1. Show that the following TRIANGLE decision problem belongs to P.

   Input: An undirected graph $G = (V, E)$.

   Question: Does G contain a "triangle", i.e., a subset of three vertices with all edges between them present in the graph?

   Solution: The following algorithm decides TRIANGLE.

   On input G:

   For each triplet of vertices (u, v, w) in G:

   Return True if G contains all edges $(u, v), (v, w), (w, u)$.

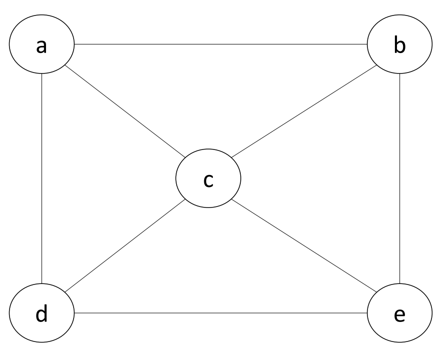
   Return False.

   By definition of TRIANGLE, the algorithm will return True iff G contains a triangle.
   Let $n = |V|$ (number of vertices) and $m = |E|$ (number of edges) in G. There are $\binom{n}{3} = \Theta(n^3)$ many triplets of vertices in G, and it is possible to enumerate them one by one in time $\mathcal{O}(n^3)$. For each triplet, it takes time $\mathcal{O}(m)$ to verify the presence of the three edges (depending on how G is encoded, this could be reduced). So the algorithm runs in time $\mathcal{O}(mn^3)$.

2. Show that the following CLIQUE decision problem belongs to NP.

   Input: An undirected graph $G = (V, E)$ and a positive integer k.

   Question: Does G contain a k-clique, *i.e.*, a subset of k vertices with all edges between them present in the graph?



   For example, the shown graph contains a 3-clique (there are sets of 3 vertices with all edges between them, e.g., $\{a, b, c\}$), but it does not contain a 4-clique (every set of 4 vertices is missing at least one edge, e.g., $\{a, b, c, d\}$ is missing (b, d)).

   Solution:

   Verifier for CLIQUE:

   On input $< G, k, c >$, where c is a subset of vertices:

   Return True if c contains k vertices and G contains edges between all pairs of vertices in c; return False otherwise.

Verifier runs in polytime (where $n = |V|, m = |E|$): checking all pairs of vertices in c takes time $\mathcal{O}(k^2 m)$ ($\mathcal{O}(k^2)$ pairs in c, times $\mathcal{O}(m)$ for each one).
If $< G, k > \in$ CLIQUE, then verifier returns True when $c =$ a k-clique of G;
if verifier returns True for some c, then $< G, k > \in$ CLIQUE (c is a k-clique).

CLIQUE $\in$ P? Unknown (checking all possible subsets not polytime because k not fixed, part of input).

Contrast CLIQUE with TRIANGLE: TRIANGLE $\in$ NP (on input $< G, c >$, check c encodes a triangle in G), but TRIANGLE $\in$ P as well.

What's the difference? Same algorithm to decide CLIQUE takes time $\mathcal{O}(n^{k+1})$, except that k is part of the input (instead of being fixed) so this could be as bad as, *e.g.*, $\mathcal{O}(n^{n/2})$ – not polytime.

3. Show that the following IndependentSet (IS) decision problem belongs to NP.

   Input: An undirected graph $G = (V, E)$ and a positive integer k.

   Question: Does G contain an *independent set* of size at least k, *i.e.*, a subset of vertices $I \subseteq V$ such that $|I| \geqslant k$ and G contains **no** edge between any two vertices in I?

   Solution: Verifier for IS:

   On input (G, k, c), where c is a subset of k vertices of G:

   Return True if G does not contain any one of the edges between vertices in c; return False otherwise.

   This takes time $\mathcal{O}(k^2 m)$: there are $\mathcal{O}(k^2)$ pairs of vertices in c and $\mathcal{O}(m)$ edges to check for each one.

   Also, if there is some value of c such that the verifier returns True for (G, k, c), then G contains an independent set of size k or more (c is such an independent set), and if G contains an independent set of size k or more, then there is some value of c such that the verifier returns True for (G, k, c) (let c be the independent set).

   It does not appear likely that IS $\in$ P, because checking every subset of k vertices takes more than polynomial time (time $\Omega(n^k)$ where k can depend on n), and there is no obvious way to speed this up.

4. Show that the following UNARY-PRIMES decision problem belongs to P.

   Input: 1^n (*i.e.*, a string of '1's of length n).

   Question: Is n prime?

   Solution: The following algorithm decides UNARY-PRIMES:

   On input 1^n:
   For $k = 2, 3, ..., n - 1$:
   If k divides n, return False
   Return True if no value of k worked.

   The algorithm returns True iff n is prime, by definition. The division can be carried out by repeated subtraction, which takes time $\mathcal{O}(n^2)$ for each value of k, so the entire algorithm runs in time $\mathcal{O}(n^3)$.

   NOTE: This works because n is the **size** of the input at the same time as the **value** of the input. For any other base, this would **NOT** work because the value m would be represented using $n = \log m$ many digits so the size would be proportional to $n = \log m$ and the running time would become **exponential** (as a function of n).