STA 437/1005, Fall 2009 — Assignment #1

Due on October 8, during office hours, or you can slip it under my door (SS6016A) if I'm not there. You might want to hand it in during the lecture on October 5, however. Please hand it in on 8 1/2 by 11 inch paper, stapled in the upper-left corner, without any folder or other packaging around it. If really necessary, you can submit it by email (to radford@stat.utoronto.ca*), but please do this only if you can't easily hand in a paper copy.*

This assignment is worth 6% of the course grade. It is to be done by each student individually. You may discuss this assignment in general terms with other students, but the work you hand in should be your own. In particular, you should not leave any discussion of this assignment with any written notes or other recordings, nor receive any written or other material from anyone else by other means such as email.

For all questions, show both the final answer and how you obtained it.

Question 1: Let U_1, U_2, and U_3 be independent random variables, all with the normal distribution with mean zero and standard deviation one. We do not directly observe these random variables, however. Instead, we observe X_1, X_2, and X_3, which are related to U_1, U_2, and U_3 as follows:

$$\begin{aligned} X_1 &= U_1 + U_2 + \epsilon_1 \\ X_2 &= U_2 + U_3 + \epsilon_2 \\ X_3 &= U_3 + U_1 + \epsilon_3 \end{aligned}$$

where ϵ_1, ϵ_2, and ϵ_3 are independent random variables having the normal distribution with mean zero and standard deviation two.

a) Find the mean vector, μ, and covariance matrix, Σ, for $[X_1,\, X_2,\, X_3]'$.

b) Find Σ^{-1}. Hint: From symmetry, the diagonal elements of Σ^{-1} must all be the same, and also the off-diagonal elements must all be the same.

c) Find the conditional distribution of X_3 given $X_1 = x_1$ and $X_2 = x_2$.

Question 2: Recall the spectral decomposition theorem: If A is a $k \times k$ symmetric real matrix, it is possible to find a set of k eigenvectors of A that are orthogonal and have length one, and if $e_1, \ldots, e_k$ are any such set of eigenvectors, with eigenvalues $\lambda_1, \ldots, \lambda_k$, then $A = \lambda_1 e_1 e_1' + \cdots + \lambda_k e_k e_k'$.

a) Use the Spectral Decomposition theorem to prove that the trace of a symmetric real matrix (the sum of its diagonal elements) is equal to the sum of its eigenvalues (with each eigenvalue appearing in the sum as many times as there are orthogonal eigenvectors that are associated with that eigenvalues).

b) Let Q be any orthogonal matrix (for which $QQ' = I$), and let Σ_X be the covariance matrix of the random vector X. Define $Y = QX$, and let Σ_Y be the covariance matrix of Y. Prove that the trace of Σ_X is equal to the trace of Σ_Y.

Question 3: The following experiment is performed to estimate the average fuel efficiency of a particular make and model of car, expressed as litres of gas used per 100 kilometres of travel (with some standard for driving conditions). For the purpose of this question, assume that the fuel used per kilometre is always the same, for any particular car (ie, it does not vary during a trip).

For the experiment, 10 new cars of the given make and model were randomly chosen from among those produced. Each car was then driven for 100 kilometres, starting with a full tank of gas. After car j (where $j = 1, \ldots, 10$) had driven 10 kilometers, its gas tank was refilled, with the quantity of gas needed being measured. Call the value of this measurement X_{j1}. Car j was then driven for for the remaining 90 kilometers, and its gas tank was again refilled, with the quantity of gas needed again being measured. Call the value of this second measurement X_{j2}.

Suppose that it is known that the fuel efficiency of cars of a given make and model varies randomly, according to a normal distribution with standard deviation 0.5. The mean, μ, of this distribution is the average fuel efficiency that we wish to estimate. Suppose also that when a gas tank is refilled, the error in measuring how much gas is required to refill the tank is normally distributed with mean zero and standard deviation 0.1. Assume that such measurement errors are all independent.

a) Find the mean vector and covariance matrix of $[X_{j1} \;\; X_{j2}]'$.

b) Find the mean vector and covariance matrix for the vector of sample means, $[\overline{X}_1 \;\; \overline{X}_2]'$.

c) Consider estimating μ by a linear combination of $\overline{X}_1$ and $\overline{X}_2$, of the form

$$a\overline{X}_1 \;+\; b\overline{X}_2$$

Find the values of a and b that make this estimate be unbiased and have the smallest variance of all unbiased estimates of this form.

For all the questions above, you should find actual values (as fractions or decimal numbers), not just formulas that could be used to compute the numbers.