CSC 2427 Fall 2014, Assignment #2

Due: Thur, Nov 27 at the **beginning** of class

You can work with other students. You must write your own solutions. List everyone that you worked with.

1. **15 pts** Choose some $\alpha < \frac{1}{2}$ and prove that w.h.p. a random 3-regular graph has no independent set of size greater than αn.

2. **30 pts** Here you will write a relatively short proof that w.h.p. $G_{n,p=c/n}$ has a large 3-core if c is a lot bigger than the 3-core threshold. Let $G = G_{n,p=100/n}$. (Remark, if you wish, you can use a larger constant value for c, and you can change some of the constants below.)

   (a) Prove that w.h.p. for all $a \leq \frac{n}{10}$, every subgraph $S \subset G$ with a vertices has fewer than $20a$ edges. **Hint:** Bound the expected number of subgraphs on a vertices and with $20a$ edges. And recall that the version of the Chernoff bound that I gave in class only holds for $t \leq E(X)$, so if you wish to bound a binomial variable, you should either do it by a direct combinatorial count, or find a stronger version of Chernoff.

   (b) Prove that w.h.p. G has fewer than $\frac{n}{20}$ vertices of degree at most 50.

   Consider the following procedure:

   Let Z be the set of all vertices of degree at most 50 in G.
   While there are any vertices outside of Z that have at least 40 neighbours in Z, add one such vertex to Z.

   (c) Prove that, when the procedure terminates, any vertex not in Z must be in the 3-core of G.

   (d) Prove that w.h.p., when the procedure terminates, $|Z| \leq \frac{n}{10}$. **Hint:** To the contrary, consider a step at which $|Z|$ is exactly $\frac{n}{10}$; show that this would contradict part (a).

3. **30 pts** Choose a random graph in the following manner: Start with a random 4-regular graph on n vertices. Then remove each edge, independently, with probability p.

   Find a constant p^* such that for $p > p^*$, the graph w.h.p. has a linear-sized 3-core, and for $p < p^*$, the graph w.h.p. has an empty 3-core. **Hint:** p^* is rational.