Tutorial 1. Linear Regression

January 11, 2017

1 Tutorial: Linear Regression

Agenda: 1. Spyder interface 2. Linear regression running example: boston data 3. Vectorize cost function 4. Closed form solution 5. Gradient descent

```
In [1]: import matplotlib
        import numpy as np
        import matplotlib.pyplot as plt
        %matplotlib inline

In [2]: from sklearn.datasets import load_boston
        boston_data = load_boston()
        print(boston_data['DESCR'])
```

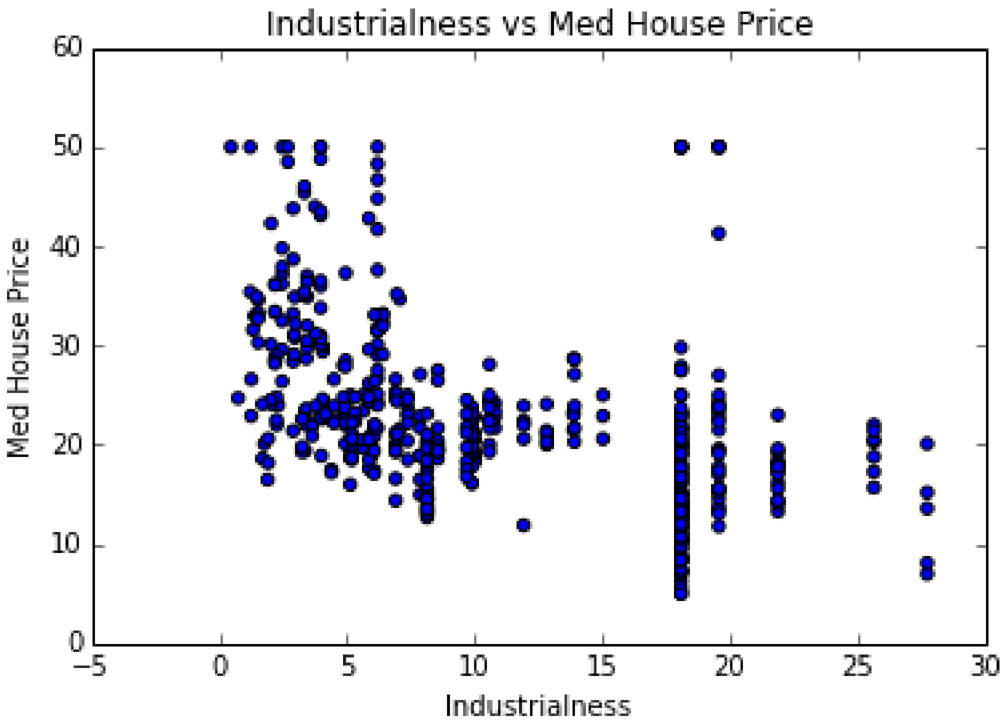
```
Boston House Prices dataset

Notes
------
Data Set Characteristics:

    :Number of Instances: 506

    :Number of Attributes: 13 numeric/categorical predictive

    :Median Value (attribute 14) is usually the target

    :Attribute Information (in order):
        - CRIM     per capita crime rate by town
        - ZN       proportion of residential land zoned for lots over 25,000 sq.ft.
        - INDUS    proportion of non-retail business acres per town
        - CHAS     Charles River dummy variable (= 1 if tract bounds river; 0 other
        - NOX      nitric oxides concentration (parts per 10 million)
        - RM       average number of rooms per dwelling
        - AGE      proportion of owner-occupied units built prior to 1940
        - DIS      weighted distances to five Boston employment centres
        - RAD      index of accessibility to radial highways
        - TAX      full-value property-tax rate per $10,000
```

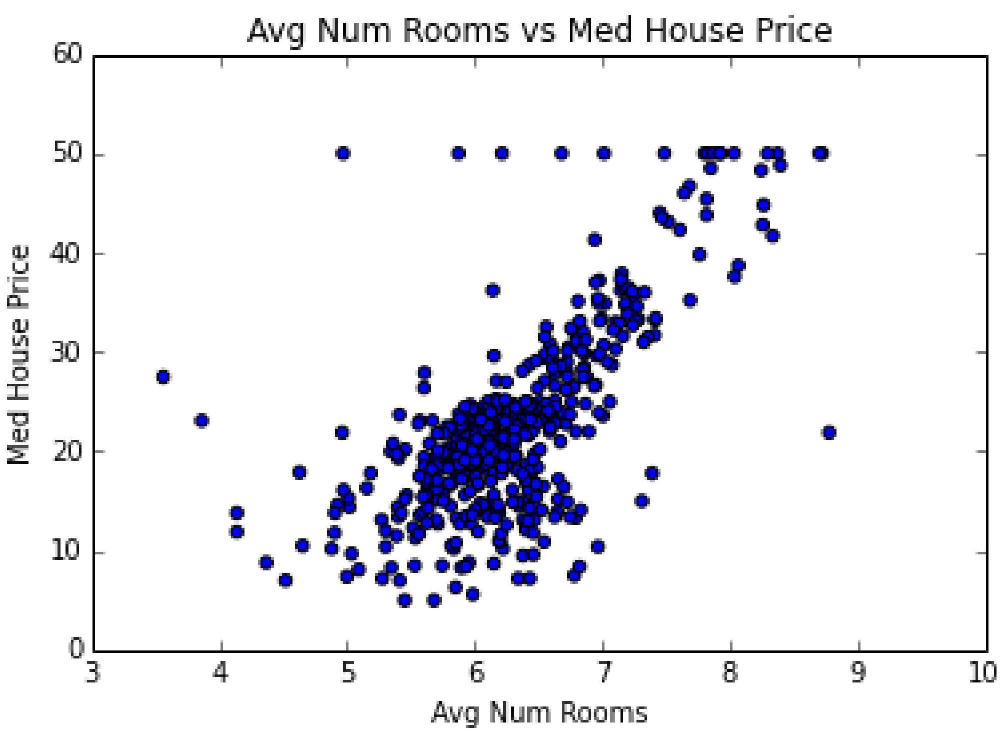
```
        - PTRATIO  pupil-teacher ratio by town
        - B        1000(Bk - 0.63)^2 where Bk is the proportion of blacks by town
        - LSTAT    % lower status of the population
        - MEDV     Median value of owner-occupied homes in $1000's

    :Missing Attribute Values: None

    :Creator: Harrison, D. and Rubinfeld, D.L.

This is a copy of UCI ML housing dataset.
http://archive.ics.uci.edu/ml/datasets/Housing


This dataset was taken from the StatLib library which is maintained at Carnegie Mel

The Boston house-price data of Harrison, D. and Rubinfeld, D.L. 'Hedonic
prices and the demand for clean air', J. Environ. Economics & Management,
vol.5, 81-102, 1978.   Used in Belsley, Kuh & Welsch, 'Regression diagnostics
...', Wiley, 1980.   N.B. Various transformations are used in the table on
pages 244-261 of the latter.

The Boston house-price data has been used in many machine learning papers that addr
problems.

**References**

   - Belsley, Kuh & Welsch, 'Regression diagnostics: Identifying Influential Data a
   - Quinlan,R. (1993). Combining Instance-Based and Model-Based Learning. In Proce
   - many more! (see http://archive.ics.uci.edu/ml/datasets/Housing)
```

In [3]:
```
# take the boston data
data = boston_data['data']
# we will only work with two of the features: INDUS and RM
x_input = data[:, [2,5]]
y_target = boston_data['target']
```

In [5]:
```
# Individual plots for the two features:
plt.title('Industrialness vs Med House Price')
plt.scatter(x_input[:, 0], y_target)
plt.xlabel('Industrialness')
plt.ylabel('Med House Price')
plt.show()

plt.title('Avg Num Rooms vs Med House Price')
plt.scatter(x_input[:, 1], y_target)
plt.xlabel('Avg Num Rooms')
```

```
plt.ylabel('Med House Price')
plt.show()
```





1.1 Define cost function

$$\mathcal{E}(y,t) = \frac{1}{2N}\sum_{i=1}^{N}(y^{(i)} - t^{(i)})^2$$

$$\mathcal{E}(y,t) = \frac{1}{2N}\sum_{i=1}^{N}(w_1 x_1^{(i)} + w_2 x_2^{(i)} + b - t^{(i)})^2$$

```
In [6]: def cost(w1, w2, b, X, t):
            '''
            Evaluate the cost function in a non-vectorized manner for
            inputs `X` and targets `t`, at weights `w1`, `w2` and `b`.
            '''
            costs = 0
            for i in range(len(t)):
                y_i = w1 * X[i, 0] + w2 * X[i, 1] + b
                t_i = t[i]
                costs += 0.5 * (y_i - t_i) ** 2
            return costs / len(t)
```

```
In [7]: cost(3, 5, 20, x_input, y_target)
```

```
Out[7]: 2241.1239166749006
```

```
In [8]: cost(3, 5, 0, x_input, y_target)
```

```
Out[8]: 1195.1098850543478
```

1.2 Vectorizing the cost function:

$$\mathcal{E}(y,t) = \frac{1}{2N}\|\mathbf{Xw} + \mathbf{b1} - \mathbf{t}\|^{\mathbf{2}}$$

```
In [9]: def cost_vectorized(w1, w2, b, X, t):
            '''
            Evaluate the cost function in a vectorized manner for
            inputs `X` and targets `t`, at weights `w1`, `w2` and `b`.
            '''
            N = len(y_target)
            w = np.array([w1, w2])
            y = np.dot(X, w) + b * np.ones(N)
            return np.sum((y - t)**2) / (2.0 * N)
```

```
In [10]: cost_vectorized(3, 5, 20, x_input, y_target)
```

```
Out[10]: 2241.1239166749015
```

```
In [11]: cost(3, 5, 0, x_input, y_target)
```

```
Out[11]: 1195.1098850543478
```

1.3 Comparing speed of the vectorized vs unvectorized code

We'll see below that the vectorized code already runs ~2x faster than the non-vectorized code!
Hopefully this will convince you to always vectorized your code whenever possible

```
In [12]: import time

         t0 = time.time()
         print cost(4, 5, 20, x_input, y_target)
         t1 = time.time()
         print t1 - t0
```

```
3182.40634167
0.00229597091675
```

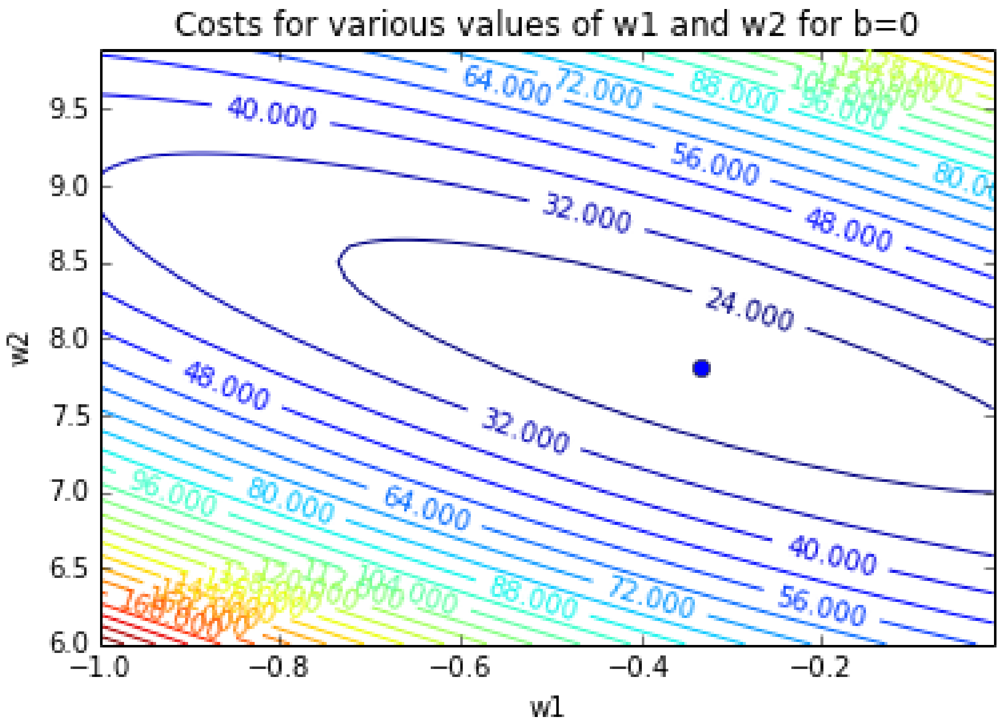
```
In [13]: t0 = time.time()
         print cost_vectorized(4, 5, 20, x_input, y_target)
         t1 = time.time()
         print t1 - t0
```

```
3182.40634167
0.000537872314453
```

1.4 Plotting cost in weight space

We'll plot the cost for two of our weights, assuming that bias = -22.89831573.
We'll see where that number comes from later.
Notice the shape of the contours are ovals.

```
In [15]: w1s = np.arange(-1.0, 0.0, 0.01)
         w2s = np.arange(6.0, 10.0, 0.1)
         z_cost = []
         for w2 in w2s:
             z_cost.append([cost_vectorized(w1, w2, -22.89831573, x_input, y_target
         z_cost = np.array(z_cost)
         np.shape(z_cost)
         W1, W2 = np.meshgrid(w1s, w2s)
         CS = plt.contour(W1, W2, z_cost, 25)
         plt.clabel(CS, inline=1, fontsize=10)
         plt.title('Costs for various values of w1 and w2 for b=0')
         plt.xlabel("w1")
         plt.ylabel("w2")
         plt.plot([-0.33471389], [7.82205511], 'o') # this will be the minima that
         plt.show()
```



2 Exact Solution

Work this out on the board:

1. ignore biases (add an extra feature & weight instead)
2. get equations from partial derivative
3. vectorize
4. write code.

```
In [16]: # add an extra feature (column in the input) that are just all ones
         x_in = np.concatenate([x_input, np.ones([np.shape(x_input)[0], 1])], axis=
         x_in
```

```
Out[16]: array([[  2.31 ,   6.575,   1.   ],
                [  7.07 ,   6.421,   1.   ],
                [  7.07 ,   7.185,   1.   ],
                ...,
                [ 11.93 ,   6.976,   1.   ],
                [ 11.93 ,   6.794,   1.   ],
                [ 11.93 ,   6.03 ,   1.   ]])
```

```
In [17]: def solve_exactly(X, t):
             '''
```

```
        Solve linear regression exactly. (fully vectorized)

        Given `X` - NxD matrix of inputs
              `t` - target outputs
        Returns the optimal weights as a D-dimensional vector
        '''
        N, D = np.shape(X)
        A = np.matmul(X.T, X)
        c = np.dot(X.T, t)
        return np.matmul(np.linalg.inv(A), c)
```

In [18]:
```
solve_exactly(x_in, y_target)
```

Out[18]:
```
array([ -0.33471389,   7.82205511, -22.89831573])
```

In [19]:
```
# In real life we don't want to code it directly
np.linalg.lstsq(x_in, y_target)
```

Out[19]:
```
(array([ -0.33471389,   7.82205511, -22.89831573]),
 array([ 19807.614505]),
 3,
 array([ 318.75354429,   75.21961717,    2.10127199]))
```

2.1 Implement Gradient Function

$$\frac{\partial \mathcal{E}}{\partial w_j} = \frac{1}{N} \sum_i x_j^{(i)} (y^{(i)} - t^{(i)})$$

In [20]:
```
# Vectorized gradient function
def gradfn(weights, X, t):
    '''
    Given `weights` - a current "Guess" of what our weights should be
          `X` - matrix of shape (N,D) of input features
          `t` - target y values
    Return gradient of each weight evaluated at the current value
    '''
    N, D = np.shape(X)
    y_pred = np.matmul(X, weights)
    error = y_pred - t
    return np.matmul(np.transpose(x_in), error) / float(N)
```

In [23]:
```
def solve_via_gradient_descent(X, t, print_every=5000,
                               niter=100000, alpha=0.005):
    '''
    Given `X` - matrix of shape (N,D) of input features
          `t` - target y values
    Solves for linear regression weights.
    Return weights after `niter` iterations.
    '''
```

```
        N, D = np.shape(X)
        # initialize all the weights to zeros
        w = np.zeros([D])
        for k in range(niter):
            dw = gradfn(w, X, t)
            w = w - alpha*dw
            if k % print_every == 0:
                print 'Weight after %d iteration: %s' % (k, str(w))
        return w
```

In [24]:
```
solve_via_gradient_descent( X=x_in, t=y_target)
```

```
Weight after 0 iteration: [ 1.10241186  0.73047508  0.11266403]
Weight after 5000 iteration: [-0.48304613  5.10076868 -3.97899253]
Weight after 10000 iteration: [-0.45397323  5.63413678 -7.6871518 ]
Weight after 15000 iteration: [ -0.43059857   6.06296553 -10.66851736]
Weight after 20000 iteration: [ -0.41180532   6.40774447 -13.06553969]
Weight after 25000 iteration: [ -0.39669551   6.68494726 -14.9927492 ]
Weight after 30000 iteration: [ -0.38454721   6.90781871 -16.54222851]
Weight after 35000 iteration: [ -0.37477995   7.08700769 -17.78801217]
Weight after 40000 iteration: [ -0.36692706   7.23107589 -18.78962409]
Weight after 45000 iteration: [ -0.36061333   7.34690694 -19.59492155]
Weight after 50000 iteration: [ -0.35553708   7.44003528 -20.24238191]
Weight after 55000 iteration: [ -0.35145576   7.5149106  -20.762941  ]
Weight after 60000 iteration: [ -0.34817438   7.57511047 -21.18147127]
Weight after 65000 iteration: [ -0.34553614   7.62351125 -21.51797024]
Weight after 70000 iteration: [ -0.343415     7.66242555 -21.78851591]
Weight after 75000 iteration: [ -0.34170959   7.69371271 -22.00603503]
Weight after 80000 iteration: [ -0.34033844   7.71886763 -22.18092072]
Weight after 85000 iteration: [ -0.33923604   7.73909222 -22.32152908]
Weight after 90000 iteration: [ -0.3383497    7.75535283 -22.4345784 ]
Weight after 95000 iteration: [ -0.33763709   7.76842638 -22.52547023]
```

Out[24]:
```
array([ -0.33706425,   7.77893565, -22.59853432])
```

In [25]:
```
# For comparison, this was the exact result:
np.linalg.lstsq(x_in, y_target)
```

Out[25]:
```
(array([ -0.33471389,   7.82205511, -22.89831573]),
 array([ 19807.614505]),
 3,
 array([ 318.75354429,   75.21961717,    2.10127199]))
```