Total Colouring With Δ+poly($\log \Delta$) Colours*

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Abstract

We provide a polynomial time algorithm which finds a total colouring of any graph with maximum degree Δ, Δ sufficiently large, using at most $\Delta + 8\log^8 \Delta$ colours. This improves the best previous upper bound on the total chromatic number of $\Delta + 18\Delta^{1/3}\log(3\Delta)$.

1 Introduction

A *total colouring* of a graph G is an assignment of colours to its vertices and edges so that no two adjacent vertices have the same colour, no two adjacent edges have the same colour, and no edge has the same colour as one of its endpoints. A k total colouring is a total colouring which uses at most k

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colours. The *total chromatic number*, $\chi^{''}(G)$ is the least number of colours required for a total colouring of G.

This concept was introduced independently by Behzad [3] and Vizing [15], who each conjectured that any graph with maximum degree Δ has a $\Delta+2$ total colouring. Note that if true, this conjecture is tight as every such graph requires at least $\Delta+1$ colours and there are some graphs such as $K_{\Delta+1}$, Δ odd, which require $\Delta+2$ colours. Kilakos and Reed [11] have shown that the fractional total chromatic number of such a graph is at most $\Delta+2$.

The first $\Delta + \mathrm{o}(\Delta)$ bound on the total chromatic number of such a graph was $\Delta+2\sqrt{\Delta}$, due to Hind [7]. More recently, Hǎggkvist and Chetwynd (see [10]) have reported a bound of $\Delta+18\Delta^{1/3}\log(3\Delta)$. In this paper we tighten the bound to $\Delta+8\log^8\Delta$ (all logarithms have base e).

Theorem 1 *For sufficiently large Δ, if G has maximum degree Δ then* $\chi^{''}(G) \leq \Delta+8\log^8\Delta$.

Our proof is probabilistic and makes use of the Lovász Local Lemma. The proof can be made constructive, providing an $\mathrm{O}(n^3\log^{\mathrm{O}(1)} n)$ randomized algorithm, and a polytime deterministic algorithm to find such a total colouring.

The total chromatic number conjecture is reminiscent of Vizing's Theorem which states that if G has maximum degree Δ then the edge chromatic number of G, $\chi^{'}(G)$ is either Δ or $\Delta+1$. It is also reminiscent of the list colouring conjecture. In fact, a slightly weaker form of the total colouring conjecture follows from the list colouring conjecture.

The *list edge chromatic number* of a graph G, $\chi^{'}_{\ell}(G)$, is the minimum number

r with the following property: For any mapping $f : E(G) \rightarrow \mathcal{S}$ where $\mathcal{S}$ is a collection of sets of colours each of size r, G has a proper edge-colouring where for each edge e, the colour of e lies in $f(e)$. The list colouring conjecture is that $\chi'_\ell(G) = \chi'(G)$.

Recall that for any graph G, $\chi(G) \leq \Delta+1$. Now consider any $\Delta+1$ colouring $c : V(G) \rightarrow \{1, \ldots, \Delta+1\}$. For each edge $e = (u, v)$ define $f(e)$ to be the set $\{1, \ldots, \Delta+3\} - \{c(u), c(v)\}$. Now the size of $f(e)$ is $\Delta+1$ for each e, and so if the list colouring conjecture holds then we can use such a colouring to provide a $\Delta+3$ total colouring of G. Therefore, the list colouring conjecture implies $\chi''(G) \leq \Delta+3$.

Inspired by this implication, we say that a proper vertex colouring is *extendible* to a t total colouring if there is a total colouring of size t whose restriction to $V(G)$ is that vertex colouring. Thus, we have seen that the list colouring conjecture implies that every $\Delta+1$ vertex colouring of G is extendible to a $\Delta+3$ total colouring of G. Hind [8] has shown that there exist graphs having a $\Delta+1$ vertex colouring which is not extendible to a $\Delta+2$ total colouring.

In [9] we define a proper vertex colouring to be *β-frugal* if no vertex has more than β members of any colour class in its neighbourhood. We prove the following:

Theorem 2 *Every graph G with maximum degree $\Delta \geq \Delta_0 = \mathrm{e}^{10^7}$ has a $\log^5 \Delta$-frugal $(\Delta+1)$ vertex colouring.*

In this paper, we show that every $\log^5 \Delta$-frugal $(\Delta+1)$ vertex colouring is extendible to an $\Delta + 8\log^8 \Delta$ total colouring, thus proving Theorem 1.

The idea behind our proof is simple. We begin by presenting the basic ideas. For ease of exposition, let us assume for now that G is Δ-regular. Consider any $\log^5 \Delta$-frugal $(\Delta+1)$ vertex colouring of G with colour classes $S_1, \ldots, S_{\Delta+1}$. If we could find an edge disjoint sequence of matchings $M_1, \ldots, M_{\Delta+1}$ such that M_i misses all of S_i and covers all of $V(G) - S_i$ (i.e. M_i is a perfect matching of $G - S_i$) then this would give us a $\Delta + 1$ total colouring. (Note that since every vertex is missed by exactly one matching here, then $\cup M_i = E(G)$.) Of course, this is not always possible as there are some graphs with $\chi'' \geq \Delta + 2$. For example, we will fail if $|V(G) - S_i|$ is odd for any i. Thus we will have to allow our matchings to miss a few more vertices. Essentially, we will show that we can find sets $X_1, \ldots, X_{\Delta+1}$ with the following two properties:

1. each vertex lies in at most $\log^8 \Delta$ of these sets, and

2. for each $1 \leq i \leq \Delta + 1$, we can find a matching M_i in $G_i = G - \cup_{1 \leq j \leq i-1} M_i$ which misses all of S_i and meets all of $V(G) - S_i - X_i$.

Therefore the colour classes $C_i = S_i \cup M_i$ will provide a total colouring of all but $E(G_{\Delta+2})$. By Condition 1, $G_{\Delta+2}$ has maximum degree at most $\log^8 \Delta$ and so it can be edge coloured with at most $\log^8 \Delta + 1$ colours thus providing a $\Delta + \log^8 \Delta + 2$ total colouring of G.

We have oversimplified things here. In fact, our argument is more intricate. In the next section we will fill in the details, including the manner in which we choose our sets X_i. For now, we will simply say that we choose them randomly and make use of the following two tools. The first is due to Lovász and appears in [4]. The second can be found in [6].

The Local Lemma *Suppose* $\mathcal{A} = A_1, \ldots, A_n$ *is a list of random events such that for each* i, $\mathbf{Pr}(A_i) \le p$ *and* A_i *is mutually independent of all but at most* d *other events in* $\mathcal{A}$. *If* $\mathrm{e}p(d+1) < 1$ *then* $\mathbf{Pr}(\wedge_{i=1}^{n} \bar{A}_i) > 0$.

The Chernoff Bounds *Suppose* $B(n,p)$ *is the sum of* n *independent Bernoulli variables each equal to 1 with probability* p. *Then for any* $a > 0$ *we have the following:*

$$\mathbf{Pr}(B(n,p) - np > a) < \mathrm{e}^{-a^2/3np},$$

and

$$\mathbf{Pr}(B(n,p) - np < -a) < \mathrm{e}^{-a^2/2np}.$$

For the remainder of this paper, we assume $\Delta \ge \mathrm{e}^{10^7}$. For each vertex v, $N(v)$ denotes the neighbourhood of v. We usually omit all $\lfloor, \rfloor$ and $\lceil, \rceil$ signs.

2 The Details

Our main task will be to prove the following.

Lemma 1 *Suppose* G *is a graph with maximum degree at most* $D \ge 8\log^8 \Delta$. *Suppose further that we are given* $S_1, S_2, \ldots, S_{\frac{D}{2}} \subseteq V(G)$ *such that for all* $v \in V(G), 1 \le i \le \frac{D}{2}$, $|N(v) \cap S_i| \le \log^5 \Delta$. *Then there exists a sequence of edge-disjoint matchings in* G, $M_1, \ldots, M_{\frac{D}{2}}$ *such that*

1. M_i *misses* S_i;
2. $G' = G - \cup_{i=1}^{\frac{D}{2}} M_i$ *has maximum degree at most* $\frac{D}{2} + 2\log^7 \Delta$.

Repeated iterations of Lemma 1 will prove Theorem 1:

Proof of Theorem 1 Take $S_1, ..., S_{\Delta+1}$ to be the colour classes of any $\log^5 \Delta$-frugal $\Delta + 1$ colouring of G, as guaranteed by Theorem 2. Set $G_0 = G$, $\Delta_0 = \Delta$, and repeatedly apply Lemma 1 until $\Delta_j < 8\log^8 \Delta$, setting $G_{j+1} = G'$, $\Delta_{j+1} = \Delta_j/2 + 2\log^7 \Delta \leq \frac{\Delta}{2^j} + 4\log^7 \Delta$, and choosing $S_1^{(j)}, \ldots, S_{\Delta_j/2}^{(j)}$ from previously unused members of $\{S_1, \ldots, S_{\Delta+1}\}$, all the while forming colour classes from the pairs $S_i \cup M_i$. As there are at most $\log \Delta$ iterations, $\sum \Delta_i/2 \leq \Delta - 4\log^8 \Delta + \log \Delta(4\log^7 \Delta) < \Delta + 1$, and so we will have produced fewer than $\Delta + 1$ colour classes. Therefore, a $8\log^8 \Delta$ edge-colouring of the final G' will provide our $\Delta + 8\log^8 \Delta$ total colouring. $\square$

We prove Lemma 1 by choosing random sets X_i which we allow M_i to miss as described in the introduction. As mentioned earlier, it is important that no vertex appears in very many sets X_i, as that will cause its degree in G' to be too high. In order to ensure this, we will divide $\{1, \ldots, D/2\}$ into $\log^7 \Delta$ subsequences, and we will insist that no vertex falls into two sets from the same subsequence.

Specifically, we set $\alpha_j = \rfloor j \times \frac{D}{\log^7 \Delta} \lfloor$, $j = 0, \ldots, \lceil \frac{1}{2}\log^7 \Delta \rceil$, and we set $A_j = \{\alpha_{j-1} + 1, \ldots, \alpha_j\}$. We will choose $X_1, \ldots, X_{D/2}$ such that for $i_1, i_2 \in A_j$, $X_{i_1} \cap X_{i_2} = \emptyset$.

For $i \in A_j$, we define $D_i = D - (i-1) + 2(j-1)$. We set $G_1 = G$, and for $1 \leq i \leq D/2$ we will find a matching M_i in $G_i = G - \cup_{j=1}^{i-1} M_i$ such that

1. M_i misses S_i;

2. M_i meets every vertex of degree at least D_i in $V(G) - S_i - X_i$.

We will see how to find X_i and M_i later, but first note that this will be enough to prove Lemma 1.

Claim 1 *For each $i \geq 2$, if M_{i-1} exists, then G_i has maximum degree at most $D_i + 2$.*

Proof In what follows, we only discuss $k \leq i$ and j such that $\alpha_{j-1} \leq i$. Consider any vertex v. We denote by $\deg_k(v)$ the degree of v in G_k. We will see by induction on j that $\deg_{\alpha_{j-1}+1}(v) \leq D_{\alpha_{j-1}+1}$ and that for all k with $\alpha_{j-1}+1 \leq k \leq \alpha_j$, $\deg_k(v) \leq D_k + 2$. The first condition holds for $j = 1$. To see that for each j, the first condition implies the second condition as well as the first condition for $j+1$, consider the first (if any) $k \in A_j$ such that $\deg_k(v) = D_k$, and note that $\deg_{k'}(v) \geq D_{k'}$ for all $k \leq k^{'} \leq \alpha_j$, and so v can be missed by at most two matchings M_{k_1}, M_{k_2} before $\alpha_j + 1$, corresponding to $v \in X_{k_1}$, and $v \in S_{k_2}$. Therefore, $\deg_{k'}(v) \leq D_{k'} + 2$ for each $k \leq k^{'} \leq \alpha_j$, and $\deg_{\alpha_j+1}(v) \leq D_{a_j+1}$ as $D_{a_j+1} = D_{a_j} + 1$. □

Therefore, $G_{D/2}$ has maximum degree at most $D_{D/2} + 2 \leq D/2 + 2\log^7 \Delta$ as required.

It only remains to choose X_i and M_i. This is done via the following two lemmas.

Lemma 2 *Suppose G_i has maximum degree at most $D_i + 2$ where $D_i \geq \log^8 \Delta$, and suppose further that there exists a set $R \subseteq V(G)$ such that for any $v \in V(G)$, $|N(v) \cap R| \leq \frac{3D}{\log \Delta}$. Then there exists $X_i \subseteq V(G) - R$ such that for all $v \in V(G)$*

1. *$|N(v) - X_i| \leq D_i - \log^6 \Delta$;*

2. $|N(v) \cap X_i| \leq 3\log^6 \Delta$.

Lemma 3 *Suppose G_i has maximum degree at most $D_i + 2$ and suppose further that we have $S_i, X_i \subseteq V(G)$ such that for each $v \in V(G)$*

1. $|N(v) \cap S_i| \leq \log^5 \Delta$*;*

2. $|N(v) - X_i| \leq D_i - \log^6 \Delta$*;*

3. $|N(v) \cap X_i| \leq 3\log^6 \Delta$*.*

Then there exists a matching M_i such that

1. M_i misses S_i;

2. M_i meets every vertex of degree at least D_i in $V(G) - S_i - X_i$.

Using these lemmas, along with Claim 1, it is now straightforward to prove Lemma 1 in the manner discussed earlier.

Proof of Lemma 1 For $i = 1, \ldots, D/2$, we choose X_i via Lemma 2 by setting $R = \cup_{k \in A_j, k<i} X_k$ where $i \in A_j$, noting that $|N(v) \cap R| \leq |A_j| \times 3\log^6 \Delta \leq \frac{3D}{\log \Delta}$, and we choose M_i via Lemma 3. The result now follows as in the earlier discussion. □

We now complete the proof of Theorem 1 by proving Lemmas 2 and 3.

Proof of Lemma 2 We will choose X_i randomly. For each $v \in V(G) - R$, we place v in X_i with probability $p_i = \frac{2\log^6 \Delta}{D_i+2}$. For each $v \in V(G)$ define E_v to be the event that v violates one of the required conditions. Note that by the Chernoff bounds,

$$
\begin{aligned}
\mathbf{Pr}(E_v) &\leq \mathbf{Pr}(|B(D_i+2,p_i)-2\log^6\Delta| > \log^6\Delta) \\
&\qquad +\mathbf{Pr}(|B(D_i-\tfrac{3D}{\log\Delta},p_i)-p_i\times(D_i-\tfrac{3D}{\log\Delta})| > \frac{1}{2}\log^6\Delta) \\
&\leq 2\mathrm{e}^{-\log^6\Delta/6}+2\mathrm{e}^{-\log^6\Delta/24} \\
&< \Delta^{-3}.
\end{aligned}
$$

Furthermore, each event E_v is mutually independent of all but at most Δ^2 other events. Therefore our result follows from the Local Lemma as $\mathrm{e}\Delta^{-3}(\Delta^2+1)<1$. □

Proof of Lemma 3 Suppose that such a matching does not exist. Then by a well-known extension of Tutte's Theorem, there exist disjoint $T,Q \subset V(G_i)-S_i$ with $T\cap X_i=\emptyset$, such that each $v\in T$ has degree at least D_i in G, the subgraph induced by T has at least $|Q|+1$ odd components $C_1,C_2,\ldots,C_{|Q|+1}$, and there are no edges from T to $G-(Q\cup S_i)$. (One way to see this is form G_i' by deleting S_i from G_i and then adding an edge between every pair of non-adjacent vertices which each have degree less than D_i in G_i. If $|G_i'|$ is odd, then add a vertex which is adjacent to every vertex of degree less than D_i in G_i. Now apply Tutte's Theorem to G_i'.)

For any disjoint $A,B\subseteq V(G_i)$, denote by $E(A,B)$ the set of edges with one endpoint in each of A,B.

Claim 2 *For each* $1\leq i\leq |Q|+1$, $|E(C_i,Q)|\geq D_i-\log^5\Delta$.

Proof

Case 1: $|C_i|\leq D_i/3$.

$|E(C_i, Q \cup S_i)| \geq D_i|C_i| - \binom{|C_i|}{2}$, and $|E(C_i, S_i)| \leq |C_i| \log^5 \Delta$. Therefore $|E(C_i, Q)| \geq |C_i|(D_i - \log^5 \Delta) - \binom{|C_i|}{2} \geq D_i - \log^5 \Delta$.

Case 2: $|C_i| > D_i/3$.

For each $v \in C_i$, $|E(\{v\}, X_i)| \geq \log^6 \Delta$ and $|E(\{v\}, S_i)| \leq \log^5 \Delta$. Therefore $|E(C_i, Q)| \geq |C_i|(\log^6 \Delta - \log^5 \Delta) \geq D_i - \log^5 \Delta$.

□

Claim 3: *For each $v \in Q$, $|E(\{v\}, T)| \leq D_i - \log^6 \Delta$.*

Proof This follows from the fact that $T \cap X_i = \emptyset$. □

Therefore, we have $(|Q| + 1)(D_i - \log^5 \Delta) \leq |E(T, Q)| \leq |Q|(D_i - \log^6 \Delta)$. Thus $D_i \leq \log^5 \Delta$ which contradicts $D \geq 8 \log^8 \Delta$. □

3 An Efficient Algorithm

We note here that our proof can be made algorithmic using the techniques developed by Beck [2], at the price of increasing our lower bound on Δ. Set $n = |V(G)|$.

[9] provides an $\mathrm{O}(n^3 \log^{\mathrm{O}(1)} n)$ randomized algorithm and a polytime deterministic algorithm to find a $\log^5 \Delta$-frugal $(\Delta+1)$-colouring of G. After doing this, we must find the set X_i guaranteed by Lemma 2 and the matching M_i guaranteed by Lemma 3 fewer than Δ times, and finally we must find the $8 \log^8 \Delta$ edge-colouring used in the proof of Theorem 1. The latter step can be done in $\mathrm{O}(n^4)$ steps, or we can find a $16 \log^8 \Delta$ edge-colouring in $\mathrm{O}(n^2)$ steps. Each M_i can be found in $\mathrm{O}(n^{2.5})$ steps, as in [5] or [13]. It only remains

to find X_i.

This can be done in $O(n^2 \log^{O(1)} n)$ steps using an algorithm essentially the same as that in Section 4 of [2]. The only modification required is to allow for sampling with probability p_i here rather than with probability $\frac{1}{2}$ as in [2]. This can be done in a straightforward manner as described in [9].

Thus we have a $O(n^{2.5} \log^{O(1)} n)$ time randomized algorithm and a polytime deterministic algorithm for finding a $\Delta + 16 \log^8 \Delta$ total colouring of G.

4 Remarks

It is worth noting that by being more careful with our calculations, both here and in [9], and raising our lower bound on Δ, we can find a $\log^3 \Delta$-frugal $(\Delta + 1)$-colouring for any graph with maximum degree Δ sufficiently large, and find a $\Delta + \log^4 \Delta$ total colouring. However, these techniques do not appear to be sufficient to get a bound any lower than $\Delta + \mathrm{poly}(\log \Delta)$.

Alon [1] has shown how to modify the technique of [2] to produce parallel algorithms. This does not seem to apply here.

Recently, Molloy and Reed [14] have improved the upper bound on the total chromatic number to $\Delta + C$ for a large constant C. The proof uses a different, much more complicated technique, and does not appear to yield an efficient algorithm for finding such a colouring.

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