Example: Show 3SAT $\leqslant_p$ 3-coloring.

(1)- Transformation: Given a formula F, we create an instance of 3-coloring problem (that is a graph $G_F = (V, E)$):

- For each variable create a node (Figure 1).
- For the negation of each variable create a node (Figure 1).
- Connect each variable to its negation (Figure 1).
- Create 3 nodes T, F, B and connect them in a triangle (since they are all connected, these nodes should get different colors so each one can represent a color.) (Figure 1).
- Connect the nodes corresponding to any variable or negated variable to B (Figure 1).
- For each clause in F, add an extension (a subgraph) with 6 nodes and 13 edges (Figure 2).

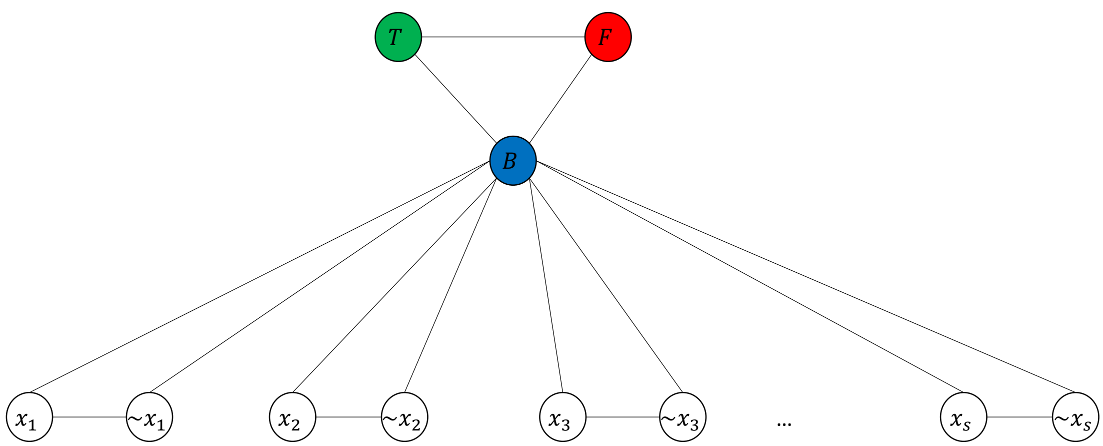

Figure 1: The subgraph containing all literals and 3 nodes T, F, B

(2)- This transformation is polytime; the size of G_F is comparable to the size of F.

(3)- F is satisfiable if and only if G_F is 3-colorable. (prove both $\Rightarrow$ and $\Leftarrow$)

Example: Show Subset-Sum $\leqslant_p$ Knapsack1.

Knapsack1:

Input items $I_i = (w_i, v_i)$ where w_i is the weight of the item and v_i is its value; number k; capacity C

Output A subset S of these items such that the weight of all items in S is less than C and sum of the value of all items in S is EXACTLY k.

Transformation: Having an instance of Subset-Sum (SS) that is a set of numbers $\{a_1, ..., a_n\}$ and a number t, we create an instance of Knapsack1.

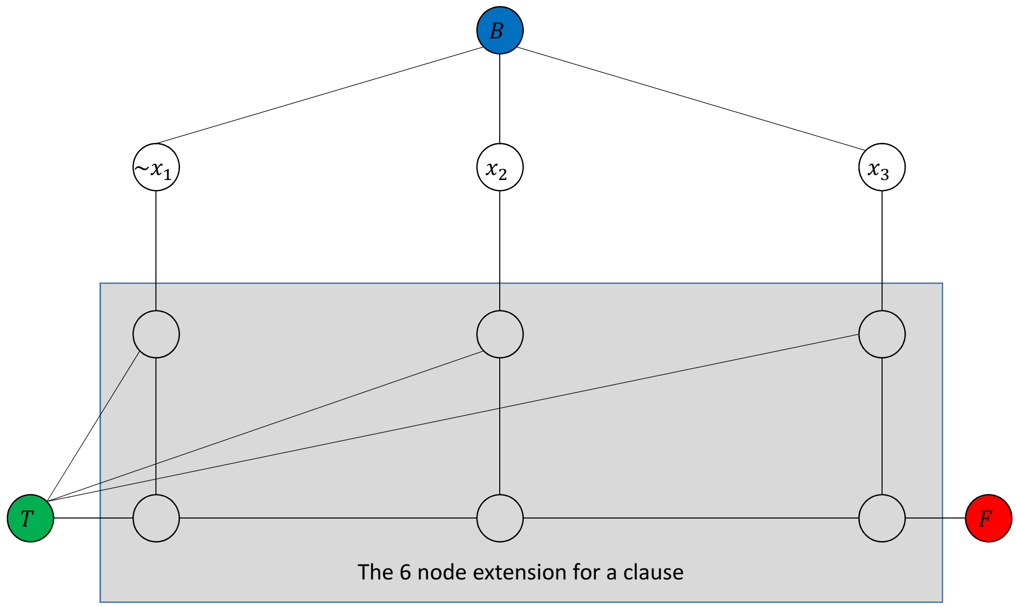

Figure 2: The subgraph added for the clause $(\sim x_1 \vee x_2 \vee x_3)$

- For each number a_i, create an item I_i with $w_i = 0$ and $v_i = a_i$.
- Set $C = 1$ and $k = t$.

This transformation is polytime. Show that the answer to the instance of SS is YES iff the answer to the created instance of Knapsack1 is YES.

Example: Show Subset-Sum $\leqslant_p$ Knapsack2.

Knapsack2:

Input items $I_i = (w_i, v_i)$ where w_i is the weight of the item and v_i is its value; number k; capacity C

Output A subset S of these items such that the weight of all items in S is less than C and sum of the value of all items in S is AT LEAST k.

Transformation: Having an instance of Subset-Sum (SS) that is a set of numbers $\{a_1, ..., a_n\}$ and a number t, we create an instance of Knapsack2.

- For each number a_i, create an item I_i with $w_i = a_i$ and $v_i = a_i$.
- Set $C = t$ and $k = t$.

This transformation is polytime. Show that the answer to the instance of SS is YES iff the answer to the created instance of Knapsack2 is YES. (Remember that this is true because in the Knapsack2 instance we conclude (1) sum of the weights of all items in the subset S is less than capacity t and (2) sum of their values is at least t. Since both the weight and value of item I_i is a_i, this means that sum of all a_i in S is exactly t.)

Example: Show Knapsack2 $\leqslant_p$ The following Scheduling problem.

Scheduling:

Input n jobs $J_i = (d_i, p_i, v_i)$ where d_i is its deadline, p_i is its processing time, and v_i is its value and they are all integers. Number V

Output Can we schedule the jobs (non-conflicting) so that sum of the value of scheduled jobs is at least V?

Transformation: Having an instance of Knapsack2 that is a set of items $I_i = (w'_i, v'_i)$ and two numbers C and k, we create an instance of Scheduling problem.

- For each item I_i, create a job $J_i = (d_i = C, p_i = w'_i, v_i = v'_i)$
- Set $V = k$.

This transformation is polytime. Show that the answer to the instance of Knapsack2 is YES iff the answer to the created instance of scheduling is YES.