Which is the form required for the mean.

$$\begin{aligned}
\mu_{\mathbf{n}} &= \Sigma_0\left(\Sigma_0+\frac{1}{n}\Sigma\right)^{-1}\left(\frac{1}{n}\sum_{i=1}^{n}\mathbf{x_i}\right) + \frac{1}{n}\Sigma\left(\Sigma_0+\frac{1}{n}\Sigma\right)^{-1}\mu_0 \\
\mathbf{\Sigma_n} &= \Sigma_0\left(\Sigma_0+\frac{1}{n}\Sigma\right)^{-1}\frac{1}{n}\Sigma
\end{aligned}$$

1.4 4. Expectation Maximization

Fit a mixture (of Gaussians) by alternating between two steps:

```
* Expectation: Compute posterior expectations of latent variable z
* Maximization: Solve ML parameters given full set of x's and z's
```

http://www.cs.toronto.edu/~rgrosse/courses/csc321_2017/slides/lec18.pdf

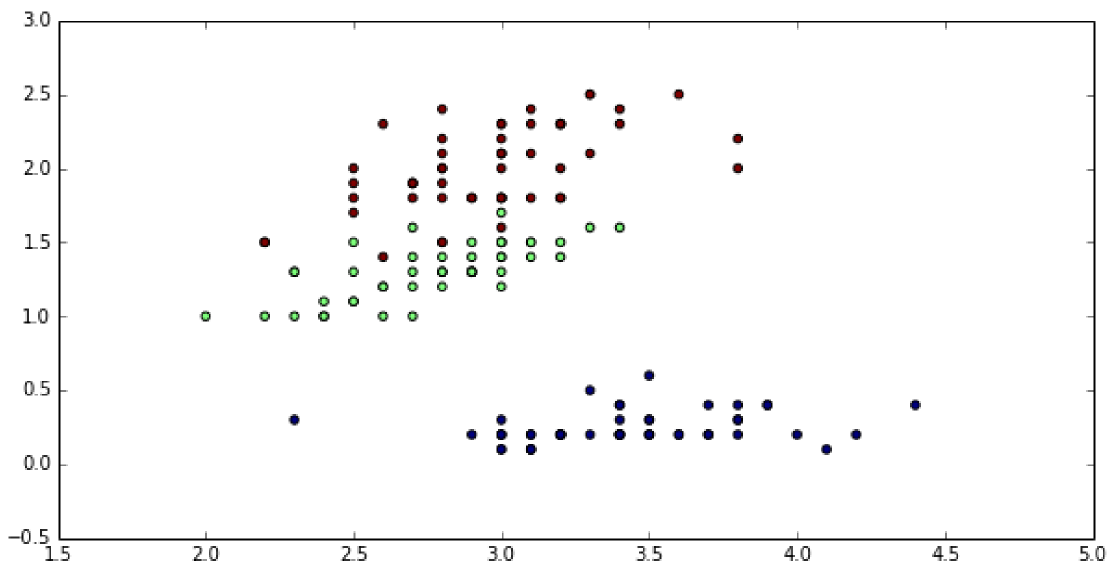
```
In [19]: from sklearn import datasets
         iris = datasets.load_iris()
```

```
In [29]: #print iris['DESCR'] # commented to prevent cutoff
```

```
In [21]: iris_data = iris['data'][:,[1,3]]
         target = iris['target']
```

```
In [23]: plt.figure(figsize=(10,5))
         plt.scatter(iris_data[:,0], iris_data[:, 1], c=target)
```

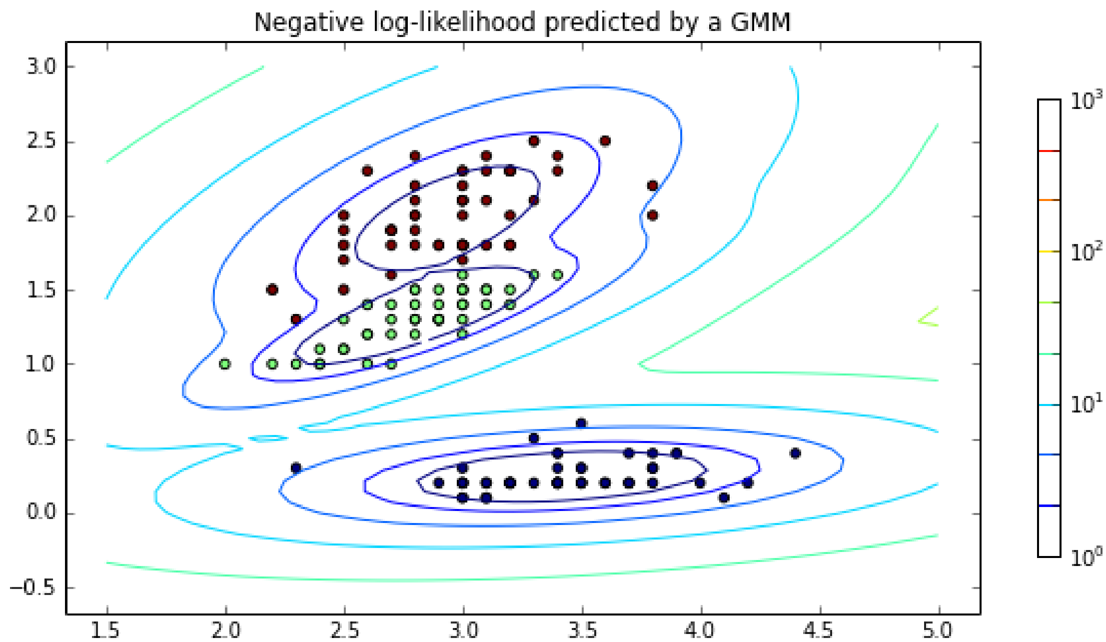
```
Out[23]: <matplotlib.collections.PathCollection at 0x10770d0d0>
```



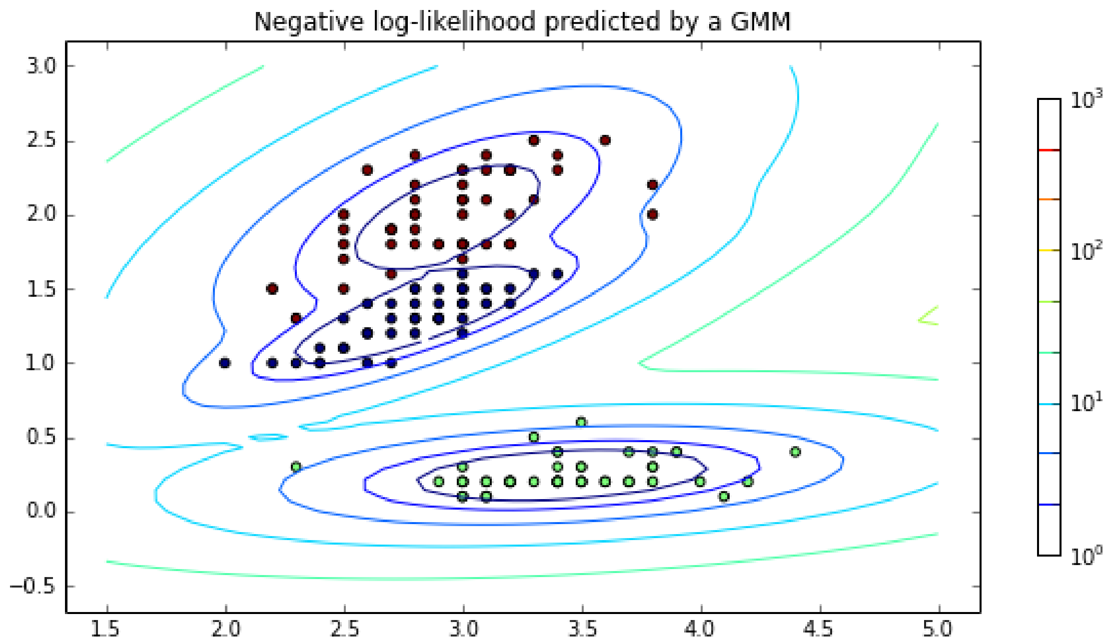

```
In [24]: from matplotlib.colors import LogNorm
         from sklearn import mixture

         def plot_clf(clf, input_data, max_iter=0):
             # display predicted scores by the model as a contour plot
             x = np.linspace(1.5, 5.0)
             y = np.linspace(-0.5, 3.0)
             X, Y = np.meshgrid(x, y)
             XX = np.array([X.ravel(), Y.ravel()]).T
             Z = -clf.score_samples(XX)
             Z = Z.reshape(X.shape)

             plt.figure(figsize=(10,5))
             CS = plt.contour(X, Y, Z, norm=LogNorm(vmin=1.0, vmax=1000.0),
                                  levels=np.logspace(0, 3, 10))
             CB = plt.colorbar(CS, shrink=0.8, extend='both')
```

```
    plt.scatter(input_data[:, 0], input_data[:, 1], c=clf.predict(input_da
    plt.title('Negative log-likelihood predicted by a GMM')
    plt.axis('tight')
```

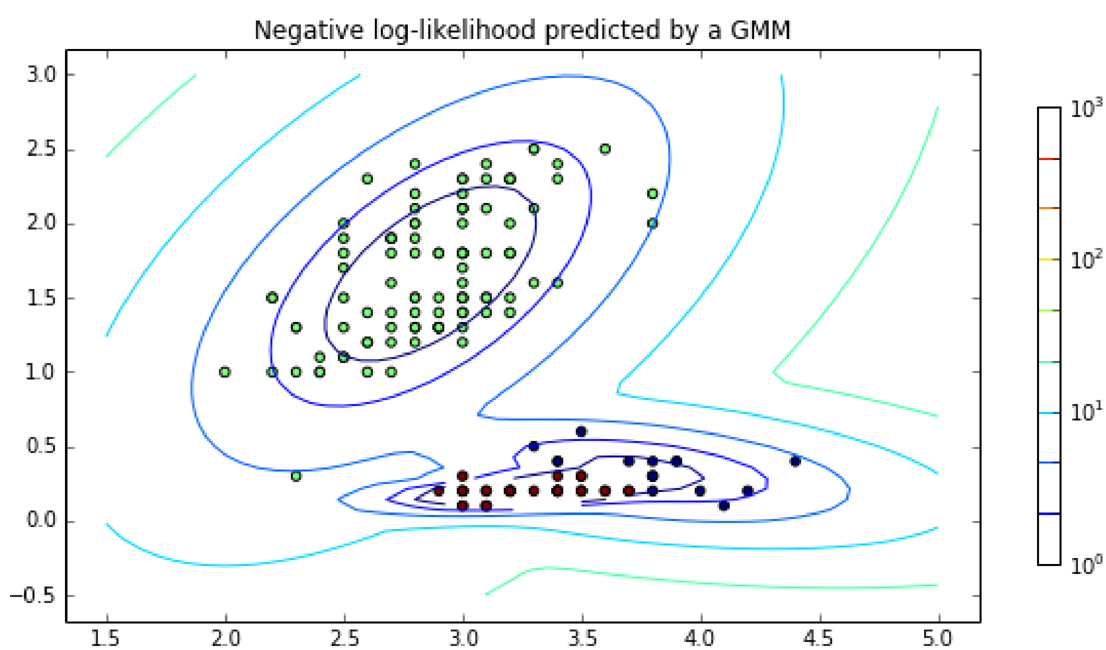
In [30]:
```
# fit a Gaussian Mixture Model with two components
clf = mixture.GaussianMixture(n_components=3, covariance_type='full')
clf.fit(iris_data)
plot_clf(clf, iris_data)
```



In [27]:
```
# Run again: same result or different?
clf = mixture.GaussianMixture(n_components=3, covariance_type='full')
clf.fit(iris_data)
plot_clf(clf, iris_data)
```




```
In [28]: # Example of a degenerate case
         np.random.seed(2)
         clf = mixture.GaussianMixture(n_components=3, covariance_type='full')
         clf.fit(iris_data)
         plot_clf(clf, iris_data)
```



Expectation Maximization Demo:
https://www.youtube.com/watch?v=v-pq8VCQk4M