A bound on the chromatic number of the square of a planar graph

Michael Molloy *
Department of Computer Science, University of Toronto,
10 King's College Rd., Toronto, M5S 3G4, Canada
E-mail: molloy@cs.toronto.edu

Mohammad R. Salavatipour †
Department of Computer Science, University of Toronto,
10 King's College Rd., Toronto, M5S 3G4, Canada
E-mail: mreza@cs.toronto.edu

Abstract

Wegner conjectured that the chromatic number of the square of any planar graph G with maximum degree $\Delta \geq 8$ is bounded by $\chi(G^2) \leq \lfloor \frac{3}{2}\Delta \rfloor + 1$. We prove the bound $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 78$. This is asymptotically an improvement on the previously best known bound. For large values of Δ we give the bound of $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 25$. We generalize this result to $L(p,q)$-labeling of planar graphs, by showing that $\lambda_q^p(G) \leq q\lceil \frac{5}{3}\Delta \rceil + 18p + 77q - 18$. For each of the results, the proof provides a quadratic time algorithm.

1 Introduction

In this paper by *graph* we mean a simple graph. The vertex set and edge set of a graph G are denoted by $V(G)$ and $E(G)$, respectively. The length of a path between two vertices is the number of edges on that path. We define the distance between two vertices to be the length of the shortest path between them. The square of a graph G, denoted by G^2, is a graph on the same vertex set such that two vertices are adjacent in G^2 iff their distance in G is at most 2. The degree of a vertex v is the number of edges incident with v and is denoted by $d_G(v)$ or simply $d(v)$ if it is not confusing. We denote the maximum degree of a graph G by $\Delta(G)$ or simply Δ. If the degree of v is i, at least i, or at most i we call it an i-vertex, a $\geq i$-vertex, or a $\leq i$-vertex, respectively. By $N_G(v)$, we mean the open neighborhood of v in G, which contains all those vertices that are adjacent to v in G. The closed neighborhood of v, which is denoted by $N_G[v]$, is $N_G(v) \cup \{v\}$. We usually use $N(v)$ and $N[v]$ instead of $N_G(v)$ and $N_G[v]$, respectively.

A vertex k-coloring of a graph G is a mapping $C : V \longrightarrow \{1, \ldots, k\}$ such that any two adjacent vertices u and v are mapped to different integers. The minimum k for which a coloring exists is called the chromatic number of G and is denoted by $\chi(G)$. The well known result of Appel and Haken [2] states that:

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Theorem 1.1 (The Four Color Theorem) *For every planar graph G: $\chi(G) \leq 4$.*

The question of finding the best possible upper bound for the chromatic number of the square of a planar graph seems to first have been asked by Wegner [21]. He posed the following conjecture:

Conjecture 1.2 *For a planar graph G:*

$$\chi(G^2) \leq \begin{cases} \Delta + 5 & \text{if } 4 \leq \Delta \leq 7, \\ \lfloor \frac{3}{2}\Delta \rfloor + 1 & \text{if } \Delta \geq 8. \end{cases}$$

He gave examples illustrating that these bounds are best possible. He also showed that if $\Delta = 3$ then G^2 can be 8-colored and conjectured that 7 colors would be enough. Very recently, Thomassen [18] has solved this conjecture for $\Delta = 3$, by showing that the square of every cubic planar graph is 7-colorable, but the conjecture for general planar graphs remains open.

This conjecture is mentioned in Jensen and Toft [14], Section 2.18, followed by a brief history of it. One might think that since every planar graph has a $\leq$5-vertex then this trivially implies a greedy algorithm for $(5\Delta+1)$-coloring of G^2. See [19] why this straightforward argument doesn't work. Jonas [13] in his Ph.D. thesis proved $\chi(G^2) \leq 8\Delta - 22$. This bound was later improved by Wong [23] to $\chi(G^2) \leq 3\Delta + 5$. Then Van den Heuvel and McGuinness [19] proved $\chi(G^2) \leq 2\Delta + 25$. For large values of Δ, Agnarsson and Halldórsson [1] have a better asymptotic bound. They showed that if G is a planar graph with $\Delta \geq 749$, then $\chi(G^2) \leq \lfloor \frac{9}{5}\Delta \rfloor + 2$. Recently, Borodin et al. [4, 5] have been able to extend this result further by proving $\chi(G^2) \leq \lceil \frac{9}{5}\Delta \rceil + 1$ for planar graphs with $\Delta \geq 47$. We improve these results asymptotically by showing that:

Theorem 1.3 *For a planar graph G, $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 78$.*

Theorem 1.4 *For a planar graph G, if $\Delta \geq 241$, then $\chi(G^2) \leq \lceil \frac{5}{3}\Delta \rceil + 25$.*

Remark: The constants 78 and 25 in the above theorems can be improved. For example with an extra page of proof the first constant can be brought down to 61 but we don't know how to bring it down to a number close to 1, using this proof.

The technique we use is inspired by that used by Sanders and Zhao [17] to obtain a similar bound on the cyclic chromatic number of planar graphs.

A generalization of ordinary vertex coloring is $L(p,q)$-labeling. Let $dist(u,v)$ denote the distance between u and v. For integers $p,q \geq 0$, an $L(p,q)$-labeling of a graph G is a mapping $L : V(G) \longrightarrow \{0,\ldots,k\}$ such that

- $|L(u) - L(v)| \geq p$ if $dist(u,v) = 1$, and
- $|L(u) - L(v)| \geq q$ if $dist(u,v) = 2$.

The p,q-span of G, denoted by $\lambda_q^p(G)$, is the minimum k for which an $L(p,q)$-labeling exists. It is easy to see that for any graph G: $\chi(G^2) = \lambda_1^1(G) + 1$. The problem of determining $\lambda_q^p(G)$ has been studied for some specific classes of graphs [3, 6, 7, 8, 9, 10, 11, 15, 16, 20, 22]. The motivation for this problem comes from the channel assignment problem in radio and cellular phone systems, where each vertex of the graph corresponds to a transmitter location, with the label assigned to it determining the frequency channel on which it transmits. In applications, because of possible interference between neighboring transmitters, the channels assigned to them must have a certain distance from each other. A similar requirement arises from transmitters that are not neighbors but are close, i.e at distance 2. This problem is also known as the Frequency Assignment Problem. Because of the motivating application for this problem, it is quite natural to consider it on planar graphs. Since the case $q = 0$ corresponds to labeling the vertices of a graph with integers such that adjacent vertices receive labels at least p apart, the upper bound $3p$ for λ_0^p of planar graphs follows from the Four Color Theorem (if we use colors from $\{0, p, 2p, 3p\}$). So let's assume that $q \geq 1$. For any planar graph G, a straightforward argument shows that $\lambda_q^p(G) \geq q\Delta + p - q + 1$. There are planar graphs G for which $\lambda_q^p(G) \geq \frac{3}{2}q\Delta + O(p, q)$. The best known upper bound for $\lambda_q^p(G)$, for a planar graph G, is proved in [19].

Theorem 1.5 *[19] For any planar graph G and positive integers p and q, such that $p \geq q$: $\lambda_q^p(G) \leq (4q-2)\Delta + 10p + 38q - 24$.*

We sharpen the gap between this result and the best possible bound asymptotically, by showing that:

Theorem 1.6 *For any planar graph G and positive integers p and q: $\lambda_q^p(G) \leq q\lceil\frac{5}{3}\Delta\rceil + 18p + 77q - 18$.*

Sections 2 and 3 contain the proof of Theorem 1.3. In Section 4 we show how to modify the proof of Theorem 1.3 to prove Theorem 1.4. In Section 5 we explain why any modifications of the lemmas used in the proof of Theorem 1.3 are not sufficient to improve this theorem, asymptotically, and one has to come up with a new configuration. These arguments will be cleared later. We generalize the proof of Theorem 1.3 in Section 6 to prove Theorem 1.6. Finally, in Section 7 we describe an $O(n^2)$ time algorithm for finding a coloring as described in Theorems 1.3, 1.4, and 1.6.

2 Preliminaries

A vertex v is called *big* if $d_G(v) \geq 47$, otherwise we call it a *small* vertex. From now on we assume that G is a counter-example to Theorem 1.3 with the minimum number of vertices. By a coloring we mean a coloring in which vertices at distance at most two from each other get different colors. Trivially G is connected.

Lemma 2.1 *For every vertex v of G, if there exists a vertex $u \in N(v)$, such that $d_G(v) + d_G(u) \leq \Delta + 2$ then $d_{G^2}(v) \geq \lceil\frac{5}{3}\Delta\rceil + 78$.*

Proof: Assume that v is such a vertex. Contract v on edge uv. The resulting graph has maximum degree at most Δ and because G was a minimum counter-example, the new graph can be colored with $\lceil\frac{5}{3}\Delta\rceil + 78$ colors. Now consider this coloring induced on G, in which every vertex other than v is colored. If $d_{G^2}(v) < \lceil\frac{5}{3}\Delta\rceil + 78$ then we can assign a color to v to extend the coloring to v, which contradicts the definition of G. ■

Observation 2.2 *We can assume that $\Delta \geq 160$, otherwise $2\Delta + 25 \leq \lceil\frac{5}{3}\Delta\rceil + 78$.*

Lemma 2.3 *Every $\leq$5-vertex in G must be adjacent to at least two big vertices.*

Proof: By way of contradiction assume that this is not true. Then there is a $\leq$5-vertex v which is adjacent to at most one big vertex and all its other neighbors are $\leq$46-vertices. Then, using Observation 2.2, v along with one of these small vertices will contradict Lemma 2.1. ■

Corollary 2.4 *Every vertex of G is a $\geq$2-vertex.*

Lemma 2.5 *G is 2-connected.*

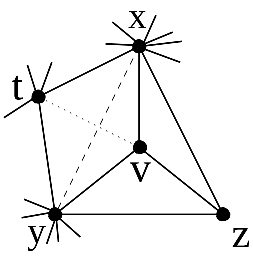

Figure 1: The switching operation

Proof: By contradiction, let v be a cut-vertex of G and let $C_1, \ldots, C_t$ $(t \geq 2)$ be the connected components of $G - \{v\}$. By the definition of G, for each $1 \leq i \leq t$, there is a coloring φ_i of $G_i = C_i \cup \{v\}$ with $\lceil \frac{5}{3}\Delta \rceil + 78$ colors. We can permute the colors in each φ_i (if needed) such that v has the same color in all φ_i's, and the sets of colors appearing in $N_{G_i}(v)$, $1 \leq i \leq t$, are all disjoint. Now the union of these colorings will be a coloring of G, a contradiction. ■

The proof of Theorem 1.3 becomes significantly simpler if we can assume that the underlying graph is a triangulation, i.e. all faces are triangles, and has minimum degree at least 4. To be able to make these assumptions, we begin by modifying graph G in two phases.

Phase 1: In this phase we transform G into a (simple) triangulated graph G', by adding edges to every non-triangle face of G. Let G' be initially equal to G. Consider any non-triangle face $f = v_1, v_2, \ldots, v_k$ of G'. Because G is 2-connected, we cannot have both $v_1v_3 \in E(G')$ and $v_2v_4 \in E(G')$ at the same time since they both have to be outside of f. So we can add at least one of these edges to $E(G')$ inside f, without creating any multiple edges. We follow this procedure to reduce the faces' sizes as long as we have any non-triangle face in G'. At the end we have a triangulated graph G' which contains G as a subgraph.

Observation 2.6 *For every vertex v, $N_G(v) \subseteq N_{G'}(v)$.*

Lemma 2.7 *All vertices of G' are $\geq$3-vertices.*

Proof: By Corollary 2.4 and Observation 2.6 all the vertices of G' are $\geq$2-vertices. Suppose that we have a 2-vertex v in G' having neighbors x and y. Since G' is triangulated, the faces on each side of edge vx must be triangles, call them f_1 and f_2. So we must have $xy \in f_1$ and also $xy \in f_2$. Since G' has at least 4 vertices, $f_1 \neq f_2$ and so we have a multiple edge. But G' is simple. ■

Lemma 2.8 *Each $\geq$4-vertex v in G' can have at most $\frac{d(v)}{2}$ neighbors which are 3-vertices.*

Proof: Let $x_0, x_1, \ldots, x_{d_{G'}(v)-1}$ be the sequence of neighbors of v in G', in clockwise order. We show that we cannot have two consecutive 3-vertices in this sequence. If there are two consecutive 3-vertices, say $d(x_i) = d(x_{i+1}) = 3$, where addition is in mod $d_{G'}(v)$, then there is a face containing $x_{i-1}, x_i, x_{i+1}, x_{i+2}$. But G' is a triangulated graph. ■

Phase 2: In this phase we transform graph G' into another triangulated graph G'', whose minimum degree is at least 4. Initially G'' is equal to G'. As long as there is any 3-vertex v we do the following *switching* operation: let x, y, z be the three neighbors of v. At least two of them, say x and y, are big in G' by Lemma 2.3 and Observation 2.6. Remove edge xy. Since G' (and also G'') is triangulated this leaves a face of size 4, say x, v, y, t. Add edge vt to G'' (see Figure 1). This way, the graph is still triangulated.

Observation 2.9 *If v is not a big vertex in G then $N_G(v) \subseteq N_{G''}(v)$.*

Lemma 2.10 *If v is a big vertex in G then $d_{G''}(v) \geq 24$.*

Proof: Follows easily from Lemma 2.8 and the definition of the switching operation. ■

So a big vertex v in G will not be a ≤ 23-vertex in G''. Let v be a big vertex in G and $x_0, x_2, \ldots, x_{d_{G''}(v)-1}$ be the neighbors of v in G'' in clockwise order. We call $x_a, \ldots, x_{a+b}$ (where addition is in mod $d_{G''}(v)$) a *sparse segment* in G'' iff:

- $b \geq 2$,
- Each x_i is a 4-vertex.

In the next two lemmas, we assume that $x_a, \ldots, x_{a+b}$ is a maximal sparse segment of v in G'', which is not equal to the whole neighborhood of v. Also, we assume that x_{a-1} and x_{a+b+1} are the neighbors of v right before x_a and right after x_{a+b}, respectively.

Lemma 2.11 *There is a big vertex in G other than v, that is connected to all the vertices of $x_{a+1}, \ldots, x_{a+b-1}$, in G'' (and in G).*

Proof: Follows easily from Observation 2.9, Lemma 2.3, and the definition of a sparse segment. ■

We use u to denote the big vertex, other than v, that is connected to all $x_{a+1}, \ldots, x_{a+b-1}$.

Lemma 2.12 *All the vertices $x_{a+1}, \ldots, x_{a+b-1}$ are connected to both u and v in G. If x_{a-1} is not big in G then x_a is connected to both u and v in G. Otherwise it is connected to at least one of them. Similarly if x_{a+b+1} is not big in G, x_b is connected to both u and v in G, and otherwise it is connected to at least one of them.*

Proof: Since the only big neighbors of $x_{a+1}, \ldots, x_{a+b-1}$ in G'' are v and u, by Lemma 2.3 they must be connected to both of them in G as well. For the same reason x_a and x_{a+b} will be connected to u and v in G, if x_{a-1} and x_{a+b-1} are not big. ■

We call $x_{a+1}, \ldots, x_{a+b-1}$ the *inner* vertices of the sparse segment, and x_a and x_{a+b} the *end* vertices of the sparse segment. Consider vertex v and let us denote the maximal sparse segments of $N(v)$ by $Q_1, Q_2, \ldots, Q_m$ in clockwise order, where $Q_i = q_{i,1}, q_{i,2}, q_{i,3}, \ldots$. The next two lemmas are the key lemmas in the proofs of Theorems 1.3 and 1.4. They provide two reducible configurations for a graph that is a minimum counter-example to theorem.

Lemma 2.13 $|Q_i| \leq d_G(v) - \lceil\frac{2}{3}\Delta\rceil - 73$*, for* $1 \leq i \leq m$*.*

Proof: We prove this by contradiction. Assume that for some i, $|Q_i| > d_G(v) - \lceil\frac{2}{3}\Delta\rceil - 73$. Let u_i be the big vertex that is adjacent to all the inner vertices of Q_i (in both G and G''). See Figure 2. For an inner vertex of Q_i, say $q_{i,2}$, we have:

$$\begin{aligned} d_{G^2}(q_{i,2}) &\leq d_G(u_i) + d_G(v) + 2 - (|Q_i| - 3) \\ &\leq \Delta + d_G(v) - |Q_i| + 5 \\ &< \lceil\frac{5}{3}\Delta\rceil + 78. \end{aligned}$$

If $q_{i,2}$ is adjacent to $q_{i,1}$ or $q_{i,3}$ in G then it is contradicting Lemma 2.1. Otherwise it is only adjacent to v and u_i in G, therefore has degree 2, and so along with v or u_i contradicts Lemma 2.1. ■

Lemma 2.14 *Consider G and suppose that u_i and u_{i+1} are the big vertices adjacent to all the inner vertices of Q_i and Q_{i+1}, respectively. Furthermore assume that t is a vertex adjacent to both u_i and u_{i+1} but not adjacent to v (see Figure 3) and there is a vertex $w \in N_G(t)$ such that $d_G(t) + d_G(w) \leq \Delta + 2$.*

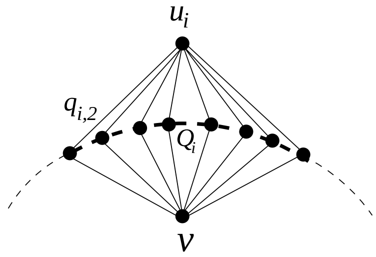

Figure 2: The configuration of Lemma 2.13

Let $X(t)$ be the set of vertices at distance at most 2 of t that are not in $N_G[u_i] \cup N_G[u_{i+1}]$. If $|X(t)| \le 6$ then:

$$|Q_i| + |Q_{i+1}| \le \lfloor \frac{1}{3}\Delta \rfloor - 67.$$

Proof: Again we use contradiction. Assume that $|Q_i| + |Q_{i+1}| \ge \lfloor \frac{1}{3}\Delta \rfloor - 66$. Using the argument of the proof of Lemma 2.1 we can color every vertex of G other than t. Note that $d_{G^2}(t) \le d_G(u_i) + d_G(u_{i+1}) + |X(t)| \le 2\Delta + 6$. If all the colors of the inner vertices of Q_i have appeared on the vertices of $N_G[u_{i+1}] \cup X(t) - Q_{i+1}$ and all the colors of inner vertices of Q_{i+1} have appeared on the vertices of $N_G[u_i] \cup X(t) - Q_i$ then there are at least $|Q_i| - 2 + |Q_{i+1}| - 2$ repeated colors at $N_{G^2}(t)$. So the number of colors at $N_{G^2}(t)$ is at most $2\Delta + 6 - |Q_i| - |Q_{i+1}| + 4 \le \lceil \frac{5}{3}\Delta \rceil + 76$ and so there is still one color available for t, which is a contradiction.

Therefore, without loss of generality, there exists an inner vertex of Q_{i+1}, say $q_{i+1,2}$, whose color is not in $N_G[u_i] \cup X(t) - Q_i$. If there are less than $\lceil \frac{5}{3}\Delta \rceil + 77$ colors at $N_{G^2}(q_{i+1,2})$ then we could assign a new color to $q_{i+1,2}$ and assign the old color of it to t and get a coloring for G. So there must be $\lceil \frac{5}{3}\Delta \rceil + 77$ or more different colors at $N_{G^2}(q_{i+1,2})$.

From the definition of a sparse segment $N_G(q_{i+1,2}) \subseteq \{v, u_{i+1}, q_{i+1,1}, q_{i+1,3}\}$. There are at most $d_G(u_{i+1}) + 7$ colors, called the *smaller* colors, at $N_G[u_{i+1}] \cup X(t) \cup N_G[q_{i+1,1}] \cup N_G[q_{i+1,3}] - \{v\} - \{q_{i+1,2}\}$ (note that t is not colored). So there must be at least $\lceil \frac{2}{3}\Delta \rceil + 70$ different colors, called the *larger* colors, at $N_G[v] - Q_{i+1}$. Since $|N_G[v]| - |Q_i| - |Q_{i+1}| \le \Delta + 1 - \lfloor \frac{1}{3}\Delta \rfloor + 66 \le \lceil \frac{2}{3}\Delta \rceil + 67$, one of the *larger* colors must be on an inner vertex of Q_i, which without loss of generality, we can assume is $q_{i,2}$. Because t is not colored, we must have all the $\lceil \frac{5}{3}\Delta \rceil + 78$ colors at $N_{G^2}(t)$. Otherwise we could assign a color to t. As there are at most $\Delta + 6$ colors, all from the *smaller* colors, at $N_G[u_{i+1}] \cup X(t)$, all the *larger* colors must be in $N_G[u_i]$, too. Let L be the number of larger colors. Therefore, the number of forbidden colors for $q_{i,2}$ that are not from the larger colors, is at most $d(u_i) - L + d(u_{i+1}) - L \le 2\Delta - 2L$. By considering the vertices at distance exactly two of $q_{i,2}$ that have a larger color and noting that $q_{i,2}$ has a larger color too, the total number of forbidden colors for $q_{i,2}$ is at most $2\Delta - L \le \lfloor \frac{4}{3}\Delta \rfloor - 70$, and so we can assign a new color to $q_{i,2}$ and assign the old color of $q_{i,2}$, which is one of the *larger* colors and is not in $N_{G^2}(t) - \{q_{i+1,2}\}$, to t and extend the coloring to G, a contradiction. ■

3 Discharging rules

We give an initial charge of $d_{G''}(v) - 6$ units to each vertex v. Using Euler's formula, $|V| - |E| + |F| = 2$, and noting that $3|F(G'')| = 2|E(G'')|$, it is straightforward to check that:

$$\sum_{v \in V} (d_{G''}(v) - 6) = 2|E(G'')| - 6|V| + 4|E(G'')| - 6|F(G'')| = -12. \tag{1}$$

By these initial charges, the only vertices that have negative charges are 4- and 5-vertices, which have charges -2 and -1, respectively. The goal is to show that, based on the assumption that G is a minimum counter-example, we can send charges from other vertices to $\le$5-vertices such that all the

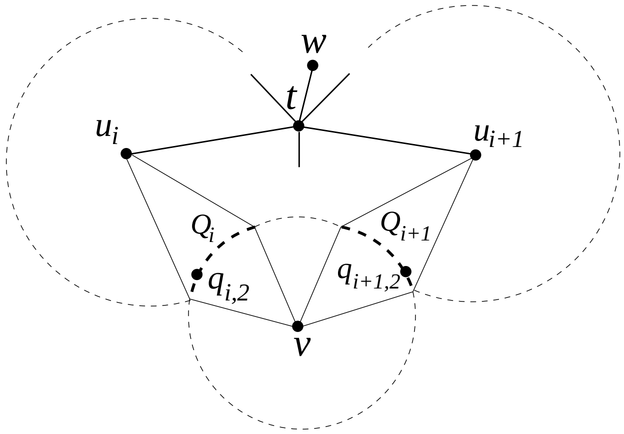

Figure 3: Configuration of Lemma 2.14

vertices have non-negative charge, which is of course a contradiction since the total charge must be negative by equation (1).

We call a vertex v *pseudo-big* (in G'') if v is big (in G) and $d_{G''}(v) \geq d_G(v) - 11$. Note that a pseudo-big vertex is also a big vertex, but a big vertex might or might not be a pseudo-big vertex. Before explaining the discharging rules, we need a few more notations.

Suppose that $v, x_1, x_2, \ldots, x_k, u$ is a sequence of vertices such that v is adjacent to x_1, x_i is adjacent to x_{i+1}, $1 \leq i < k$, and x_k is adjacent to u.

Definition: By *"v sends c units of charge through $x_1, \ldots, x_k$ to u"* we mean v sends c units of charge to x_1, it passes the charge to x_2... etc, and finally x_k passes the charge to u. In this case, we also say *"v sends c units of charge through x_1"* and *"u gets c units of charge through x_k"*. In order to simplify the calculations of the total charges on vertex x_i, $1 \leq i \leq k$, we do not take into account the charges that only pass through x_i.

In discharging phase, a big vertex v of G:

1) Sends 1 unit of charge to each 4-vertex u in $N_{G''}(v)$.

2) Sends $\frac{1}{2}$ unit of charge to each 5-vertex u in $N_{G''}(v)$.

In addition, if v is a big vertex and u_0, u_1, u_2, u_3, u_4 are consecutive neighbors of v in clockwise or counter-clockwise order, where $d_{G''}(u_0) = 4$, then:

3) If $d_{G''}(u_1) = 5$, u_2 is big, $d_{G''}(u_3) = 4$, $d_{G''}(u_4) \geq 5$, and the neighbors of u_1 in clockwise or counter-clockwise order are v, u_0, x_1, x_2, u_2 then v sends $\frac{1}{2}$ to x_1 through u_2, u_1.

4) If $d_{G''}(u_1) = 5$, $5 \leq d_{G''}(u_2) \leq 6$, $d_{G''}(u_3) \geq 7$, and the neighbors of u_1 in clockwise or counter-clockwise order are v, u_0, x_1, x_2, u_2 then v sends $\frac{1}{2}$ to x_1 through u_3, u_2, u_1.

5) If $d_{G''}(u_1) = 5$, u_2 is big, $d_{G''}(u_3) \geq 5$, and the neighbors of u_1 in clockwise or counter-clockwise order are v, u_0, x_1, x_2, u_2 then v sends $\frac{1}{4}$ to x_1 through u_2, u_1.

6) If $d_{G''}(u_1) = 6$, $d_{G''}(u_2) \leq 5$, $d_{G''}(u_3) \geq 7$, and the neighbors of u_1 in clockwise or counter-clockwise order are $v, u_0, x_1, x_2, x_3, u_2$ then v sends $\frac{1}{2}$ to x_1 through u_1.

7) If $d_{G''}(u_1) = 6$, $d_{G''}(u_2) \geq 6$, and the neighbors of u_1 in clockwise or counter-clockwise order are $v, u_0, x_1, x_2, x_3, u_2$ then v sends $\frac{1}{4}$ to x_1 through u_1.

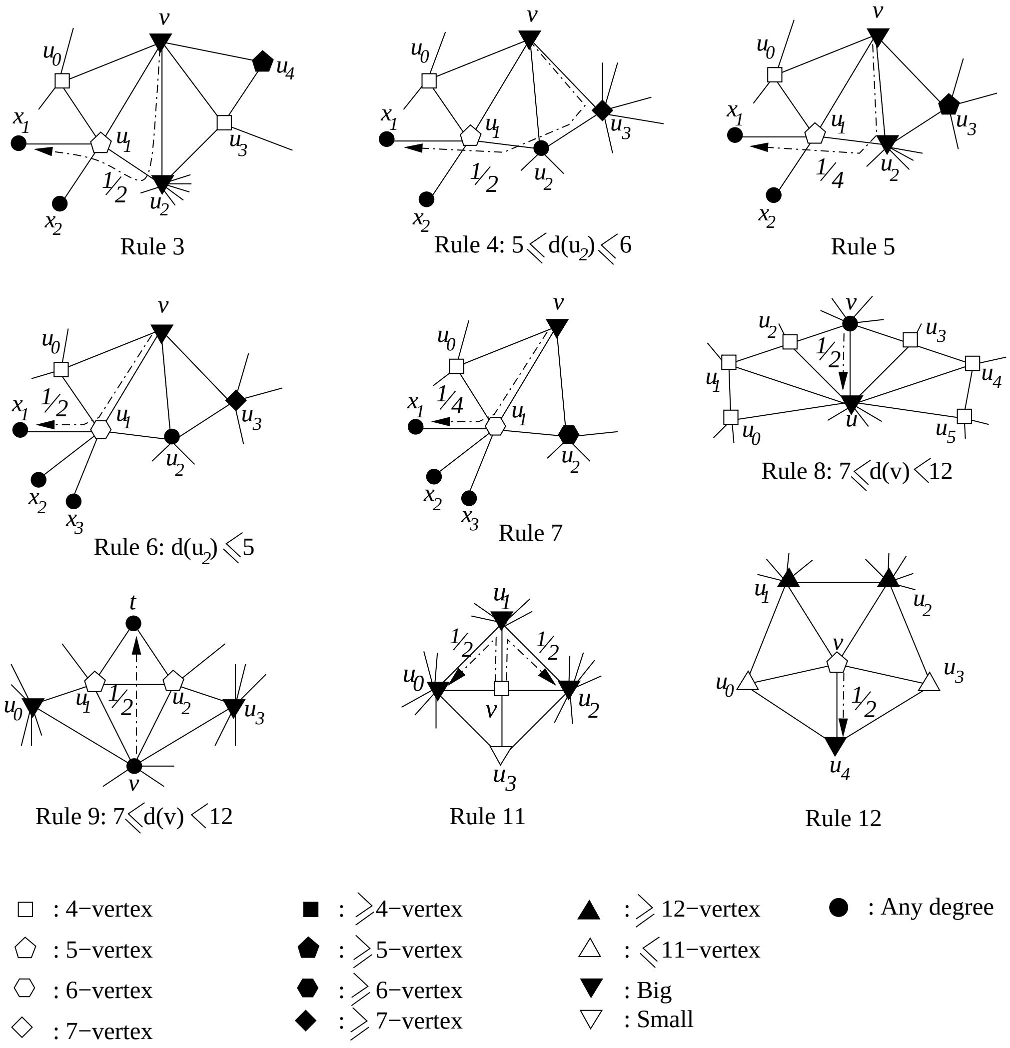

Figure 4: Discharging rules

If $7 \leq d_{G''}(v) < 12$ then:

8) If u is a big vertex and $u_0, u_1, u_2, v, u_3, u_4, u_5$ are consecutive neighbors of u where all $u_0, u_1, u_2, u_3, u_4, u_5$ are 4-vertices then v sends $\frac{1}{2}$ to u.

9) If u_0, u_1, u_2, u_3 are consecutive neighbors of v, such that $d_{G''}(u_1) = d_{G''}(u_2) = 5$, u_0 and u_3 are big, and t is the other common neighbor of u_1 and u_2 (other than v), then v sends $\frac{1}{2}$ to t.

Every $\geq$12-vertex v of G'' that was not big in G:

10) Sends $\frac{1}{2}$ to each of its neighbors.

A $\leq$5-vertex v sends charges as follows:

11) If $d_{G''}(v) = 4$ and its neighbors in clockwise order are u_0, u_1, u_2, u_3, such that u_0, u_1, u_2 are big in G and u_3 is small, then v sends $\frac{1}{2}$ to each of u_0 and u_2 through u_1.

12) If $d_{G''}(v) = 5$ and its neighbors in clockwise order are u_0, u_1, u_2, u_3, u_4, such that $d_{G''}(u_0) \leq 11$, $d_{G''}(u_1) \geq 12$, $d_{G''}(u_2) \geq 12$, $d_{G''}(u_3) \leq 11$, and u_4 is big, then v sends $\frac{1}{2}$ to u_4.

From now on, by "the total charge sent from v to one of its neighbors u", we mean the total charge sent from v to u or through u. Similarly, by "the total charge v received from u", we mean the total charge sent from or through u to v.

Lemma 3.1 *Every big vertex v sends at most $\frac{1}{2}$ to every $5-$ or 6-vertex in $N_{G''}(v)$.*

Proof: For any $5-$ or 6-vertex u, v sends charges to u by at most one rule. ■

Lemma 3.2 *If v is big and u_0, u_1, u_2, u_3, u_4 are consecutive neighbors of v in counter-clockwise order, such that $d_{G''}(u_2) \geq 7$ then v sends at most $\frac{1}{2}$ through u_2, or sends 1 through u_2 and $d_{G''}(u_0) = d_{G''}(u_4) = 5$ and u_1 and u_3 are $5-$ or 6-vertices.*

Proof: If u_2 is big and one of rules 3 or 5 applies then it is easy to verify that it is the only rule by which u_2 gets charge from v. If u_1 and u_3 are both 5-vertices then rule 5 may apply twice, one for sending charge to a neighbor of u_1 and one for sending charge to a neighbor of u_3, so overall u_2 gets at most $\frac{1}{2}$ from v. It is straightforward to check that there is no configuration in which we can apply rule 3 twice.

The only other way for v to send charge to u_2 is by rule 4. Note that if this rule applies then none of the other rules apply. Also, v can send charge to u_2 twice by rule 4 since it might apply under clockwise and counter-clockwise orientations of neighbors of v. This happens if $d_{G''}(u_0) = 5$, $5 \leq d_{G''}(u_1) \leq 6$, $5 \leq d_{G''}(u_3) \leq 6$, $d_{G''}(u_4) = 5$, v, u_1, x_2, x_1, x_0 are neighbors of u_0 in clockwise order where $d_{G''}(x_0) = 4$, and y_0, y_1, y_2, u_3, v are neighbors of u_4 in clockwise order where $d_{G''}(y_0) = 4$. In this case v sends $\frac{1}{2}$ to x_1 through u_2, u_1, u_0 and sends $\frac{1}{2}$ to y_1 through u_2, u_3, u_4, and this is the only configuration in which v sends charge to u_2 twice. This proves the lemma. ■

Lemma 3.3 *Every vertex v that is not big in G will have non-negative charge.*

Proof: By Lemma 2.3 every 4-vertex gets a total of at least 2 units of charge by rule 1 and each 5-vertex gets a total of at least 1 unit of charge by rule 2. Also, the $\leq$5-vertices that send charges by rules 11 and 12 will have non-negative charges, since they are adjacent to at least three $\geq$12-vertices. If $d_{G''}(v) \geq 12$ then it sends $\frac{1}{2} d_{G''}(v) \leq d_{G''}(v) - 6$ by rule 10 and so will have non-negative charge. It is straightforward to verify that there is no configuration in which a 7-vertex v sends more than 1 unit of charge in rules 8 or 9. Finally, it is not difficult to see that by rules 8 and 9, a vertex sends at most $\frac{1}{2}$ for every two neighbors that it has. So if $8 \leq d_{G''}(v) < 12$ it sends at most $\frac{d_{G''}(v)}{4} \leq d_{G''}(v) - 6$, and therefore it will have non-negative charge in any of these cases. Finally, rules 3 to 7 do not apply to the vertices that are not big in G. ■

Lemma 3.4 *Every big vertex v that is not pseudo-big will have non-negative charge.*

Proof: Suppose that v is such a vertex. So $d_{G''}(v) \leq d_G(v) - 12$ and therefore v was involved in at least 12 switching operations, in each of which the edge between v and another big vertex of G was removed. Since G' is simple, these big vertices are distinct. Call them $y_1, y_2, \ldots, y_k$, where $k \geq 12$, in clockwise order. Let $x_i z_i$ be the edge that was added during the switching operation that removed $v y_i$, and the order of x_i's and z_i's is such that x_i comes before z_i in clockwise order. Note that all x_i's and all z_i's are neighbors of v in G'' (see Figure 5).

Let us call the vertices between z_i and x_{i+1}, $u_{i,1}, u_{i,2}, \ldots, u_{i,l_i}$, starting from z_i. For consistency, let us relabel temporarily z_i and x_{i+1} to $u_{i,0}$ and u_{i,l_i+1}, respectively. Recall that $k \geq 12$ and v sends a total of no more than 1 to any vertex. Thus, in order to show that v sends no more than its initial charge of $d_{G''}(v) - 6$, it is enough to show that for each $1 \leq i \leq k$, either

(a) v sends a total of at most $\frac{1}{2}$ to a vertex from z_i to x_{i+1}; or

(b) v sends a total of at most $l_{i+1} + 1$ to the $l_{i+1} + 2$ vertices from z_{i+1} to x_{i+2}.

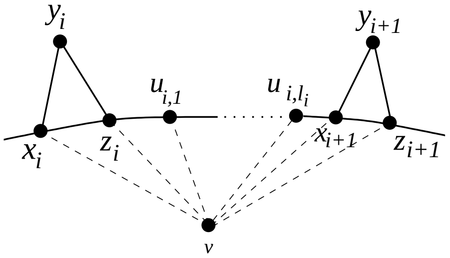

Figure 5: Configuration of Lemma 3.4

First we show that there is at least one $\geq$5-vertex in $u_{i,0}, \ldots, u_{i,l_i+1}$, for each $1 \leq i \leq k$. If $u_{i,0}$ is a 4-vertex we must have $y_i u_{i,1} \in G''$, because G'' is a triangulation. Assuming that $u_{i,1}$ is a 4-vertex we must have $y_i u_{i,2} \in G''$ and so on, until we have $y_{i+1} u_{i,l_i+1} \in G''$ and so u_{i,l_i+1} will be a $\geq$5-vertex. So for every $1 \leq i \leq k$, there is a $\geq$5-vertex between z_i and x_{i+1}; take any such vertex and call it u_{i,j_i}. By Lemmas 3.1 and 3.2 and rule 10, it can be seen that v sends a total of at most $\frac{1}{2}$ to u_{i,j_i}, unless $7 \leq d_{G''}(u_{i,j_i}) \leq 11$.

So assume that $7 \leq d_{G''}(u_{i,j_i}) \leq 11$ and v sends 1 through u_{i,j_i}. By Lemma 3.2 both of the neighbors of v before and after u_{i,j_i} are 5$-$ or 6-vertices and so to each of them v sends a total of at most $\frac{1}{2}$. If $z_i \neq x_{i+1}$ then at least one of these lies between z_i and x_{i+1} and therefore we satisfy (a) above.

So we can assume $z_i = x_{i+1}$. Thus $u_{i,j_i} = z_i = x_{i+1}$, and so (i) $5 \leq d_{G''}(z_{i+1}) \leq 6$, and (ii) $d_{G''}(u_{i+1,1}) = 5$ if $z_{i+1} \neq x_{i+2}$, or $d_{G''}(z_{i+2}) = 5$ otherwise. First assume that $z_{i+1} = x_{i+2}$. Now if $d_{G''}(z_{i+1}) = 5$ then v gets back $\frac{1}{2}$ from z_{i+1} by rule 12 and so sends a total of at most 0 to it. If $d_{G''}(z_{i+1}) = 6$ then it is easy to verify that v sends nothing to z_{i+1} by any rule and so sends a total of at most 0 to it. Either way, we satisfy (b), above.

Otherwise if $z_{i+1} \neq x_{i+2}$ then there are at least two vertices between $z_{i+1}, \ldots, x_{i+2}$, that are 5$-$ or 6-vertices and so to each of them v sends a total of at most $\frac{1}{2}$. Therefore we satisfy (b), above. ■

So the only vertices that may have negative charges are pseudo-big vertices in G''. Assume that v is a pseudo-big vertex of G'' whose neighborhood sequence in clockwise order is $x_1, \ldots, x_k$. Let m be the number of maximal sparse segments of the neighborhood of v and call these segments $Q_1, Q_2, \ldots, Q_m$ in clockwise order. Also, let R_i be the sequence of neighbors of v between the last vertex of Q_i and the first vertex of Q_{i+1}, where $Q_{m+1} = Q_1$. If $m = 0$ then we define R_1 to be equal to $N_{G''}(v)$

Lemma 3.5 *Let $R = x_a, \ldots, x_b$, where R is one of $R_1, \ldots, R_m$. Then v sends at total of at most $\lceil \frac{5|R|}{6} \rceil$ to the vertices of R.*

Proof: Since R does not overlap with any maximal sparse segment, from every three consecutive vertices x_i, x_{i+1}, x_{i+2} in R (where we consider the neighbors cyclicly if $R = N_{G''}(v)$), at least one of them is a $\geq$5-vertex. Either v sends a total at most $\frac{1}{2}$ to this vertex, or v sends 1 and by lemma 3.2 the two vertices before that and the two vertices after that are 5$-$ or 6-vertices and so v sends to each of them a total of at most $\frac{1}{2}$. Thus in either case v sends a total of at most $\frac{5}{2}$ to every three consecutive vertices of R and so sends at most $\lceil \frac{5}{6}(b - a + 1) \rceil = \lceil \frac{5|R|}{6} \rceil$ to the vertices of R. ■

Lemma 3.6 *Suppose that $m \geq 4$. Then for every $1 \leq i \leq m$ either v sends at most $|R_i| - \frac{3}{2}$ to R_i, or v sends at most $|R_i| - 1$ to R_i and*

$$|Q_i| + |Q_{i+1}| \leq \lfloor \frac{1}{3} \Delta \rfloor - 67. \tag{2}$$

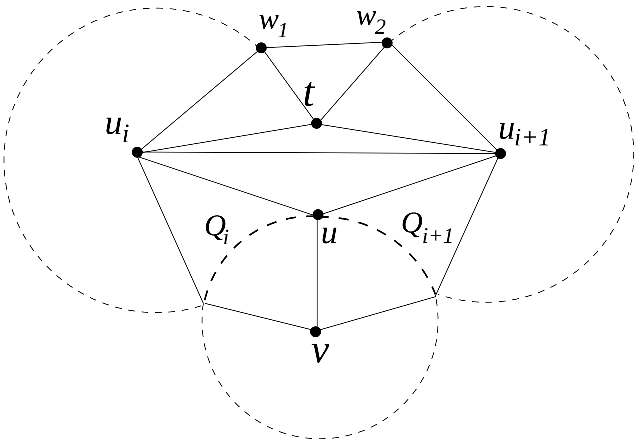

Figure 6: The first configuration in Lemma 3.6

Proof: We consider different cases based on $|R_i|$:

$|R_i| = 1$: Assume that $R_i = u$. Since u is the only vertex between two maximal sparse segments, $d_{G''}(u) \geq 5$. First let $d_{G''}(u) = 5$. Since Q_i and Q_{i+1} are sparse segments there must be two big vertices u_i and u_{i+1} that are connected to all the vertices of Q_i and Q_{i+1}, respectively. Also, u must be connected to these two vertices, because G'' is a triangulation (see Figure 6).

Note that by rule 12 v gets back the $\frac{1}{2}$ charge it had sent to u. So v is sending a total of at most 0, so far. Let t be the other vertex that makes a triangle with edge u_iu_{i+1}. Assume that $d_{G''}(t) = 4$, and w_1, w_2 are the two neighbors of t other than u_i and u_{i+1}. If $d_{G''}(w_1) \leq 4$ and $d_{G''}(w_2) \leq 4$ then since Q_i and Q_{i+1} are sparse segments and u_i and u_{i+1} are big vertices in G, by Lemma 2.14 Equation (2) holds. Otherwise, assume that $d_{G''}(w_1) \geq 5$. Then by rule 3 u_i will be sending extra $\frac{1}{2}$ to v through u. So overall, v sends a total of $-\frac{1}{2}$ to u. If $d_{G''}(t) \geq 5$ then each of u_i and u_{i+1} will send an extra $\frac{1}{4}$ to v through u by rule 5 and therefore v sends a total of $-\frac{1}{2}$ to u.

Now assume $d_{G''}(u) = 6$ and that the neighbors of u are v, u_i, u_{i+1}, t and the end vertices of Q_i and Q_{i+1}. Note that in this case v will send nothing to u. Assume that $d_{G''}(t) = 4$ and its other neighbor is w. If $d_{G''}(w) \leq 6$ then by Lemma 2.14 Equation (2) holds. Otherwise, $d_{G''}(w) \geq 7$ and so each of u_i and u_{i+1} sends an extra $\frac{1}{2}$ to v through u by rule 6 and so v sends a total of -1 to u. If $d_{G''}(t) = 5$ and its other neighbors are w_1 and w_2 then either $d_{G''}(w_1) \leq 6$ and $d_{G''}(w_2) \leq 6$ and we can apply Lemma 2.14 to get Equation (2), or at least one of w_1 and w_2 has degree ≥ 7 and so one of u_i or u_{i+1} will send an extra $\frac{1}{2}$ unit of charge to v through u by rule 6 and so v sends a total of $-\frac{1}{2}$ to u. If $d_{G''}(t) \geq 6$ then both u_i and u_{i+1} send an extra $\frac{1}{4}$ charge to v through u by rule 7. So v sends a total of $-\frac{1}{2}$ to u.

If $7 \leq d_{G''}(u) \leq 11$, or $12 \leq d_{G''}(u)$ and u was not big in G, then u sends $\frac{1}{2}$ to v by rules 8 or 10 and so v sends a total of $-\frac{1}{2}$ to u.

If u was big in G then by rule 11 v gets back $\frac{1}{2}$ through u for each of the end vertices of Q_i and Q_{i+1} that are adjacent to u, and so v sends a total of at most -1 to u.

$|R_i| = 2$: Assume that $R_i = v_1, v_2$. If $d_{G''}(v_1) \geq 6$ or $d_{G''}(v_2) \geq 6$ then it is easy to check that v sends nothing to one of v_1, v_2 and sends at most $\frac{1}{2}$ to the other one, or sends $\frac{1}{4}$ to each, and so sends at most $\frac{1}{2}$ to R_i. So let us assume that $d_{G''}(v_1) = d_{G''}(v_2) = 5$ and let t be the other vertex which makes a triangle with v_1, v_2. Note that v sends only $\frac{1}{2}$ to each of v_1 and v_2.

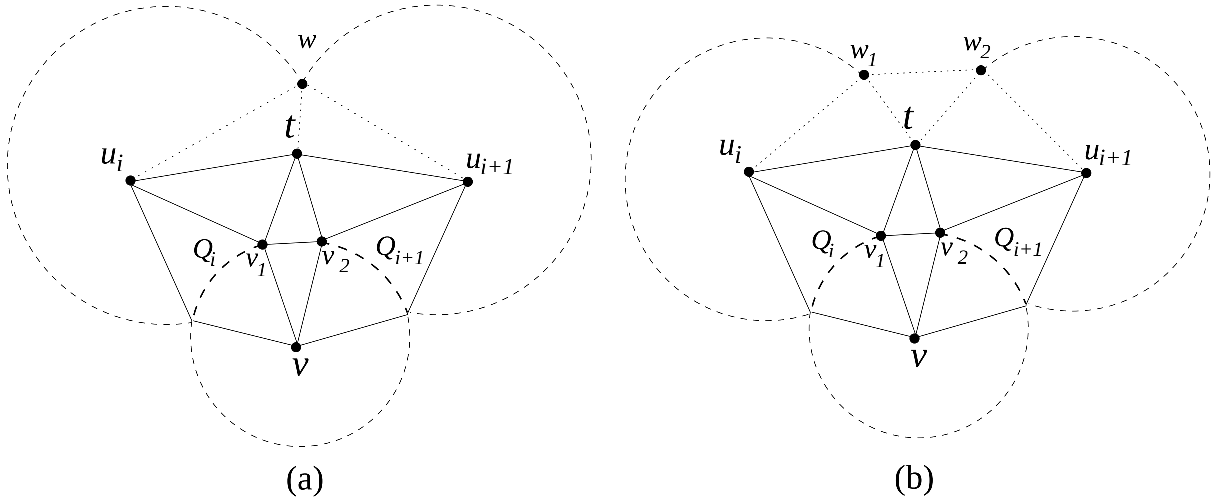

Figure 7: Two other configurations for Lemma 3.6

If $d_{G''}(t) = 4$ then we can apply Lemma 2.14 and get Equation (2). Let $d_{G''}(t) = 5$ and call the other neighbor of t (other than u_i, v_1, v_2, u_{i+1}), w (see Figure 7(a)). If $d_{G''}(w) \leq 6$ then we can apply Lemma 2.14 to get Equation (2). Otherwise $d_{G''}(w) \geq 7$ and by rule 4 u_i and u_{i+1} each send an extra $\frac{1}{2}$ to v (through v_1 and v_2, respectively) and therefore v sends a total of at most 0 to R_i. Now assume that $d_{G''}(t) = 6$ and its neighbors are $w_1, w_2, u_i, u_{i+1}, v_1, v_2$ (see Figure 7(b)). If $d_{G''}(w_1) \leq 6$ and $d_{G''}(w_2) \leq 6$ then by Lemma 2.14 we have Equation (2). Otherwise, at least one of w_1 or w_2 is a $\geq$7-vertex and so one of u_i or u_{i+1} sends an extra $\frac{1}{2}$ to v (through v_1 or v_2) by rule 4 and therefore v sends a total of at most $\frac{1}{2}$ to R_i. If $7 \leq d_{G''}(t) < 12$ then t sends $\frac{1}{2}$ to v by rule 9 and so v sends a total of at most $\frac{1}{2}$ to R_i. If $12 \leq d_{G''}(t)$ then v gets back the $\frac{1}{2}$ it had sent to each of v_1 and v_2 by rule 12 and so sends a total of at most o to R_i.

$|R_i| \geq 3$: If there is no 4-vertex in R_i then they are all $\geq$5-vertices and by Lemmas 3.1 and 3.2 v sends a total of at most $|R_i| - \frac{3}{2}$ to R_i. If $|R_i| \geq 5$, since R_i cannot have three consecutive 4-vertices, we must have at least three $\geq$5-vertices and again by Lemmas 3.1 and 3.2 v sends a total of at most $|R_i| - \frac{3}{2}$. So consider the case that $R_i = v_1, v_2, v_3, v_4$, $d_{G''}(v_1) \geq 5$, $d_{G''}(v_4) \geq 5$, and $d_{G''}(v_2) = d_{G''}(v_3) = 4$ (exactly the same argument works for the case that $|R_i| = 3$ and $v_2 = v_3$). There must be a big vertex w, other than v, connected to all the vertices of R_i. If $d_{G''}(v_1) = 5$ then v gets back $\frac{1}{2}$ from v_1 by rule 12 and so sends a total of at most 0 to v_1. If $d_{G''}(v_1) \geq 6$ it can be verified that v sends nothing to v_1 by any rule. Since v sends a total of at most $\frac{1}{2}$ to v_2 and at most 1 to any vertex, it sends a total of at most $|R_i| - \frac{3}{2}$ to R_i.

■

Lemma 3.7 *Every pseudo-big vertex v has non-negative charge.*

Proof: Recall that the initial charge of v was $d_{G''}(v) - 6$ and that v sends a total of at most 1 to any neighbor. We will show that v sends a total of less than 1 to each of several neighbors, enough so that the total charge that v loses is at most $d_{G''}(v) - 6$. We consider different cases based on the value of m, the number of maximal sparse segments of v. Recall that by Observation 2.2 we can assume that $\Delta \geq 160$.

$m = 0$: Since v is pseudo-big $d_{G''}(v) \geq d_G(v) - 11 \geq 36$. Using Lemma 3.5 v will send at most $\lceil \frac{5}{6} d_{G''}(v) \rceil \leq d_{G''}(v) - 6$ and therefore will have non-negative charge.

$1 \leq m \leq 3$: By lemma 2.13 and definition of a pseudo-big vertex:

- $m = 1$: Then:

$$\begin{aligned} |R_1| &= d_{G''}(v) - Q_1 \\ &\geq d_{G''}(v) - d_G(v) + \lceil \frac{2}{3}\Delta \rceil + 73 \\ &\geq \lceil \frac{2}{3} \times 160 \rceil + 62 \\ &\geq 36. \end{aligned}$$

  So by Lemma 3.5 v sends a total of at most $|R_1| - 6$ to R_1.

- $m = 2$: Then:

$$\begin{aligned} \sum_{1 \leq i \leq 2} |R_i| = d_{G''}(v) - \sum_{1 \leq i \leq 2} |Q_i| \\ \geq d_{G''}(v) - 2d_G(v) + 2 \times \lceil \frac{2}{3}\Delta \rceil + 146 \\ \geq \lceil \frac{1}{3}\Delta \rceil + 135 \\ \geq 36. \end{aligned}$$

  So by Lemma 3.5 v sends a total of at most $|R_1 \cup R_2| - 6$ to $R_1 \cup R_2$.

- $m = 3$: Then:

$$\begin{aligned} \sum_{1 \leq i \leq 3} |R_i| = d_{G''}(v) - \sum_{1 \leq i \leq 3} |Q_i| \\ \geq d_{G''}(v) - 3d_G(v) + 3 \times \lceil \frac{2}{3}\Delta \rceil + 219 \\ \geq 208. \end{aligned}$$

  Therefore by Lemma 3.5 v sends at most $|R_1 \cup R_2 \cup R_3| - 6$ to $R_1 \cup R_2 \cup R_3$.

$m = 4$: If v sends a total of at most $|R_i| - \frac{3}{2}$ to each R_i then we are done. Otherwise by Lemma 3.6, we can assume without loss of generality that v sends a total of $|R_1| - 1$ to R_1 and that Equation (2) holds for Q_1 and Q_2. Therefore using Lemma 2.13:

$$\begin{aligned} |R_2| + |R_3| + |R_4| &\geq d_{G''}(v) - (|Q_1| + |Q_2|) - |Q_3| - |Q_4| \\ &\geq d_{G''}(v) - \lfloor \frac{1}{3}\Delta \rfloor + 67 - 2(d_G(v) - \lceil \frac{2}{3}\Delta \rceil - 73) \\ &\geq \Delta - 2d_G(v) + d_{G''}(v) + 213 \\ &\geq 202. \end{aligned}$$

Thus by Lemma 3.5, v sends a total of at most $|R_2 \cup R_3 \cup R_4| - 5$ to $R_2 \cup R_3 \cup R_4$.

$m = 5$: v sends a total of at most $|R_i| - 1$ to each R_i, by Lemma 3.6. If there are at least two values of i such that v sends a total of at most $|R_i| - \frac{3}{2}$ to R_i then we are done. Otherwise there is at most one R_i, say R_5, to which v sends a total of at most $|R_i| - \frac{3}{2}$. Therefore Equation (2) must hold for $|Q_1| + |Q_2|$ and $|Q_3| + |Q_4|$, i.e:

$$|Q_1| + |Q_2| + |Q_3| + |Q_4| \leq 2 \times \lfloor \frac{1}{3}\Delta \rfloor - 134.$$

Then using Lemma 2.13:

$$\begin{aligned}\sum_{1\le i\le 5}|R_i| &\ge d_{G''}(v)-d_G(v)+\lceil\frac{2}{3}\Delta\rceil+73-2\times\lfloor\frac{1}{3}\Delta\rfloor+134\\ &\ge 196.\end{aligned}$$

Therefore by Lemma 3.5, v sends a total of at most $|R_1\cup R_2\cup R_3\cup R_4\cup R_5|-6$ to $R_1\cup R_2\cup R_3\cup R_4\cup R_5$.

$m\ge 6$: v sends at most $|R_i|-1$ to each R_i, by lemma 3.6. So we are done.

■

Proof of Theorem 1.3: By Lemmas 3.3, 3.4, and 3.7 every vertex of G'' will have non-negative charge, after applying the discharging rules. Therefore the total charge over all the vertices of G'' will be non-negative, but this is contradicting equation (1). This disproves the existence of G, a minimum counter-example to the theorem.

Remark: Using a more careful analysis one can prove the bound $\lfloor\frac{4|R|}{5}\rfloor$ in Lemma 3.5 which in turn can be used to prove $\chi(G^2)\le\lfloor\frac{5}{3}\Delta\rfloor+61$. By being even more careful throughout the analysis one can probably prove the bound $\chi(G^2)\le\lfloor\frac{5}{3}\Delta\rfloor+51$ or even maybe with 30 or 20 instead of 51.

4 A better bound for graphs with large Δ

The steps of the proof of Theorem 1.4 are very similar to those of Theorem 1.3, we only have to modify a few lemmas and redo the calculations. For these lemmas, since the proofs are almost identical and do not need any new ideas, we only state the lemmas without giving further proofs. Let G be a minimum counter-example to Theorem 1.4 such that $\Delta\ge 241$.

Lemma 4.1 *For every vertex v of G, if there exists a vertex $u\in N(v)$, such that $d_G(v)+d_G(u)\le\Delta+2$ then $d_{G^2}(v)\ge\lfloor\frac{5}{3}\Delta\rfloor+25$.*

We construct the triangulated graphs G' and then G'' exactly in the same way. Lemmas 2.3 to 2.12 are still valid. The analogous of Lemmas 2.13 and 2.14 will be as follows.

Lemma 4.2 $|Q_i|\le d_G(v)-\lceil\frac{2}{3}\Delta\rceil-20$, *for $1\le i\le m$.*

Lemma 4.3 *Under the same assumption as in Lemma 2.14, we have:*

$$|Q_i|+|Q_{i+1}|\le\lfloor\frac{1}{3}\Delta\rfloor-14.$$

We apply the same initial charges and discharging rules. Again, all Lemmas 3.1 to 3.5 hold. The analogue of lemma 3.6 will be:

Lemma 4.4 *Suppose that $m\ge 4$. Then for every $1\le i\le m$ either v sends a total of at most $|R_i|-\frac{3}{2}$ to R_i, or v sends a total of at most $|R_i|-1$ to R_i and*

$$|Q_i|+|Q_{i+1}|\le\lfloor\frac{1}{3}\Delta\rfloor-14.$$

Now it is straightforward to do the calculations of Lemma 3.7 with the above values to see that it holds in this case too. This will complete the proof of Theorem 1.4.

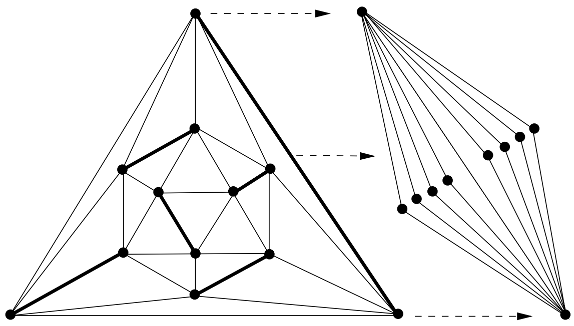

Figure 8: The icosahedron and the modified graph

5 On possible asymptotic improvement of Theorem 1.3

In this section, we only focus on the asymptotic order of the bounds, i.e. the coefficient of Δ. The results of [1] and [4, 5] are essentially based on showing that in a planar graph G, there exists a vertex v such that $d_{G^2}(v) \leq \lceil \frac{9}{5}\Delta \rceil + O(1)$ ([5] actually obtains a slightly weaker, but still sufficient bound.) However, as pointed out in [1], this is the best possible bound on the minimum degree of a vertex in G^2. That is, there are 2-connected planar graphs in which every vertex v satisfies $d_{G^2}(v) \geq \lceil \frac{9}{5}\Delta \rceil$. One of these extremal graphs can be obtained from a icosahedron, by taking a perfect matching of it, adding $k-1$ paths of length two parallel to each edge of the perfect matching, and replacing every other edge of the icosahedron by k parallel paths of length two (see Figure 8).

Therefore, by only bounding the minimum degree of G^2 we cannot improve the bound $\frac{9}{5}\Delta + O(1)$, asymptotically. This is the reason we introduced the reducible configuration of Lemma 2.14. We proved that any planar graph G either has a cut-vertex, or a vertex v such that $d_{G^2}(v) \leq \frac{5}{3}\Delta + O(1)$, or has the configuration of Lemma 2.14.

But there are graphs that are extremal for this new set of reducible configurations in the following sense: these graphs do not have a cut-vertex, do not have a vertex v with $d_{G^2}(v) \leq \frac{5}{3}\Delta$, and do not have the configuration of Lemma 2.14. For an odd value of k, one of these graphs is shown in Figure 9. To interpret this figure, we have to join the three copies of v_8 and remove the multiple edges (we draw the graph in this way for clarity). Also, the dashed lines represent sequences of consecutive 4-vertices. Around each of $v_1, \ldots, v_4$ there are $3k-6$ such vertices. So, $d(v_1) = d(v_2) = d(v_3) = d(v_4) = 3k$, $d(v_5) = d(v_6) = d(v_7) = d(v_8) = 3k+3$, $\Delta = 3k+3$, and for any vertex $v \in G$: $d_{G^2}(v) \geq 5k+3$ (with equality holding for $v \in \{v_1, \ldots, v_4\}$). The minimum degree of G^2 is $\frac{5}{3}\Delta + O(1)$ and it is easy to see that G does not have the configuration of Lemma 2.14. Therefore, using reducible configurations similar to those of Section 2 the best asymptotic bound that we can achieve is $\frac{5}{3}\Delta + O(1)$. So we need another reducible configuration to improve the multiplicative constant $\frac{5}{3}$.

6 Generalization to $L(p,q)$-labeling

In this section we prove Theorem 1.6. As we said before, the upper bound $3p$ for λ_0^p of a planar graph follows from the Four Color Theorem (if we use colors from $\{0, p, 2p, 3p\}$). So let's assume that $q \geq 1$. We prove the following theorem:

Theorem 6.1 *For any planar graph G and positive integer p:*

$$\lambda_1^p(G) \leq \lceil \frac{5}{3}\Delta \rceil + 18p + 59.$$

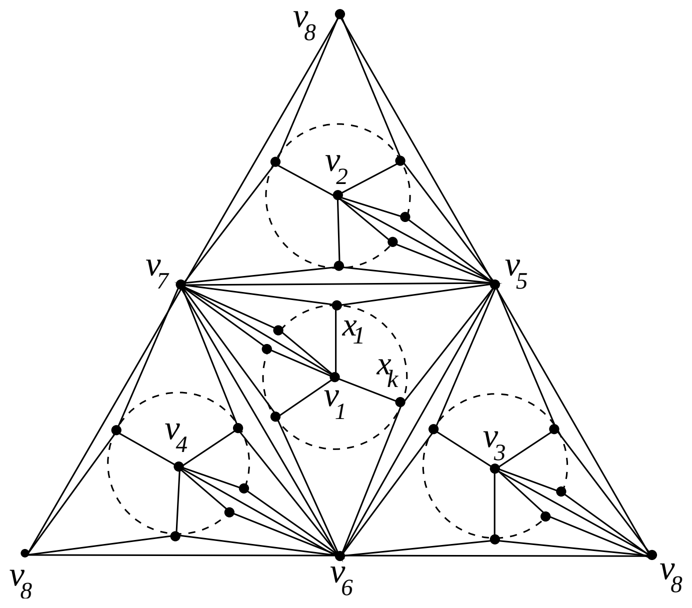

Figure 9: The graph obtained based on a tetrahedron

Assuming Theorem 6.1, we can prove Theorem 1.6 as follows:

Proof of Theorem 1.6: Let $c = \lceil \frac{5}{3}\Delta \rceil + 18\lceil \frac{p}{q} \rceil + 60$. By Theorem 6.1, there is an $L(\lceil \frac{p}{q} \rceil, 1)$-labeling of G with colors in $\{0, \ldots, c-1\}$. Consider such a labeling and multiply every color by q. This yields an $L(p, q)$-labeling of G with colors in $\{0, \ldots, q(c-1)\}$. Noting that $q(c-1) \leq q\lceil \frac{5}{3}\Delta \rceil + 18p + 77q - 18$ completes the proof. ■

In the rest of this section we give the proof of Theorem 6.1. The steps of the proof are very similar to those of proof of Theorem 1.3. Let G be a planar graph which is a counter-example to Theorem 6.1 with the minimum number of vertices. We set

$$C = \lceil \frac{5}{3}\Delta \rceil + 18p + 60$$

and throughout this section we use colors from $\{0, \ldots, C-1\}$. Recall that a vertex is said to be big if $d_G(v) \geq 47$.

Lemma 6.2 *Suppose that v is a $\leq$5-vertex in G. If there exists a vertex $u \in N(v)$, such that $d_G(v) + d_G(u) \leq \Delta + 2$ then $d_{G^2}(v) \geq d_G(v) + \lceil \frac{5}{3}\Delta \rceil + 73$.*

Proof: Assume that v is such a vertex and assume that $d_{G^2}(v) < d_G(v) + \lceil \frac{5}{3}\Delta \rceil + 73$. Contract v on edge vu. The resulting graph has maximum degree at most Δ and because G was a minimum counter-example, the new graph has an $L(p, 1)$-labeling with at most c colors. Now consider such a labeling induced on G, in which every vertex other than v is colored. Every vertex at distance (exactly) two of v in G forbids 1 color for v, and every vertex in $N(v)$ forbids at most $2p-1$ colors for v. So the total number of forbidden colors for v, i.e. the colors that we cannot assign to v, is at most:

$$\begin{array}{rcl} d_G(v)(2p-1) + d_{G^2}(v) - d_G(v) & < & 10p - 5 + \lceil \frac{5}{3}\Delta \rceil + 73 \\ & = & \lceil \frac{5}{3}\Delta \rceil + 10p + 68 \\ & \leq & C. \end{array}$$

The last inequality follows from the assumption that $p \geq 1$. Therefore, there is still at least one color available for v whose absolute difference from its neighbors in G^2 is large enough and so we can extend the coloring to G. ■

Observation 6.3 *By Theorem 1.5 we can assume that* $\Delta \geq 162$, *otherwise* $(4q-2)\Delta+10p+38q-23 \leq C$.

Lemma 6.4 *Every* $\leq$*5-vertex must be adjacent to at least 2 big vertices.*

Proof: By way of contradiction assume that there is a $\leq$5-vertex v which is adjacent to at most one big vertex and so all its other neighbors are $\leq$46-vertices. Then, using Observation 6.3, v along with one of these small vertices will contradict Lemma 6.2. ■

Now construct graph G' from G and then G'' from G' in the same way we did in the proof of Theorem 1.3. Also, we define the sparse segments in the same way. Consider vertex v and let's call the maximal sparse segments of it $Q_1, Q_2, \ldots, Q_m$ in clockwise order, where $Q_i = q_{i,1}, q_{i,2}, q_{i,3}, \ldots$.

Lemma 6.5 $|Q_i| \leq d_G(v) - \lceil \frac{2}{3}\Delta \rceil - 69$.

Proof: Analogous to the proof of Lemma 2.13. ■

The next lemma is analogous to Lemma 2.14. The key difference is that we require a bound on the degree of t. This is because each vertex adjacent to t can forbid for t up to $2p-1$ colors. Thus we have to be more careful about controlling the number of such vertices.

Lemma 6.6 *Suppose that* u_i *and* u_{i+1} *are the big vertices adjacent to all the vertices of* Q_i *and* Q_{i+1}, *respectively. Furthermore assume that* t *is a* $\leq$*6-vertex adjacent to both* u_i *and* u_{i+1} *but not adjacent to* v *(see Figure 3) and there is a vertex* $w \in N(t)$ *such that* $d_G(t) + d_G(w) \leq \Delta + 2$. *Let* $X(t)$ *be the set of vertices at distance at most two of* t *that are not in* $N[u_i] \cup N[u_{i+1}]$. *If* $|X(t)| \leq 6$ *then:*

$$|Q_i| + |Q_{i+1}| \leq \lfloor \frac{1}{3}\Delta \rfloor - 60. \tag{3}$$

Proof: Again, by way of contradiction, assume that $|Q_i| + |Q_{i+1}| \geq \lfloor \frac{1}{3}\Delta \rfloor - 59$. Using the same argument as at the beginning of the proof of Lemma 6.2, we can color every vertex of G other than t using colors in $\{0, \ldots, C-1\}$ such that the vertices that are adjacent receive colors that are at least p apart and the vertices at distance two receive distinct colors. Consider such a coloring.

Note: We often focus on the inner vertices of Q_i. So recall that there are exactly $|Q_i| - 2$ such vertices (similarly for Q_{i+1}).

Claim 1: There are at least $\lceil \frac{5}{3}\Delta \rceil + 78$ colors in $N_{G^2}(t)$ and they forbid all the C colors for t.

Proof: Trivially, if there is a non-forbidden color for t then we can extend the coloring to t, which contradicts the minimality of G.

If there are at most $\lceil \frac{5}{3}\Delta \rceil + 77$ colors in $N_{G^2}(t)$ then (because t is not colored and has degree at most 6) they forbid at most $\lceil \frac{5}{3}\Delta \rceil + 71 + 6(2p-1) = \lceil \frac{5}{3}\Delta \rceil + 12p + 65 < C$ colors for t, which contradicts what we proved in the previous paragraph. ■

Claim 2: There exists an inner vertex of Q_i or Q_{i+1} whose color is distinct from the color of every other vertex in $N_{G^2}(t)$ and differs from the color of every vertex in $N(t)$ by at least p.

Proof: By way of contradiction assume the above statement is false. Let us count the number of forbidden colors for t. The neighbors of t forbid at most $d_G(t) \times (2p-1)$ colors for t. Let's denote this set of forbidden colors by R. The vertices at distance exactly two of t are in $N(u_i) \cup N(u_{i+1}) \cup X(t) - N(t)$, and each of them forbids its own color for t. However, at least $|Q_i| - 2 + |Q_{i+1}| - 2$ of these forbidden colors (for t) are counted twice. This is because we assumed the claim is false; i.e. for every color α that appears on an inner vertex of Q_i or Q_{i+1} there is a neighbor of t whose color differs from α by less than p (and so $\alpha \in R$) or there is another vertex in $N_{G^2}(t)$ with color α. Since $d_{G^2}(t) \leq d_G(u_i) + d_G(u_{i+1}) + |X(t)| \leq 2\Delta + 6$, the total number of forbidden colors for t is at most $d_G(t) \times (2p-1) + 2\Delta + 6 - d_G(t) - |Q_i| - |Q_{i+1}| + 4 \leq \lceil \frac{5}{3}\Delta \rceil + 6(2p-1) + 63 \leq \lceil \frac{5}{3}\Delta \rceil + 12p + 57 < C$. This contradicts Claim 1. ■

Without loss of generality, assume there exists an inner vertex of Q_{i+1}, say $q_{i+1,2}$, whose color is different from the color of every vertex in $N_{G^2}(t)$ and differs from the color of every vertex in $N(t)$ by at least p.

Claim 3: There are at least $\lceil \frac{5}{3}\Delta \rceil + 77$ colors in $N_{G^2}(q_{i+1,2})$ and they forbid for $q_{i+1,2}$, $C-1$ colors (all the colors except the one that appears on $q_{i+1,2}$).

Proof: First we show that the vertices in $N_{G^2}(q_{i+1,2})$ must forbid all the colors (except the one that appears on $q_{i+1,2}$) for $q_{i+1,2}$. Otherwise, we can remove the color of $q_{i+1,2}$ and assign it without any conflict to t (because Claim 2 holds), and assign a new color (from the colors that are not forbidden) to $q_{i+1,2}$. Hence, the number of forbidden colors for $q_{i+1,2}$ is $C-1$.

If there are fewer than $\lceil \frac{5}{3}\Delta \rceil + 77$ different colors in $N_{G^2}(q_{i+1,2})$ then, since $d_G(q_{i+1,2}) \leq 4$, the vertices in $N_{G^2}(q_{i+1,2})$ forbid fewer than $4(2p-1) + \lceil \frac{5}{3}\Delta \rceil + 73 = \lceil \frac{5}{3}\Delta \rceil + 8p + 69 \leq C-1$ colors for $q_{i+1,2}$. This contradicts what we proved in the previous paragraph. ■

From the definition of a sparse segment $N(q_{i+1,2}) \subseteq \{v, u_{i+1}, q_{i+1,1}, q_{i+1,3}\}$. Let's denote the set of colors on the vertices in $N[u_{i+1}] \cup N(t) \cup X(t) \cup N[q_{i+1,1}] \cup N[q_{i+1,3}]$ by S and call it the set of *smaller colors*.

Claim 4: $|S| \leq d_G(u_{i+1}) + 14$.

Proof: Follows from the definition of S. ■

Let us call the set of colors that are forbidden for t or $q_{i+1,2}$ by the smaller colors the *smaller forbidden* colors, and denote them by SF. Since $d(t) \leq 6$ and $d(q_{i+1,2}) \leq 4$ and u_{i+1} is a common neighbor of t and $q_{i+1,2}$,

$$|SF| \leq 9(2p-1) + |S| - 9 = |S| + 18p - 18. \tag{4}$$

So, SF contains S along with at most $18(p-1)$ colors which differ from the color of some neighbor of t or some neighbor of $q_{i+1,2}$ by at most $p-1$.

Claim 5: Every color that is not in SF differs from every color in $N(t) \cup N(q_{i+1,2})$ by at least p.

Proof: By the definition of SF, every color which differs from the color of a vertex in $N(t) \cup N(q_{i+1,2})$ by less than p is in SF. ■

We will use Claim 5 at the end of the proof of this Lemma. By Claim 3, there are at least $C-1-|SF|$ colors, different from the smaller forbidden colors, in $N(v) - Q_{i+1}$. We call this set the *larger* colors and denote it by L.

Claim 6: $|L| \geq \lceil \frac{5}{3}\Delta \rceil - |S| + 77 \geq \lceil \frac{5}{3}\Delta \rceil - d_G(u_{i+1}) + 63$.

Proof: Follows from the definition of L, Claim 4, and the bound on $|SF|$ (Inequality 4). ■

Since $|N(v)| - (|Q_i| - 2) - |Q_{i+1}| \leq \Delta - \lfloor \frac{1}{3}\Delta \rfloor + 61 < |L|$, one of the *larger* colors must be on an inner vertex of Q_i, which without loss of generality, we can assume is $q_{i,2}$.

Claim 7: The vertices in $N(v) - Q_{i+1} - \{q_{i,2}\}$ forbid for $q_{i,2}$ all the colors in L, except the one that appears on $q_{i,2}$.

Proof: All the larger colors appear in $N(v) - Q_{i+1}$ and so they are at distance at most two of $q_{i,2}$. ■

Claim 8: The number of forbidden colors for $q_{i,2}$ is at most $\lfloor \frac{4}{3}\Delta \rfloor + 8p - 68 < C$.

Proof: By noting that $d(q_{i,2}) \leq 4$, neighbors of $q_{i,2}$ forbid at most $4(2p-1)$ colors for $q_{i,2}$. Now let's count the number of forbidden colors for $q_{i,2}$ by the vertices at distance exactly two of it.

Since the colors in $N[u_{i+1}] \cup N(t) \cup X(t)$ are smaller colors and forbid for t only colors that are in SF, by Claim 1, all the larger colors must appear in $N[u_i] - N(t)$. Remember that the larger colors appear in $N(v) - Q_{i+1}$, too. Therefore, the number of colors that are not in L and are forbidden for $q_{i,2}$ by the vertices at distance exactly 2 of $q_{i,2}$ is at most: $d(u_i) - 1 - (|L| - 1) + d(v) - 1 - (|L| - 1) \leq 2\Delta - 2|L|$. By considering the vertices at distance exactly two of $q_{i,2}$ that have a larger color and noting that $q_{i,2}$ has a larger color too, and using Claim 6, the total number of forbidden colors for $q_{i,2}$ is at most:

$$\begin{aligned} 4(2p-1) + (2\Delta - 2|L|) + (|L| - 1) &\leq \lfloor \frac{1}{3}\Delta \rfloor + d_G(u_{i+1}) + 8p - 68 \\ &\leq \lfloor \frac{4}{3}\Delta \rfloor + 8p - 68. \end{aligned}$$

■

By Claim 8, we can assign the color of $q_{i,2}$ to t (because it is a larger color and so it is different from the colors in $X(t)$ and, by Claim 5, differs from all the colors in $N(t)$ by at least p) and find a new color for $q_{i,2}$ that is not forbidden for it. This completes the proof of Lemma 6.6. ■

The rest of the proof is almost identical to that of Theorem 1.3. We use Lemmas 6.4, 6.5, and 6.6, instead of Lemmas 2.3, 2.13, and 2.14, respectively. The initial charges and the discharging rules are the same. Without any modifications, Lemmas 3.1 to 3.5 hold in this case, too. In Lemma 3.6 we should replace Equation (2) with Equation (3) and use Lemma 6.6 instead of Lemma 2.14. To do so, it is important to note that whenever we used Lemma 2.14 in the proof of Lemma 3.6, the degree of t was at most 6; thus, we can use Lemma 6.6, instead. After doing these modifications, the calculations for the proof of this revised version of Lemma 3.6 are fairly straightforward.

7 An $O(n^2)$ time algorithm

In this section we show how to transform the proof of Theorem 1.3 into a coloring algorithm which uses at most $\lceil \frac{5}{3}\Delta \rceil + 78$ colors. With some minor modifications in the algorithm, we can obtain coloring algorithms for Theorems 1.4 and 1.6.

Consider a planar graph G. We may assume that $\Delta \geq 160$ since for smaller values of Δ it is straightforward to obtain an algorithm based on the result of [19] that uses at most $\lceil \frac{5}{3}\Delta \rceil + 78$ colors. Also, we assume that the input to our algorithm is connected, since for a disconnected graph it is enough to color each connected component, separately. One iteration of the algorithm either finds a cut-vertex and breaks the graph into smaller subgraphs, or reduces the size of the problem by contracting a suitable edge of G. Then it colors the new smaller graph(s) recursively, and extends the coloring(s) to G. More specifically, we do the following steps, as long as the graph has at least one vertex:

1. Check to see whether G has a cut-vertex. If v is a cut-vertex and $C_1, \ldots, C_k$ are the connected components of $G - v$ then color each $G_i = C_i \cup \{v\}$, independently. The union of these colorings, after permuting the colors in some of them, will be a coloring of G.

2. Else, check to see whether there is a $\leq$5-vertex adjacent to at most one big vertex. If such a vertex exists, then that vertex along with one of its small neighbours will be the suitable edge to be contracted.

3. Else, construct the triangulated graph G''.

4. Apply the initial charges and the discharging rules.

5. As the total charge is negative, we can find a vertex v with negative charge. This vertex must be a $\leq$5-vertex violating Lemma 2.3, or lies in one of the reducible configurations described in Lemmas 2.13 or 2.14.

   If it is $\leq$5-vertex violating Lemma 2.3, then we continue as explained in the second step. If we find the reducible configuration of Lemma 2.13 around v then one of the inner vertices of the sparse segment along with one of its two big neighbours will be the suitable edge to contract. Finally, if we find the reducible configuration of Lemma 2.14 around v then we can contract edge tw (recall the specification of t and w from Lemma 2.14).

6. Color the new graph (after contracting the suitable edge), recursively.

7. This coloring can be easily extended to G by the arguments of proofs of Lemmas 2.3, 2.13 or 2.14.

That this algorithm works follows easily from the proofs of Lemmas 3.3, 3.4, and 3.7. Since in a planar graph the number of edges and faces is linear in the number of vertices we may let $n = |V|$ be the size of the graph. Finding a cut-vertex in a graph takes linear time. To see if there is a $\leq$5-vertex with less than 2 big neighbors we spend at most $O(n)$ time. Also, applying the initial charges and the discharging rules takes $O(n)$ time. After finding a vertex with negative charge, finding the suitable edge and then contracting it can be done in $O(n)$. Since there are $O(n)$ iterations of the main procedure, the total running time of the algorithm would be $O(n^2)$.

The algorithms for Theorems 1.4 and 1.6 work almost identically.

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References

[1] G. Agnarsson and M.M. Halldórsson, "Coloring powers of planar graphs", *To appear in SIAM J. Disc. Math.*, Earlier version in *SODA* (2000).

[2] K. Appel and W. Haken, "Every planar map is four colourable", *Contemp. math.* 98 (1989).

[3] H.L. Bodlaender, T. Kloks, R.B. Tan, and J. van Leeuwen, "Approximations for λ-Coloring of Graphs", *Prec. of 17th Annual Symp. on Theo. Aspc. of Comp. Sci., LNCS 1770* (2000).

[4] O. Borodin, H.J. Broersma, A. Glebov, and J. van den Heuvel, "Stars and bunches in planar graphs. Part I: Triangulations", *CDAM Research Report Series 2002-04*, 2002.

[5] O. Borodin, H.J. Broersma, A. Glebov, and J. van den Heuvel, "Stars and bunches in planar graphs. Part II: General planar graphs and colourings", *CDAM Research Report Series 2002-05*, 2002.

[6] J. Chang and Kuo, "The $L(2,1)$-labeling problem on graphs", *SIAM J. Disc. Math.*, 9 (1996), pp 309-316.

[7] D.A. Fotakis, S.E. Nikoletseas, V.G. Papadopoulou, and P.G. Spirakis, "Hardness results and efficient approximations for frequency assignment problems: radio labeling and radio coloring", *J. of Computers and Artificial intelligence*, Vol. 20 (2), 2001, pp 121-180.

[8] J.P. Georges and D.W. Mauro, "Generalized vertex labeling with a condition at distance two", *Cong. Numer.*, 109 (1995), pp 141-159.

[9] J.P. Georges and D.W. Mauro, "On the size of graphs labeled with a condition at distance two", *J. Graph Theory*, 22 (1996), pp 47-57.

[10] J.P. Georges and D. W. Mauro, "Some results on λ^i_j-numbers of the products of complete graphs", *Congr. Numer.*, 140 (1999), pp 141-160.

[11] J.R. Griggs and R.K. Yeh, "Labeling graphs with a condition at distance 2", *SIAM J. Disc. Math.*, 5 (1992), pp 586-595.

[12] W.K. Hale, "Frequency assignment: Theory and application", *Proc. IEEE*, 68 (1980), pp 1497-1514.

[13] T.K. Jonas, "Graph coloring analogues with a condition at distance two: $L(2,1)$-labelings and list λ-labelings", *Ph.D. Thesis, University of South Carolina* (1993).

[14] T.R. Jensen and B. Toft, "Graph coloring problems", *John-Wiley and Sons, New York*, (1995).

[15] S. Ramanathan and E.L. Lloyd, "On the complexity of distance-2 coloring", *In Proc. 4th Int. Conf. Comput. and Inform.*, (1992), pp 71-74.

[16] S. Ramanathan and E.L. Lloyd, "Scheduling algorithms for multi-hop radio networks", *IEEE/ACM Trans. on Networking*, 1(2) (1993), pp 166-172.

[17] D.P. Sanders and Y. Zhao, "A new bound on the cyclic chromatic number", *J. of Comb. Theory Series B* 83 (2001), pp 102-111.

[18] C. Thomassen, "Applications of Tutte cycles", *Technical report, Department of Mathematics, Technical University of Denmark*, Sept. 2001.

[19] J. van den Heuvel and S McGuinness, "Colouring the Square of a Planar Graph", *J. of Graph Theory*, 42 (2003), pp 110-124.

[20] J. van den Heuvel, R.A. Leese, and M.A. Shepherd, "Graph labeling and radio channel assignment", *J. Graph Theory*, 29 (1998), pp 263-283.

[21] G. Wegner, "Graphs with given diameter and a coloring problem", *Technical report, University of Dortmond'*, 1977.

[22] A. Whittlesey, J.P. Georges, and D.W. Mauro, "On the λ-number of Q_n and related graphs", *SIAM J. Disc. Math.*, 8 (1995), pp 499-506.

[23] S.A. Wong, "Colouring Graphs with Respect to Distance", *M.Sc. Thesis, Department of Combinatorics and Optimization, University of Waterloo*, 1996.