Problem 1. If graph G is connected and contains more than $n-1$ edges (where $n = |V|$, as usual), and if there is a unique edge e with minimum cost, then is e guaranteed to be in every MST of G? If so, give a convincing argument. If not, provide a counter-example. In this case, what other conditions can you put on G to guarantee that e will be in every MST of G?

Solution. Proof. For a contradiction, suppose T is a MST that does not contain e. Then T contains some path between the endpoints of e. Pick some edge e' on this path. Then $T' = T \cup \{e\} - \{e'\}$ is a spanning tree. But $c(T') = c(T) + c(e) - c(e') < c(T)$ because $c(e') > c(e)$ (by assumption, $c(e)$ is minimum and unique). This contradicts the fact that T is a MST.

Problem 2. If graph G is connected and contains more than $n-1$ edges (where $n = |V|$, as usual), and if there is a unique edge e with maximum cost, then is e guaranteed *not* to be in any MST of G? If so, give a convincing argument. If not, provide a counter-example. In this case, what other conditions can you put on G to guarantee that e will be in no MST of G?

Solution. Counter-example:

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G = a --1-- b --2-- c --4-- d
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Additional condition:

If e belongs to some cycle C in G, then e belongs to no MST.

Proof. For a contradiction, suppose e belongs to a MST T. Consider $T - \{e\}$. This is made up of two connected components. Because e belongs to some cycle C, there is a way to get from one endpoint of e to the other along this cycle. So there is at least one edge e' of C that connects both components. Then $T' = T \cup \{e'\} - \{e\}$ is a spanning tree, and $c(T') = c(T) + c(e') - c(e) < c(T)$ because $c(e') < c(e)$ (by assumption, $c(e)$ is maximum and unique). This contradicts the fact that T is a MST.

Problem 3. For every graph G whose edge weights are all distinct, every MST of G contains the two edges e_1, e_2 with the two smallest weights. If this is true, give a convincing argument. If not, provide a counter-example. In this case, what other conditions can you put on G to guarantee that e_1, e_2 will be in every MST of G?

Solution. TRUE. Suppose G is a graph whose edge weights are all distinct. Let e_1, e_2 be the two edges with smallest weights ($c(e_1) < c(e_2) <$ cost of every other edge).
For a contradiction, suppose T is a MST that does not contain both e_1 and e_2. WLOG, suppose T does not contain e_2. Consider the endpoints (u, v) of e_2. They are connected by a path P in T. This path contains at least two edges (it cannot contain just one as this would just be e_2 itself). Since e_1, e_2 have the two smallest edge costs, there is at least one edge e' on P with $c(e') > c(e_2)$. But then, $T' = T \cup \{e_2\} - \{e'\}$ is a spanning tree and $c(T') < c(T)$.
This contradicts the fact that T is a MST. Hence, every MST of G contains both e_1 and e_2.