This document contains the key ideas to solve A1 questions. Note that many details may be ignored in several places.

Q2.

Assume the road starts at a, ends at b. There are n houses at points x_i where $a \leq x_i \leq b$. We represent each house by an interval I_i that starts at $\max(a, x_i - 4)$ and $\min(b, x_i + 4)$. Identifying the minimum base stations that cover all houses is equivalent to maximizing the number of non-overlapping intervals (why?). Therefore the earliest finish time greedy algorithm (EFT) solves the problem. Place a base station at the end point of each selected interval.

Q3.

We need to remove 7 edges. In a graph with distinct edges, the edge with maximum weight in any cycle does not belong to the MST (why?). You can find a cycle using BFS algorithm. Thus one algorithm follows: Find a cycle by DFS ($O(n)$). Find the edge with maximum weight in that cycle ($O(n)$). Remove that edge. Continue this for 7 items. Thus the total run time is $O(n)$.

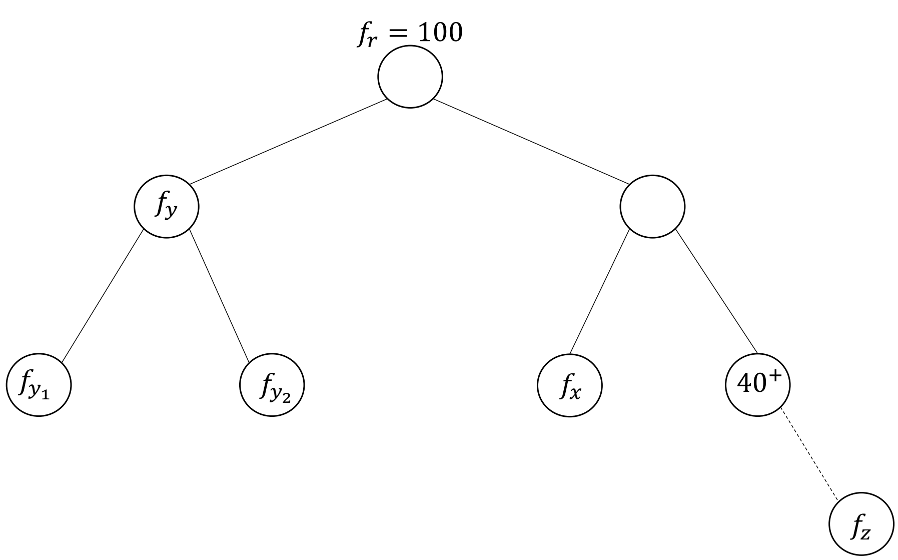
Q4.

The longest codeword (the maximum height of the huffman tree) is $n - 1$. Note that the huffman tree is a binary tree where each inner node has two children. So a huffman tree with a larger height is not possible.

Example: $f_1 = 1,\ f_2 = 1,\ f_3 = 2,\ f_4 = 4,\ f_5 = 8,\ f_6 = 16,\ \cdots,\ f_n = 2^{n-2}$. In general it happens when $f_i \geq \sum_{j=1}^{i-1} f_j$.

Q5.

Proof by contradiction. All of the numbers are in percentage.



Assume there exist one character (z) with a frequency > 40 but there is no codeword of length 1.

Note that the huffman algorithm sorts the characters according to the frequency (non-decreasing) and creates the tree based on this ordering. Assume character z has a frequency $f_z > 40$. Since there is no codeword of length 1, in the last iteration two inner nodes are merged to create the root node r. WLOG, assume character z is in the right subtree. Thus, the root of the right subtree is composed of a node with frequency greater than 40 and a node with frequency f_x. Moreover, assume the root of the left subtree has a frequency of f_y. Therefore, $f_x \leq f_y$ and $f_z \leq f_y$. Since $f_z > 40$, $f_y > 40$. Thus, $f_x < 20$ (sum of all frequencies is 100). Now, assume the children of the node with frequency f_y have frequencies f_{y_1} and f_{y_2}. Either (1) these two nodes are considered and merged after node z then they should have a frequency of more than 40 that is not possible, or (2) these nodes are considered and merged before node z so $f_{y_1} \leq f_x$ and $f_{y_2} \leq f_x$. Thus $f_y = f_{y_1} + f_{y_2} < 40$. This is in contradiction with $f_y > 40$. Hence the first assumption is incorrect.

Q6.

Assume OPT is an optimal solution and S is the result of EFT. We define a map $h : OPT \to S$. Similar to lecture, define $h(I) = J$ where $I \in OPT$; $J \in S$; J conflicts I; and among all intervals in S conflicting I, J has the earliest finish time. Note that if $h(I) = J$ then $f(J) \leq f(I)$ otherwise I should be in S not J.

First h is a function: $h(I)$ has a unique value. (why?)

Second, at most two jobs in OPT can be mapped to a single job in S. For this part we use proof by contradiction technique. Assume there are three intervals $I_1, I_2, I_3 \in OPT$ that are mapped to the same interval $J \in S$. So all of these intervals should conflict with J either by having the same type or overlapping. At most one of I_1, I_2, and I_3 can have the same type as J otherwise OPT is conflicting and not a solution. Thus, at least two intervals (say I_1 and I_2) overlap with I. Since $f(J) \leq f(I_1)$ and $f(J) \leq f(I_2)$: $f(J) \in I_1$ and $f(J) \in I_2$. Thus, I_1 and I_2 overlap and OPT is conflicting and not a solution.

In all cases OPT is not a solution. This is a contradiction. Hence EFT is a 2-approximation algorithm for JISP.

Q7.

Semantic array: $V[i]$ contains the maximum total weight of an independent set utilizing nodes 1 to i.

$V[0] = 0$.

$V[i] = \max(V[i-1], w_i + V[i-2])$.

```
V[0] = 0
for i = 1 to n do
    Used[i] = false
for i = 1 to n do
    V[i] = V[i − 1]
    if V[i] < w_i + V[i − 2] then
        V[i] = w_i + V[i − 2]
        Used[i] = true
S = {} // the optimal independent set
i = n
while i > 0 do
    if Used[i] then
        S = S ∪ {V_i}
        i = i − 2
    else
        i = i − 1
```