Integer Programming:

Integer programming: more restricted version where all constants and variables are integers. NP-complete (no efficient algorithm).
Example: Minimum Vertex Cover: Given an undirected graph $G = (V, E)$, Identify a subset of vertices C that *covers* every edge (i.e., each edge has at least one endpoint in C), with minimum size.

We represent this problem as an integer program: use variable x_i for each vertex $v_i \in V$
minimize: $x_1 + x_2 + ... + x_n$
subject to:

- $x_i + x_j \geqslant 1$ for all $(v_i, v_j) \in E$
- $x_i \in \{0, 1\}$ for all $v_i \in V$

This 0-1 integer program is completely equivalent to original problem, through correspondence: v_i in cover iff $x_i = 1$. In more detail:

- Any vertex cover C yields feasible solution $x_i = 1$ if $v_i \in C$, 0 if $v_i \notin C$ because each constraint $x_i + x_j \geqslant 1$ satisfied (C must include one endpoint of each edge).
- Any feasible solution to LP yields vertex cover $C = \{v_i \in V : x_i = 1\}$ because for each edge (v_i, v_j), constraint $x_i + x_j \geqslant 1$ ensures C contains at least one of v_i, v_j.

Unfortunately, Integer Programming (IP) is NP-hard, so the problem cannot be solved in polytime this way. In the next lecture we will propose an algorithm to approximate the solution.

Linear relaxation method: remove restriction of x_i to integer values
minimize: $x_1 + x_2 + ... + x_n$ subject to:

- $x_i + x_j \geqslant 1$ for all $(v_i, v_j) \in E$
- $0 \leqslant x_i \leqslant 1$ for all $v_i \in V$

Solution can be found in polytime, but may include fractional values of x_i's.
Example: $G = (V, E)$ with $V = \{v_1, v_2, v_3\}$, $E = \{(v_1, v_2), (v_2, v_3), (v_1, v_3)\}$ becomes
minimize: $x_1 + x_2 + x_3$
subject to:

- $x_1 + x_2 \geqslant 1$
- $x_2 + x_3 \geqslant 1$
- $x_1 + x_3 \geqslant 1$
- $0 \leqslant x_1, x_2, x_3 \leqslant 1$

with solution $x_1 = x_2 = x_3 = 1/2$.

Then how to solve this problem? Suppose the solution to the relaxed LP is x_i^* for each variable x_i. Create cover as follows: for each $v_i \in V$, put $v_i \in C$ iff $x_i^* \geqslant 1/2$. (C is a cover because constraint $x_i + x_j \geqslant 1$ guarantees at least one of x_i^*, and x_j^* is $\geqslant 1/2$ for each edge (v_i, v_j).)
Approximation ratio?
Consider minimum vertex cover C'. For $i = 1, ..., n$, let $x'_i = 1$ if v_i in C'; $x'_i = 0$ otherwise. $x'_1, ..., x'_n$ is a

solution to linear program that satisfies all constraints with 0-1 values so $|C'| = \sum x'_i \geqslant \sum x^*_i$ where x^*_i is optimal solution to linear program with no restriction on values, so guaranteed to be at least as small as any other solution, including those with additional restrictions. For $i = 1, ..., n$, let $\hat{x}_i = 1$ if $x^*_i \geqslant 1/2$; $\hat{x}_i = 0$ otherwise. Then, for each i, $\hat{x}_i \leqslant 2x^*_i$ so $|C| = \sum \hat{x}_i \leqslant 2 \sum x^*_i \leqslant 2|C'|$ (by equation above) Hence, $|C|$ is no more than twice the size of a minimum vertex cover.

Q: In general, how can we compute ratio without knowing OPT?
A: Use a lower bound. Find another value LB that's easy to compute and for which you can prove $LB \leqslant OPT$ and $A \leqslant r \times LB$, as in the above example for vertex cover.