CSC 2427 Fall 2014, Assignment #1

Due: Thur, Oct 2 at the **beginning** of class

You can work with other students. You must write your own solutions. List everyone that you worked with.

1. **30 pts** Consider a random k-uniform hypergraph (every hyperedge contains exactly k vertices) with n vertices and $M = cn$ hyperedges. If you prefer, you can work with the $G_{n,p}$ version of a random hypergraph model; your first step will be to define this model - you can assume that they are roughly equivalent in the same sense as the $G_{n,M}, G_{n,p}$ graph models.

   Determine a constant c_k and prove that: (i) for any constant $c < c_k$, w.h.p. the components of the random hypergraph are all small (quantify what you mean by "small"); (ii) for any constant $c > c_k$, w.h.p. the random hypergraph has a giant component.

2. **20 pts** Prove the following statement twice - once using the second moment method and once using the Simple Concentration Bound:

   Fix any tree T and any constant $c > 0$. There exists a constant $z = z(T, c)$ such that a.s. $G_{n,p=c/n}$ has $zn + o(n)$ components that are isomorphic to T. Note: T is fixed and so $|T| = O(1)$.

3. **30 pts** Consider an instance of random 2-SAT. You can work with either the $G_{n,M}$ or the $G_{n,p}$ version of random 2-SAT; in the latter, each of the $4\binom{n}{2}$ potential clauses is included with probability p. You will determine the satisfiability threshold.

   Any instance of 2-SAT gives rise to a directed graph, as follows: For each variable x, there are two vertices $x, \bar{x}$. For each clause, there are two directed edges, where $a \to b$ means that the clause implies "if a then b"; eg. the clause $(x \vee \bar{y})$ yields the edges $\bar{x} \to \bar{y}$ and $y \to x$. For a vertex u in the digraph, we use $-u$ to represent the negation of u; eg. if $u = \bar{x}$ then $-u = x$.

   (a) A *bicycle* is a path $u_1, ..., u_t$ in the directed graph, along with two additional edges: $-u_a \to u_1$ and $u_t \to -u_b$ for some $a \leq b$. We also require that no two vertices of $u_1, ..., u_t$ correspond to the same variable; i.e. we cannot have $u_i = u_j$ or $u_i = -u_j$ for any $i \neq j$.

   You can assume that the following Claim is true; try to think about how to prove it, but you don't have to submit a proof.

   Claim: The 2-SAT formula is satisfiable iff the directed graph does not have a bicycle.

   Bound the expected number of bicycles to prove that if $c < c^*$ then w.h.p. the random formula is satisfiable.

   (b) A *flower* is a bicycle in which $a = b$. So if the directed graph has a flower, then the formula is not satisfiable.

   Choose suitable numbers t, a and then apply the second moment method to prove that if $c > c^*$ then w.h.p. the directed graph contains a flower with those values of t, a.