CSC 310, Fall 2011 — Theory Assignment #3

Due in class on December 5. Worth 7% of the course grade.

Note that this assignment is to be done by each student individually. You may discuss it in general terms with other students, but the work you hand in should be your own.

Question 1 (35 marks): Consider a channel for which the input alphabet and output alphabet are both $\{0, 1, 2\}$, and for which the channel transition probabilities are given by $Q_{0|0} = Q_{2|2} = 1$, $Q_{0|1} = Q_{2|1} = 1/4$, and $Q_{1|1} = 1/2$.

a) Find the mutual information between the channel input and output if the input probabilities are $p_0 = 1/2$, $p_1 = 0$, and $p_2 = 1/2$.

b) Find the mutual information between the channel input and output if the input probabilities are $p_0 = p_1 = p_2 = 1/3$.

c) Find the capacity of this channel, and an input distribution that achieves this capacity. Once you have reduced this to a one-dimensional optimization problem, you may use a numerical method to find the solution, written in any language you choose. Hand in the command or program that you use, which may be quite short if you're using something like Maple. Giving the capacity in bits to three decimal places is sufficient, and a brute force numerical search is acceptable.

Question 2 (30 marks): Suppose that we send a two-bit message by sending five bits through a Binary Symmetric Channel with error probability 1/3, using the $[5, 2]$ linear code from the lectures, in which the codewords are 00000, 00111, 11001, and 11110. Suppose that these four codewords are equally likely to be sent, and suppose that the decoder decodes by maximum likelihood. What is the probability that the decoder will decode to the wrong codeword? Does this error probability depend on any arbitrary choices made by the decoder?

Question 3 (30 marks): Suppose that we use a linear code on a binary alphabet (ie, a linear subspace of Z_2^N) to encode messages that are sent over a Binary Symmetric Channel. Suppose also that decoding is done by maximum likelihood (and in case of ties, a codeword is selected uniformly at random from among all those with the maximum likelihood). Prove that the probability that the decoded message is not what was sent is the same regardless of which message was sent.