A CRITICAL POINT FOR RANDOM GRAPHS WITH A GIVEN DEGREE SEQUENCE

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Abstract

Given a sequence of non-negative real numbers $\lambda_0, \lambda_1, \ldots$ which sum to 1, we consider random graphs having approximately $\lambda_i n$ vertices of degree i. Essentially, we show that if $\sum i(i-2)\lambda_i > 0$ then such graphs almost surely have a giant component, while if $\sum i(i-2)\lambda_i < 0$ then almost surely all components in such graphs are small. We can apply these results to $G_{n,p}, G_{n,M}$, and other well-known models of random graphs. There are also applications related to the chromatic number of sparse random graphs.

1 Introduction and Overview

In this paper we consider two parameters of certain random graphs: the number of vertices and the number of cycles in the largest component. Of course, the behaviour of these parameters depends on the probability distribution from which the graphs are picked. In one standard model we pick a random graph $G_{n,M}$ with n vertices and M edges where each such graph is equally likely. We are interested in what happens when we choose M as a function of n and let n go to infinity. The point $M = \frac{1}{2}n$ is referred to as the *critical point* or the *double-jump threshold* because of classical results due to Erdős and Rényi [8] concerning the dramatic changes which occur to these parameters at this point. If $M = cn + o(n)$ for $c < \frac{1}{2}$ then almost surely (i.e. with probability tending to one as n tends to infinity) $G_{n,M}$ has no component of size greater than $O(\log n)$, and no component has more than one cycle. If $M = \frac{1}{2}n + o(n)$, then almost surely (a.s.) the largest component of $G_{n,M}$ has size $\Theta(n^{2/3})$. If $M = cn$ for $c > \frac{1}{2}$ then there are constants $\epsilon, \delta > 0$ dependent on c such that a.s. $G_{n,M}$ has a component on at least ϵn vertices with at least δn cycles, and no other component has more than $O(\log n)$ vertices or more than one cycle. This component is referred to as the *giant component* of $G_{n,M}$. For more specifics on these two parameters at and around $M = \frac{1}{2}n$ see [3], [11], or [14].

In this paper, we are interested in random graphs with a fixed degree sequence where each graph with that degree sequence is chosen with equal probability. Of course, we have to say what we mean by a degree sequence. If the number of vertices in our graph, n is fixed, then a degree sequence is simply a sequence of n numbers. However, we are concerned here with what happens asymptotically as n tends to infinity, so we have to look at a "sequence of sequences". Thus, we generalize the definition of degree sequence:

Definition: *An* **asymptotic degree sequence** *is a sequence of integer-valued functions* $\mathcal{D} = d_0(n), d_1(n), \ldots$ *such that*

1. $d_i(n) = 0$ *for* $i \geq n$*;*

2. $\sum_{i \geq 0} d_i(n) = n$.

Given an asymptotic degree sequence $\mathcal{D}$, we set $\mathcal{D}_n$ to be the degree sequence $\{c_1, c_2, \ldots, c_n\}$, where $c_j \geq c_{j+1}$ and $|\{j : c_j = i\}| = d_i(n)$ for each $i \geq 0$.. Define $\Omega_{\mathcal{D}_n}$ to be the set of all graphs with vertex set $[n]$ with degree sequence $\mathcal{D}_n$. A random graph on n vertices with degree sequence $\mathcal{D}$ is a uniformly random member of $\Omega_{\mathcal{D}_n}$.

Definition: *An asymptotic degree sequence $\mathcal{D}$ is* **feasible** *if $\Omega_{\mathcal{D}_n} \neq \emptyset$ for all $n \geq 1$.*

In this paper, we will only discuss feasible degree sequences.

Because we wish to discuss asymptotic properties of random graphs with degree sequence $\mathcal{D}$, we want the sequences $\mathcal{D}_n$ to be in some sense similar. We do this by insisting that for any fixed i, the proportion of vertices of degree i is roughly the same in each sequence.

Definition: *An asymptotic degree sequence $\mathcal{D}$ is* **smooth** *if there exist constants λ_i such that $\lim_{n\to\infty} d_i(n)/n = \lambda_i$.*

Throughout this paper, all asymptotics will be taken as n tends to ∞ and we only claim things to be true for sufficiently large n.

In the past, the most commonly studied random graphs of this type have been random regular graphs. Perhaps the most important recent result is by Robinson and Wormald [20], [21], who proved that if G is a random k-regular graph for any constant $k \geq 3$, then G is a.s. Hamiltonian.

Another motivation for studying random graphs on a fixed degree sequence comes from the analysis of the chromatic number of sparse random graphs. This is because a minimally $(r+1)$-chromatic graph must have minimum degree at least r. In an attempt to determine how many edges were neccessary to force a random graph to a.s. be not 3-colourable, Chvátal [7] studied the expected number of subgraphs of minimum degree three in random graphs with a linear number of edges. He showed that for $c < c^* = 1.442\ldots$, the expected number of such subgraphs in $G_{n,M=cn}$ is exponentially small, while for $c > c^*$ the expected number of such subgraphs in $G_{n,M=cn}$ is exponentially large. In the work that motivated the results of this paper, the authors used a special case of the main theorem of this paper to show that the probability that a random graph on n vertices with minimum degree three and at most $1.793n$ edges is minimally 4-chromatic is exponentially small [18]. We used this to show that for c a little bit bigger than c^*, the expected number of minimally 4-chromatic subgraphs of $G_{n,M=cn}$ is exponentially small. This suggests that determining the minimum value of c for which a random graph with cn edges is a.s. 4-chromatic may require more than a study of the subgraphs with minimum degree 3.

Recently Łuczak [14] showed (among other things) that if G is a random graph on a fixed degree sequence[1], with no vertices of degree less than 2, and at least $\Theta(n)$ vertices of degree greater than 2, then G a.s. has a unique giant component. Our main theorem also generalizes this result.

We set $Q(\mathcal{D}) = \sum_{i\geq 1} i(i-2)\lambda_i$. Essentially if $Q(\mathcal{D}) > 0$ then a random graph with degree sequence $\mathcal{D}$ a.s. has a giant component, while if $Q(\mathcal{D}) < 0$ then all the components of such a random graph are a.s. quite small. Note how closely this parallels the phenomenon in the more standard model $G_{n,M}$.

Note further that these results allow us to determine a similar threshold for any model of random graphs as long as: *(i)* we can determine the degree sequence of graphs in the model with reasonable accuracy, and *(ii)* once the degree sequence is determined, every graph on that degree sequence is equally likely. $G_{n,p}$ is such a model, and thus (as we see later), our results can be used to verify the previously known threshold for $G_{n,p}$.

Before defining the parameter precisely, we give an intuitive explanation of why it determines whether or not a giant component exists. Suppose that $\mathcal{D}_n$ has $(\lambda_i + o(1))n$ vertices of degree i for each $i \geq 0$. Pick a random vertex in our graph and expose the component in which it lies using a branching process. In other words, expose its neighbours, and then the neighbours of its neighbours, repeating until the entire component is exposed. Now when a vertex of degree i is exposed, then the number of "unknown" neighbours increases by $i-2$. The probability that a certain vertex is selected as a neighbour is proportional to its degree. Therefore the expected increase in the number of unknown neighbours is (roughly) $\sum_{i\geq 1} i(i-2)\lambda_i$. This is, of course, $Q(\mathcal{D})$.

Thus, if $Q(\mathcal{D})$ is negative then the component will a.s. be exposed very quickly. However, if it is positive then the number of unknown neighbours, and thus the size of the component, might grow quite large. This gives the main thrust of our arguments. We will now begin to state all of this more formally.

There are a few caveats, so in order for our results to hold, we must insist that the asymptotic degree sequences we consider are well-behaved. In particular, when the maximum degree in our degree sequence grows with n, we can run into some problems if things do not converge uniformly. For example, if $d_1(n) = n - \lceil n^{.9} \rceil, d_i(n) = \lceil n^{.9} \rceil$ if $i = \lceil \sqrt{n} \rceil$, and $d_i(n) = 0$

[1]He didn't use the asymptotic degree sequence introduced here, but the results translate.

otherwise, then $\lambda_1 = 1$, and $\lambda_i = 0$ for $i > 1$, and we get $Q(\mathcal{D}) = -1$. However, this is deceiving as there are enough vertices of degree $\sqrt{n}$ to ensure that a giant component containing $n - o(n)$ vertices a.s. exists.

Definition: *An asymptotic degree sequence $\mathcal{D}$ is* **well-behaved** *if:*

1. *$\mathcal{D}$ is feasible and smooth.*

2. *$i(i-2)d_i(n)/n$ tends uniformly to $i(i-2)\lambda_i$; i.e. for all $\epsilon > 0$ there exists N such that for all $n > N$ and for all $i \geq 0$:*

$$\left| \frac{i(i-2)d_i(n)}{n} - i(i-2)\lambda_i \right| < \epsilon$$

3. $$L(\mathcal{D}) = \lim_{n\to\infty} \sum_{i\geq 1} i(i-2)d_i(n)/n$$

   exists, and the sum approaches the limit uniformly, i.e.:

   (a) *If $L(\mathcal{D})$ is finite then for all $\epsilon > 0$, there exists i^*, N such that for all $n > N$:*

   $$\left| \sum_{i=1}^{i^*} i(i-2)d_i(n)/n - L(\mathcal{D}) \right| < \epsilon$$

   (b) *If $L(\mathcal{D})$ is infinite then for all $T > 0$, there exists i^*, N such that for all $n > N$:*

   $$\sum_{i=1}^{i^*} i(i-2)d_i(n)/n > T$$

We note that it is an easy exercise to show that if $\mathcal{D}$ is well-behaved then:

$$L(\mathcal{D}) = Q(\mathcal{D})$$

It is not surprising that the threshold occurs when there are a linear number of edges in our degree sequence. We define such a degree sequence as sparse:

Definition: *An asymptotic degree sequence $\mathcal{D}$ is* **sparse** *if* $\sum_{i\geq 0} id_i(n)/n = K + o(1)$ *for some constant* K.

Note that for a well-behaved asymptotic degree sequence $\mathcal{D}$, if $Q(\mathcal{D})$ is finite then $\mathcal{D}$ is sparse.

The main result in this paper is the following:

Theorem 1 *Let $\mathcal{D} = d_0(n), d_1(n), \ldots$ be a well-behaved sparse asymptotic degree sequence for which there exists $\epsilon > 0$ such that for all n and $i > n^{\frac{1}{4}-\epsilon}$, $d_i(n) = 0$. Let G be a graph with n vertices, $d_i(n)$ of which have degree i, chosen uniformly at random from amongst all such graphs. Then:*

- *(a) If $Q(\mathcal{D}) > 0$ then there exist constants $\zeta_1, \zeta_2 > 0$ dependent on $\mathcal{D}$ such that G a.s. has a component with at least $\zeta_1 n$ vertices and $\zeta_2 n$ cycles. Furthermore, if $Q(\mathcal{D})$ is finite then G a.s. has exactly one component of size greater than $\gamma \log n$ for some constant γ dependent on $\mathcal{D}$.*
- *(b) If $Q(\mathcal{D}) < 0$ and for some function $0 \leq \omega(n) \leq n^{\frac{1}{8}-\epsilon}$, $d_i(n) = 0$ for all $i \geq \omega(n)$, then for some constant R dependent on $Q(\mathcal{D})$, G a.s. has no component with at least $R\omega(n)^2 \log n$ vertices, and a.s. has fewer than $2R\omega(n)^2 \log n$ cycles. Also, a.s. no component of G has more than one cycle.*

Consistent with the model $G_{n,M}$, we call the component refered to in Theorem 1(a) a *giant component.*

Note that if $Q(\mathcal{D}) < 0$ then $Q(\mathcal{D})$ is finite.

Note also that Theorem 1 fails to cover the case where $Q(\mathcal{D}) = 0$. This is analogous to the case $c = 1$ in the model $G_{n,p=c/n}$, and would be interesting to analyze.

One immediate application of Theorem 1 is that if G is a random graph on a fixed well-behaved degree sequence with $cn + \mathrm{o}(n)$ edges for any $c > 1$ then G a.s. has a giant component, as there is no solution to
$\sum_{i\geq 1} i\lambda_i > 2, \sum_{i\geq 1} i(i-2)\lambda_i < 0, \sum_{i\geq 1} \lambda_i = 1, 0 \leq \lambda_i \leq 1.$

A major difficulty in the study of random graphs on fixed degree sequences is that it is difficult to generate such graphs directly. Instead is has become standard to study random configurations on a fixed degree sequence, and use some lemmas which allow us to translate results from one model to the other. The configuration model was introduced by Bender and Canfield[2] and refined by Bollobás[3] and also Wormald[22].

In order to generate a random configuration with n vertices and a fixed degree sequence, we do the following:

1. Form a set L containing $\deg(v)$ distinct copies of each vertex v.
2. Choose a random matching of the elements of L.

Each configuration represents an underlying multigraph whose edges are defined by the pairs in the matching. We say that a configuration has a graphical property P if its underlying multigraph does.

Using the main result in [17] it follows that the underlying multigraph of a random configuration on a degree sequence meeting the conditions of Theorem 1 is simple with probability tending to $\mathrm{e}^{-\lambda(\mathcal{D})}$, for some $\lambda(\mathcal{D}) < O(n^{1/2-\epsilon})$. The condition $d_i(n) = 0$ for all $i > n^{1/4-\epsilon}$ is needed to apply this result. If $Q(\mathcal{D})$ is finite then $\lambda(\mathcal{D})$ tends to a constant.

Also, any simple graph G can be represented by $\prod_{v \in V(G)} \deg(v)!$ configurations, which is clearly equal for all graphs on the same degree sequence and the same number of vertices.

This gives us the following very useful lemmas:

Lemma 1 *If a random configuration with a given degree sequence $\mathcal{D}$ meeting the conditions of Theorem 1 (with $Q(\mathcal{D})$ possibly unbounded) has a property P with probability at least $1 - z^n$ for some constant $z < 1$, then a random graph with the same degree sequence a.s. has P.*

Lemma 2 *If a random configuration with a given degree sequence $\mathcal{D}$ meeting the conditions of Theorem 1 a.s. has a property P, and if $Q(\mathcal{D}) < \infty$, then a random graph with the same degree sequence a.s. has P.*

Using these Lemmas, it will be enough to prove Theorem 1 for a random configuration.

The configuration model is very similar to the pseudograph model developed independently by Bollobás and Frieze [6], Flajolet, Knuth and Pittel [10], and Chvátal [7]. Both models are very useful when working with random graphs on a given degree sequence.

Having defined the precise objects that we are interested in, and the model in which we are studying them, we can now give a more formal overview of the proof. The remainder of this section is devoted to this overview. In the following two sections we give all the details of the proof. In section 4, we see some applications of Theorem 1: the aforementioned work concerning the chromatic number of sparse random graphs, and a new proof of a classical double-jump theorem, showing that this work generalizes that result. A reader who is not interested in the details of the proof might want to just finish this section and then skip ahead to the last one.

In order to examine the components of our random configuration, we will be more specific regarding the order in which we expose the pairs of the random matching.

Given $\mathcal{D}$, we will expose a random configuration F on n vertices, $d_i(n)$ of which have degree i as follows:

At each step, a vertex all of whose copies are in exposed pairs is *entirely exposed*. A vertex some but not all of whose copies are in exposed pairs is *partially exposed*. All other vertices are *unexposed*. The copies of partially exposed vertices which are not in exposed pairs are *open*.

1. Form a set L consisting of i distinct copies of each of the $d_i(n)$ vertices which have degree i.

2. Repeat until L is empty:

   (a) Expose a pair of F by first choosing any member of L, and then choosing its partner at random. Remove them from L.

   (b) Repeat until there are no partially exposed vertices:

   Choose an open copy of a partially exposed vertex, and pair it with another randomly chosen member of L. Remove them both from L.

All random choices are made uniformly.

Essentially we are exposing the random configuration one component at a time. When any component is completely exposed, we move on to a new one; i.e. we repeat step 2(a).

It is clear that every possible matching amongst the vertex-copies occurs with the same probability under this procedure, and hence this is a valid way to choose a random configuration.

Note that we have complete freedom as to which vertex we pick in Step 2(a). In a few places in this paper, it will be important that we take advantage of this freedom, but in most cases we will pick it randomly in the same manner in which we pick all the other vertex-copies, i.e. unless we state otherwise, we will always just pick a uniformly random member of L.

Now, let X_i represent the number of open vertex-copies after the ith pair is exposed. If the neighbour of v chosen in step 2(b) is of degree d, then X_i goes up by $d-2$. Each time a component is completely exposed and we repeat step 2(a) then if the pair exposed in step 2(a) involves vertices of degree d_1 and d_2 then X_i is set to a value of d_1+d_2-2.

Note that if the number of vertex-copies in L which are copies of vertices of degree d is r_d, then the probability that we pick a copy of a vertex of degree d in step 3(b) is $r_d/\sum_{i\geq 1} r_i$. Therefore initially the expected change in X_i is approximately

$$\frac{\sum_{i\geq 1} i(i-2)d_i(n)}{\sum_{j\geq 1} jd_j(n)} = \frac{Q(\mathcal{D})}{K}.$$

Therefore, at least initially, if this value is positive then X_i follows a Markov process very close in distribution to the well-studied "Drunkard's Walk", with an expected change of $\frac{Q(\mathcal{D})}{K}$. Since $X_{i+1} \geq X_i - 1$ always, a standard result of random walk theory (see for example [9]) implies that if $Q(\mathcal{D}) > 0$, then after $\Theta(n)$ steps, X_i is a.s. of order $\Theta(n)$.

It follows that our random configuration a.s. has at least one component on $\Theta(n)$ vertices. We will see that such a component a.s. has at least $\Theta(n)$ cycles in it, and this will give us the first part of Theorem 1. We will also see that if $Q(\mathcal{D})$ is bounded, then this giant component is a.s. unique.

On the other hand, if $Q(\mathcal{D}) < 0$, then X_i a.s. returns to zero fairly quickly, and this will give us the other part of Theorem 1, as the sizes of the components of F are bounded above by the distances between values of i such that $X_i = 0$.

Of course the random walk followed by X_i is not really as simple as this. There are three major complications:

1. A pure random walk can drop below 0. Whenever X_i reaches 0, it resets itself to a positive number.

2. We neglected to consider that the second vertex-copy chosen in Step 2(b) might be an open vertex-copy in which case X_i decreases by 2. We will call such a pair of vertex-copies a *backedge*.

3. As more and more vertices are exposed, the ratio of the members of L which are copies of vertices of degree d shifts, and the expected increase of X_i changes.

These complications are handled as follows:

1. This will increase the probability of X_i growing large, and so this only poses a potential problem in proving part (b). In this case, we will show that the probability of a component growing too big is of order $o(n^{-1})$, and hence even if we "try again" n times, this will a.s. never happen.

2. We will see that this a.s. doesn't happen often enough to pose a serious problem, unless the partially exposed component is already of size $\Theta(n)$.

3. In proving part (a), we look at our component at a time when the expected increase in X_i is still at least $\frac{1}{2}$ its original value. We will see that the component being exposed at this point is a.s. a giant component. In proving part (b), it is enough to consider the configuration after $o(n)$ steps. At this point, the expected increase hasn't changed significantly.

This is a rough outline of the proof. We will fill in the details in the next two sections.

2 Graphs With No Large Components

In this section we will prove that the analogue of Theorem 1(b) holds for random configurations. Lemma 1 will then imply that it holds for random graphs. We will first prove that if F is a random configuration meeting the conditions given in Theorem 1(b), then F a.s. does not have any large components.

Given $Q(\mathcal{D}) < 0$ set $\nu = -Q(\mathcal{D})/K$ and set $R = \frac{150}{\nu^2}$.

Lemma 3 *Let F be a random configuration with n vertices and degree sequence $\mathcal{D}_n$ meeting the conditions of Theorem 1. If $Q(\mathcal{D}) < 0$ and if for some function $0 \leq \omega(n) \leq n^{\frac{1}{8}-\epsilon}$ F has no vertices of degree greater than $\omega(n)$, then F a.s. has no components with more than $\alpha = \lceil R\omega(n)^2 \log n \rceil$ vertices.*

The following theorem of Azuma will play an important role:

Azuma's Inequality [1] *Let $0 = X_0, \ldots, X_n$ be a martingale with*

$$| X_{i+1} - X_i | \leq 1$$

for all $0 \leq i < n$. *Let* $\lambda > 0$ *be arbitrary. Then*

$$\Pr[\mid X_n \mid > \lambda\sqrt{n}] < e^{-\lambda^2/2}$$

This yields the following very useful standard corollary.

Corollary *Let* $\Sigma = \Sigma_1, \Sigma_2, \ldots, \Sigma_n$ *be a sequence of random events. Let* $f(\Sigma) = f(\Sigma_1, \Sigma_2, \ldots, \Sigma_n)$ *be a random variable defined by these* Σ_i. *If for each* i*:*

$$\max \mid E(f(\Sigma) \mid \Sigma_1, \Sigma_2, \ldots \Sigma_{i+1}) - E(f(\Sigma) \mid \Sigma_1, \Sigma_2, \ldots \Sigma_i) \mid \leq c_i$$

where $E(f)$ *denotes the expected value of* f, *then the probability that* $\mid f - E(f) \mid > t$ *is at most:*

$$2\exp\left(\frac{-t^2}{2\sum c_i^2}\right)$$

For more details on this corollary and an excellent discussion of martingale arguments see either [16] or [5].

In order to prove Lemma 3, we will analyze the Markov process described in Section 1.

Recall that X_i is the number of open vertex-copies after i pairs of our configuration have been exposed. Similarly, we let Y_i be the number of backedges formed, and C_i be the number of components that have been at least partially exposed during the first i steps. We also define W_i to be the sum of $\deg(v) - 2$ over all vertices v completely or partially exposed during the first i steps. We note that $W_i = X_i + 2Y_i - 2C_i$.

Now W_i "stalls" whenever a backedge is formed, and only changes whenever a new vertex is completely or partially exposed. For this reason, it is easier to analyze W_i when it is indexed not by the number of pairs exposed, but by the number of new vertices exposed. Thus we introduce another variable which does exactly this. We let Z_j be the sum of $\deg(v) - 2$ over the first j new vertices (partially or completely) exposed.

The reason that we are introducing Z_j is that it has the same initial expected increase as X_i, but behaves much more nicely. In particular, it isn't affected by the first and second complications discussed at the end of Section 1. Specifically, if after the first j vertices have been completely or partially exposed, there are exactly $r_i(j)$ unexposed vertices of degree i, then $Z_{j+1} = Z_j + (i-2)$ with probability $ir_i(j)/\sum ir_i(j)$.

Now in order to discuss X_i and Z_j at the same time, we will introduce the random variable I_j which is the number of pairs exposed by the time that the jth vertex is partially exposed; i.e. $W_{I_j} = Z_j$.

Recall that $\alpha = R\omega(n)^2 \log n$.

Lemma 4 *Suppose that F is as described in Lemma 3. Given any vertex v in F, the probability that v lies on a component of size at least α is less than n^{-2}.*

Proof:

Here we will insist that v is the first vertex chosen in Step 2(a). Therefore the probability that v lies on a component that large is at most the probability that $X_i > 0$ for all $1 \le i \le \alpha$. Thus, we will consider the probability of the latter.

Note that for any i, if $C_i = 1$ then $W_i = X_i + 2Y_i - 2 \ge X_i - 2$. In fact, we can also get $Z_i \ge X_i - 2$. This is because at each iteration we either have a backedge or expose a new vertex. Thus in iteration i, we have exposed $i - Y_i$ new vertices, therefore $W_i = Z_{i-Y_i}$. Now Z_i decreases by at most one at each step; therefore $Z_i \ge Z_{i-Y_i} - Y_i \ge W_i - Y_i \ge X_i + Y_i - 2 \ge X_i - 2$.

Now if $X_i > 0$ for all $1 \le i \le \alpha$, then $C_\alpha = 1$. Therefore, the probability that $X_i > 0$ for all $1 \le i \le \alpha$ is at most the probability that $Z_\alpha > -2$. We will concentrate on this probability, as Z_i behaves much more predictably than X_i.

Initially the expected increase in Z_j is $\sum_{i\ge1} i(i-2)d_i(n)/\sum_{i\ge1} id_i(n) = -\nu + o(1)$. We claim that for $j \le \alpha$, the expected increase in Z_j is less than $-\frac{\nu}{2}$.

This is true because the expected increase of Z_j would be highest if the first j vertex-copies chosen were all copies of vertices of degree 1. If this were the case then the expected increase in Z_j would be:

$$\begin{aligned} \frac{-(d_1(n)-j) + \sum_{i\ge2} i(i-2)d_i(n)}{(d_1(n)-j) + \sum_{i\ge2} id_i(n)} + o(1) &= -\nu + o(1) \\ &\le -\frac{\nu}{2} \end{aligned}$$

for sufficiently large n, as $j = o(n)$ and $id_i(n) \to \lambda_i$ uniformly.

Therefore, the expected value of Z_α is less than $-\frac{\nu}{2}\alpha + \deg(v) < -\frac{\nu}{3}\alpha$.

We will use the corollary of Azuma's Inequality to show that Z_α is a.s. very close to its expected value.

Σ_i will indicate the choice of the ith new vertex exposed, $i = 1, \ldots, \alpha$, and $f(\Sigma) = Z_\alpha$. We need to bound

$$\mid E(f(\Sigma) \mid \Sigma_1, \Sigma_2, \ldots \Sigma_{i+1}) - E(f(\Sigma) \mid \Sigma_1, \Sigma_2, \ldots \Sigma_i) \mid .$$

Suppose that we are choosing the $(i+1)$st vertex to be partially exposed. Let Ω be the set of unexposed vertices at this point. The size of Ω is $n - i$.

For each $x \in \Omega$, define $E_{i+1}(x)$ to be $E(Z_\alpha \mid \Sigma_1, \Sigma_2, \ldots, \Sigma_{i+1})$ where Σ_{i+1} is the event that x is the $(i+1)$st new vertex exposed.

Consider any two vertices $u, v \in \Omega$. We will bound $|E_{i+1}(u) - E_{i+1}(v)|$.

Consider the order that the vertices in $\Omega - \{u, v\}$ are exposed. Note that the distribution of this order is unaffected by the positions of u, v.

Let S be the set of the first $\alpha - 2$ vertices under this order, and let w be the next vertex. Now, $Z_\alpha = Z_{j-1} + (\sum_{x \in S} \deg(x) - 2) + \deg(y_1) - 2 + \deg(y_2) - 2$, where y_1 is the jth vertex exposed (either u or v) and y_2 is either u, v, or w. Therefore, the most that choosing between u, v can affect the conditional expected value of Z_α is twice the maximum degree, i.e. $|E_{i+1}(u) - E_{i+1}(v)| \leq 2\omega(n)$.

Since,

$$E(f(\Sigma) \mid \Sigma_1, \Sigma_2, \ldots \Sigma_i) = \sum_{x \in \Omega} \Pr\{x \text{ is chosen}\} \times E_{i+1}(x),$$

we have that

$$\mid E(f(\Sigma) \mid \Sigma_1, \Sigma_2, \ldots \Sigma_{i+1}) - E(f(\Sigma) \mid \Sigma_1, \Sigma_2, \ldots \Sigma_i) \mid \leq 2\omega(n).$$

Therefore by the corollary of Azuma's Inequality, the probability that $Z_\alpha > 0$ is at most:

$$\begin{aligned} 2\exp\left(-\frac{(\frac{\nu}{3} R\omega(n)^2 \log n)^2}{2\sum(2\omega(n))^2}\right) &= 2n^{-\frac{\nu^2}{72}R} \\ &< n^{-2}. \end{aligned}$$

□

And now Lemma 3 follows quite easily:

Proof of Lemma 3:

By Lemma 4, the expected number of vertices which lie on components of size at least α is $o(1)$. Therefore a.s. none exist. $\square$

We also get the following Corollary:

Corollary 3 *Under the same conditions as Lemma 3, a.s.* $X_i < 2\alpha$ *throughout the exposure of our configuration.*

Proof:

Because X_i drops by at most 2 at each step, if it ever got that high, it would not be able to reach 0 within $R\omega(n)^2 \log n$ steps. $\square$

We will now show that F a.s. doesn't have many cycles. First, we will see that it a.s. has no multicyclic components.

Lemma 5 *Let F be a random configuration meeting the same conditions as in Lemma 3. F a.s. has no component with at least 2 cycles.*

Proof:

Choose any vertex v. Let E_v be the event that v lies on a component of size at most α with more than one cycle, and that throughout the exposure of this component, $X_i < 2\alpha$.

We will insist that v is the first vertex examined under Step 2(a). If the size of the first component is at most α then the second backedge must be chosen within at most $\alpha + 2$ steps. Therefore the probability that E holds is less than

$$\binom{\alpha+2}{2}\left(\frac{2\alpha}{M-2\alpha-3}\right)^2 = \mathrm{o}\left(n^{-1}\right)$$

as $\omega(n) < n^{\frac{1}{8}-\epsilon}$.

Therefore the expected number of vertices for which E_v holds is $o(1)$ and so the probability that E_v holds for any v is $o(1)$.

Therefore by Lemma 3 and Corollary 3 a.s. no components of F have more than one cycle. $\square$

We can now show that F a.s. does not have many cycles, by showing that it a.s. does not have many cyclic components.

Lemma 6 *Let F be a random configuration meeting the same conditions as in Lemma 3. F a.s. has less than $2\alpha \log n$ cycles.*

Proof:

We will show that a.s. throughout the exposure of F, at most $2\alpha \log n$ back-edges are formed. The rest will then follow, since by Lemma 5, a.s. no component contains more than one cycle, and so a.s. the nunmber of cycles in F is exactly the number of backedges.

First we must define a set B_i of unmatched vertex-copies:

For each i, if there are more than 2α open vertex-copies at the ith iteration, then let B_i consist of any 2α of them. Otherwise, let B_i consist of the open vertex-copies and enough arbitrarily chosen members of L to bring the size of B_i up to α. Of course if L is too small to do this, then we will just add all of L to B_i. Let T_i be the event that a member of B_i is chosen in step i.

Clearly the number of backedges formed is at most the number of successful T_i's, plus the number of backedges formed at times when $X_i > 2\alpha$. Now by Corollary 3, we know that there are a.s. none of the latter type of backedges, so we will concentrate on the number of the former type.

Now the number of vertex-copies to choose from is $\sum_{j\geq 1} jd_j(n) - 2i + 1 = M - 2i + 1$. Therefore the probability of T_i holding is $\frac{2\alpha}{M-2i+1}$, for $M - 2i + 1 \geq 2\alpha$ and 1 otherwise.

Therefore the expected value of T, the number of successful T_i's is:

$$E(T) = 2\alpha + \sum_{i=1}^{(M-2\alpha)/2} \frac{2\alpha}{M-2i+1} = \alpha \log(M)(1+o(1))$$

Now we will use a second moment argument to show that T is a.s. not much bigger than $E(T)$.

$$E(T^2) = \sum_{i\neq j} \frac{\alpha^2}{(M-2i+1)(M-2j+1)} + E(T)$$

$$= (E(T)^2 + E(T))(1+o(1))$$

Therefore, by Chebyshev's inequality, the probability that $T > 1.5\alpha\log(M)$ is at most $1/(4E(T))(1+o(1)) = o(1)$.

Therefore, a.s. the number of backedges formed is less than $1.5\alpha\log(M) < 2\alpha\log n$, proving the result. □

And now we can prove Theorem 1(b).

Proof of Theorem 1(b):

This clearly follows from Lemmas 2, 3, 5, and 6. □

3 Graphs With Giant Components

In this section we will prove the analogue of Theorem 1(a) for random configurations. Lemmas 1 and 2 will then imply that Theorem 1(a) holds.

First we will show that a giant component exists with high probability:

Lemma 7 *Let F be a random configuration with n vertices and degree sequence $\mathcal{D}_n$ meeting the conditions of Theorem 1. If $Q(\mathcal{D}) > 0$ then there exist constants $\zeta_1, \zeta_2 > 0$ dependent on $\mathcal{D}$ such that F a.s. has a component with at least $\zeta_1 n$ vertices and $\zeta_2 n$ cycles. Moreover, the probability of the converse is at most z^n, for some fixed $0 < z < 1$.*

Throughout this section we will assume that the conditions of Lemma 7 hold.

As in Section 2, we will prove Lemma 7 by analyzing the Markov process discussed in the previous section. Again, the key will be to concentrate on the random variable Z_j.

Lemma 8 *There exists $0 < \epsilon < 1, 0 < \Delta < \min(\frac{1}{4}, \frac{K}{4})$ such that for all $0 < \delta < \Delta$ a.s. $Z_{\lceil \delta n \rceil} > \epsilon \delta n$. Moreover, the probability of the converse is at most $(z_1)^n$, for some fixed $0 < z_1 < 1$.*

Proof:

For simplicity, we will assume that δn is an integer.

Initially, the probability that a vertex-copy of degree i is chosen as a partner is $p_i(n) = i d_i(n) / \sum_{j \geq 1} j d_j(n) = i\lambda_i / K + o(1)$.

Unlike in Section 2, we have to consider the behaviour of our walk after $\Theta(n)$ steps. Thus we have to worry about the third complication described at the end of Section 2, i.e. that fact that the ratios of unexposed vertices of different degrees are shifting.

It turns out that this problem is much less serious if we can ignore vertices of high degree. So what we will do is show that we can find a value i^*, such that if we change Z_j slightly by saying that every time a vertex of degree $i > i^*$ is chosen, we subtract one from Z_j instead of adding $i - 2$ to it, then we will still have positive expected increase.

We will then show that we can find a sequence $\phi_1, \ldots, \phi_{i^*}$ summing to one, such that for each $2 \leq i \leq i^*$, ϕ_i is a little less than the initial probability of a vertex of degree i being chosen. However, if we were to adjust Z_j a little further by selecting a vertex of degree i with probability ϕ_i at each step, then we would still have a positive expected increase.

We will call this "adjusted Z_j" Z_j^*. Clearly, if we find some J such that after J steps, the probability of choosing a vertex of degree i is still at least ϕ_i for $2 \leq i \leq i^*$, then the probability that $Z_J > R$ for any R is at least as big as the probability that $Z_J^* > R$. We will concentrate on the second probability as Z_j^* is much simpler to analyze.

More formally, what we wish do is choose a sequence $\phi_1 \ldots, \phi_{i^*}$ such that

1. $\sum \phi_i = 1$;
2. $0 < \phi_i < i\lambda_i/K$, for $2 \leq i \leq i^*$, unless $0 = \phi_i = i\lambda_i/K$;
3. $\sum_{i\geq 1} i(i-2)\phi_i > 0$.

Note that:

$$\begin{aligned}\sum_{i\geq 1}(i-2)p_i(n) &= \sum_{i\geq 1}(i-1)p_i(n) - \sum_{i\geq 1} p_i(n)\\ &= \sum_{i\geq 2}(i-1)p_i(n) - 1.\end{aligned}$$

Set $p_i = i\lambda_i/K$.

Since $\mathcal{D}$ is well-behaved and $Q(\mathcal{D}) > 0$, there exists i^* such that $\sum_{i=2}^{i^*}(i-1)p_i > 1+\epsilon'$, for some $\epsilon' > 0$ and sufficiently large n.

Therefore, we can choose a sequence $\phi_1, \ldots, \phi_{i^*}$ such that for all $2 \leq i \leq i^*$, $p_i > \phi_i > 0$ unless $p_i = \phi_i = 0$, $\phi_1 = 1 - \phi_2 - \phi_3 - \ldots - \phi_{i^*}$, and $\sum_{i=2}^{i^*}(i-1)\phi_i = 1 + \frac{\epsilon'}{2}$. It follows that $\sum_{i\geq 1}(i-2)\phi_i = \frac{\epsilon'}{2}$.

Consider the random variable Z_j^* which follows the following random walk:

- $Z_0^* = 0$
- $Z_{j+1}^* = Z_j^* + (i-2)$ with probability ϕ_i, $1 \leq i \leq i^*$.

For $i = 2, \ldots, i^*$, choose any $\Delta_i > 0$ such that $\frac{i\lambda_i - \Delta_i}{K} < \pi_i$, and set $\Delta = \min\{\Delta_2, \ldots, \Delta_{i^*}, \frac{K}{4}\}$. Clearly, after at most Δ iterations, the probability of choosing a copy of a vertex of degree $i \geq 2$ is at least ϕ_i. Therefore, for $0 \leq j \leq \Delta n$, the random variable Z_j majorises Z_j^*; i.e. for any R:

$$\Pr[Z_j > R] \geq \Pr[Z_j^* > R.]$$

Now the expected increase in Z_j^* at any step is $\frac{\epsilon'}{2}$. Thus the lemma follows by letting $\epsilon = \frac{\epsilon'}{4}$, as it is well-known (see for example [9]) that $Z_{\delta n}^*$

is a.s. concentrated around its expected value which is $2\delta\epsilon n$, and that the probability of deviating from the expected value by more than $\Theta(n)$ is as low as claimed. $\square$

We have just shown that Z_j a.s. grows large. However, we really want to analyze X_j. In order to do this, recall that the random variable I_j is defined to be the number of pairs exposed by the time that the jth vertex is partially exposed; i.e. $W_{I_j} = Z_j$.

Lemma 9 *There exists $0 < \delta' < \Delta$ such that for any $0 < \delta \leq \delta'$ there a.s. exists some $1 \leq I \leq I_{\lceil \delta n \rceil}$. such that $X_I > \gamma n$, where $\gamma = \min(\frac{\epsilon\delta}{2}, \frac{1}{4})$. Moreover, the probability of the converse is at most $(z_2)^n$ for some $0 < z_2 < 1$, dependent on δ.*

Proof:

For simplicity, we will assume that δn is an integer.

We will count W, the number of backedges formed before either $X_i > \gamma n$ or $I_{\delta n}$ pairs have been exposed. We claim that we can choose δ' such that a.s. $W < \frac{\gamma}{2} n$ for $\delta \leq \delta'$.

At any step $i, 1 \leq i \leq I_{\delta n}$, the probability that an open vertex-copy is chosen is $\frac{X_i}{Kn-2i} + o(1)$, regardless of the choices made previous to that step. Now $I_j \leq j + Y_{I_j} \leq j + \frac{Z_j}{2} \leq (\delta + \epsilon\delta)n$.

Therefore, at each step, the probability that such a backedge is formed is 0 if $X_j > \gamma n$ and at most

$$p = \frac{\frac{1}{2}\delta\epsilon}{K - 2\delta - 2\delta\epsilon}$$

if $X_j \leq \gamma n$.

Thus the number of such copies chosen is majorised by the binomial variable $\mathrm{BIN}(p, I_{\delta n})$.

Therefore the lemma follows so long as $pI_{\delta n} \leq p(\delta + \epsilon\delta)n < \frac{\gamma}{2} n$, which is equivalent to $4\delta + 4\delta\epsilon < K$, yielding δ'.

Now if $X_i \leq \gamma n$ for all $1 \leq i < I_{\delta n}$ then W is equal to $Y_{I_{\delta n}}$.

Therefore $X_{I_{\delta n}} = Z_{\delta n} - 2Y_{I_{\delta n}}$ which with probability at least $1 - (z_1)^n$ is at least $\frac{\delta\epsilon}{2} n$, which yields our result. $\square$

Now that we know that X_i a.s. gets to be as large as $\theta(n)$, we can show that there is a.s. a giant component:

Lemma 10 *There exists $\zeta_1, \zeta_2 > 0$ such that the component being exposed at step $I = I_{\lceil \delta' n \rceil}$ will a.s. have at least $\zeta_1 n$ vertices and $\zeta_2 n$ cycles. Moreover, the probability of the converse is at most $(z_3)^n$, for some fixed $0 < z_3 < 1$.*

Proof:

Note that at this point there are a.s. at least $n - 2\delta' n - \gamma n > \frac{n}{5}$ unexposed vertices. Form a set β consisting of exactly one copy of each of them.

There is a set χ of X_I open vertex-copies whose partners must be exposed before this component is entirely exposed. We will show that a.s. at least $\zeta_1 n$ of these will be matched with members of β, and at least $\zeta_2 n$ of these will be matched with other open vertex-copies from χ. Clearly this will prove the lemma.

Now there are $M - 2I$ vertex-copies available to be matched. Our procedure for exposing F simply generates a random matching amongst them where each matching is equally probable. The expected number of pairs containing one vertex from each of χ, β is at least $\frac{n}{5}\left(\frac{X_I}{2M-I}\right)$, and the expected number of pairs of open vertex-copies which form an edge of F is $\left(\frac{M}{2} - I\right)\left(\frac{X_I}{M-2I}\right)^2$.

The previous lemmas give us a lower bound of $2\zeta_1 n, 2\zeta_2 n$ on these numbers, and it follows from the Chernoff bounds that these numbers are a.s. at least half of their expected values with the probability of the converse as low as claimed. Therefore the component a.s. has at least $\zeta_1 n$ vertices and at least $\zeta_2 n$ cycles. $\square$

And now Lemma 7 follows quite easily:

Proof of Lemma 7:

This is clearly a corollary of Lemma 10. $\square$

We will now see that F a.s. has only one large component.

Lemma 11 *If F is a random configuration as described in Lemma 7, then F a.s. has exactly one component on more than $T \log n$ vertices, for some constant T dependent on the degree sequence.*

Proof:

We have already shown that F a.s. has at least one giant component of size at least $\zeta_1 n$. We will see here that no other components of F are large.

Consider any ordered pair of vertices (u, v). We say that (u, v) has *property A* if u and v lie on components of size at least $\zeta_1 n$ and $T \log n$ respectively. We will show that for an appropriate choice of T, the probability that (u, v) has property A is $o(n^{-2})$, which is enough to prove the lemma.

Recall that we may choose any vertex-copy we wish to start the exposure with. We will choose u.

By Lemma 9, there a.s. exists some $I \le I_{\lceil \zeta_1 n \rceil}$, such that $X_I > \gamma n$ where

$\gamma = \min(\frac{\epsilon\delta'}{2}, \frac{\epsilon\zeta_1}{2}, \frac{1}{4})$, and so we can assume this to be the case. Note that if after I steps, we are not still exposing the first component, C_1, or if we have exposed a copy of v, then (u, v) does not have property A, so we will assume the contrary. Define χ to be the set of open vertex-copies after I steps.

Here we will break from the standard method of exposure. We will start exposing v's component, C_2, immediately, and put off the exposure of the rest of C_1 until later. We will see that if C_2 gets too big, then it will a.s. include a member of χ.

We expose C_2 in the following way. We start by picking any copy of v, and exposing its partner. We continue exposing pairs, always choosing a copy of a partially-exposed vertex which is known to be in C_2 (if one is available), and exposing its partner. We check to see if this partner lies in χ. This would imply that v lies in C_1. Once C_2 is entirely exposed, if it is disjoint from C_1 then we return to exposing the rest of C_1 and continue to expose F in the normal manner. Note that this is a valid way to expose F.

At each step, the probability that a member of χ is chosen is at least $\frac{\gamma}{K}$. Also, if v lies on a component of size greater than $T \log n$, then it must take at least $T \log n$ steps to expose this component. Therefore, the probability that v lies on a component of size greater than $T \log n$ which is not C_1 is at most:

$$\left(1 - \frac{\gamma}{K}\right)^{T \log n} = \mathrm{o}\left(n^{-2}\right)$$

for a suitable value of T.

Therefore the expected number of pairs of vertices with property A tends to zero as $n \to \infty$, so a.s. none exist. □

It only remains to be shown that F a.s. has no small components with more than one cycle.

Lemma 12 *If F is a random configuration as described in Lemma 7, then F a.s. has no multi-cyclic component on at most $T \log n$ vertices, for any constant T.*

Proof:

Consider the probability of some vertex v lying on such a component.

We will insist that we expose an edge containing v in the first execution of Step 2(a). Now if this component contains at most $T \log n$ vertices, then it is entirely exposed after at most $o(n^{1/4})$ steps as the maximum degree is $n^{1/4-\epsilon}$.

At each point during the exposure we can assume $X_i < n^{1/4}$, as otherwise X_i wouldn't be able to return to zero quickly enough.

Therefore, at each step, the probability that a backedge is formed is at most $o(1)n^{1/4}/(M - 2n^{1/4}) = o(n^{-3/4})$.

Therefore, the probability that at least 2 cycles are formed is at most:

$$o(1) \times \binom{n^{1/4}}{2} (n^{-3/4})^2$$

$$= o\left(n^{-1}\right)$$

Therefore the expected number of vertices lying on such components is $o(1)$, and hence a.s. none exist. $\square$

And now we have Theorem 1(a):

Proof of Theorem 1(a):

This clearly follows from Lemmas 1, 2, 7, 11, and 12. $\square$

It is worth noting that by analyzing the number of open vertices of each degree more carefully throughout the exposure of F, it is possible to compute the size of the giant component more precisely. In fact we can find a $\kappa(\mathcal{D})$ such that the size of the giant component is a.s $(1 + o(1))\kappa(\mathcal{D})n$. Details will appear in a future paper.

4 Applications

Here are a few applications of Theorem 1. The first is a new proof of a classical result concerning the double-jump threshold:

Theorem 2 *For $c > \frac{1}{2}$ $G_{n,M=cn}$ a.s. has a giant component, while for $c < \frac{1}{2}$, $G_{n,M=cn}$ a.s. doesn't have one.*

Proof:

It is well known (see for example [7]) that such a graph a.s. has

$$\frac{(2c)^i}{i!} e^{-2c} n + o(n^{.51})$$

vertices of degree i for each $i \leq O(\log n / \log\log n)$, and no vertices of higher degree.

Now expose G by first exposing its degree sequence, and then choosing a random graph on that degree sequence. We will a.s. get a sequence $\mathcal{D}$ which satisfies all the conditions of Theorem 1 and for which:

$$Q(\mathcal{D}) = \sum_{i\geq 1} i(i-2)\frac{(2c)^i}{i!}e^{-2c}$$

which is positive for $c > \frac{1}{2}$ and negative for $c < \frac{1}{2}$. □

Note that this only gives an upper bound of $O\left(\frac{(\log n)^3}{(\log\log n)^2}\right)$ for the size of the largest component of G for $c < \frac{1}{2}$, rather than the proper upper bound of $O(\log n)$.

As mentioned earlier, this work was motivated by the study of minimally 4-chromatic subgraphs of a random graph G. Recall that such a subgraph must have minimum degree at least 3. This is of interest mainly in the study of the chromatic number of sparse random graphs, as if $\chi(G) \geq 4$, then G must have a minimally 4-chromatic subgraph, H.

Chvatál [7] showed that if G is a random graph on n vertices and cn edges, then for $c < c^* = 1.442\ldots$, the expected number of subgraphs of G with minimum degree at least 3 tends to 0 with n, while for $c > c^*$, the expected number of such subgraphs is exponentially large in n.

The authors wished to find which such subgraphs could actually be minimally 4-chromatic graphs. We looked at the following condition of Gallai [12]:

Definition *If H is a graph with minimum degree r, then the* **low graph** *of H ($L(H)$) is the subgraph induced by the vertices of degree r.*

Theorem 3 *If H is a minimally k-chromatic graph with minimum degree $k-1$, then $L(H)$ has no even cycles whose vertices do not induce a clique.*

We used this to prove the following:

Theorem 4 *Let H be a random graph on n vertices and at most $1.793n$ edges with minimal degree 3. H is a.s. not a minimally 4-chromatic graph. Moreover, the probability of failure is at most z^n, for some fixed $0 < z < 1$.*

Outline of Proof:

$L(H)$ is a graph whose vertices are all of degree $0, 1, 2, 3$. We showed that the degree sequence of $L(H)$ could be approximately determined by the edge-density of H, and that all graphs on that degree sequence were equally likely to appear as $L(H)$. It then followed from Theorem 1 that if H has edge-density at most 1.793 then $L(H)$ a.s. has $O(n)$ cycles. We then showed that a.s. at least one of these cycles was even and of length at least 6, and the result followed from the fact that $L(H)$ has no cliques on more than 4 vertices. □

We used Theorem 4 to show:

Corollary 4 *There exists $\delta > 0$ such that if G is a random graph on n vertices and δn edges, then the expected number of minimally 4-chromatic subgraphs of G is exponentially small, while the expected number of subgraphs of G with minimum degree at least 3 is exponentially large in n.*

Outline of Proof:

It follows from the results of [7] that for c slightly larger than c^*, the expected number of such subgraphs of G with edge-density at least 1.793 is exponentially small. □

The details to Theorem 4 and Corollary 4 will appear in a future paper, and can also be found in [18].

It is worth noting that Frieze, Pittel, and the authors [18], used a different type of argument to show that for $c < 1.756$ a random graph with edge-density c a.s. has no subgraph with minimum degree at least 3, and hence is a.s. 3-colourable.

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