The Complexity of Composition

New Approaches to Depth and Space

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Composition

When can we save by doing two separate things together?

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Non-computational example: running all your errands in one trip

Composition

Composition theorem:

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- *complexity measure* $s(\cdot)$

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- functions f and g
- *composition* $f \circ g$
- *complexity measure* $s(\cdot)$

Central goal: determine whether

$$s(f \circ g) \approx s(f) + s(g)$$

or

$$s(f \circ g) \ll s(f) + s(g)$$

Composition and lower bounds

What happens if $s(f \circ g) \approx s(f) + s(g)$?

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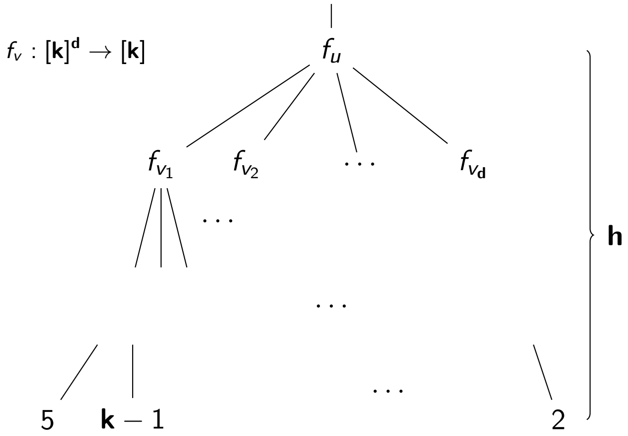
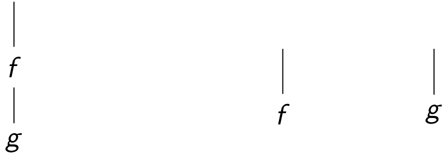
General idea: going from easier functions $f_1 \ldots f_m$ to a harder function $F = f_1 \circ \ldots \circ f_m$

Composition and lower bounds

Tree Evaluation Problem (TreeEval := $\mathsf{TreeEval}_{k,d,h}$)



Composition and lower bounds



"Two kinds of composition" at work:

- *sequential composition*: between layers
- *parallel composition*: within layers

Complexity measures

What does "save" mean? The measure matters!

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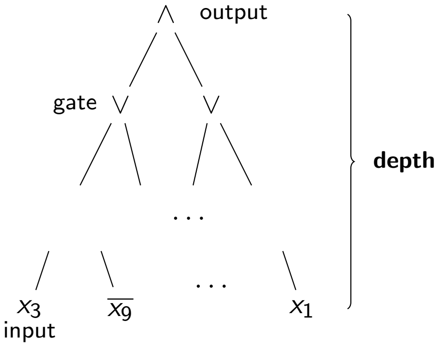
Errands example: can save yourself the *time* it takes to go downtown twice, but cannot save any *money* on the bills themselves.

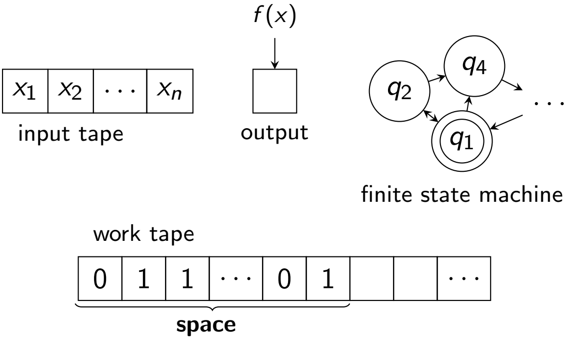
Complexity measures

Model 1: formulas



Complexity measures

Model 2: Turing Machines



Complexity measures

time efficient (P): Turing Machines with time $n^{O(1)}$

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Complexity measures

time efficient (P): Turing Machines with time $n^{O(1)}$

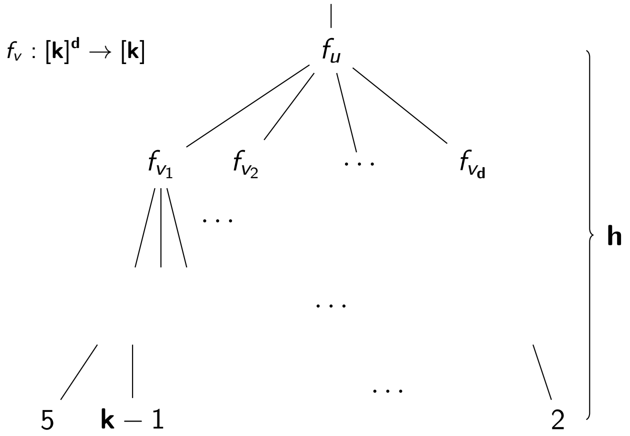
depth efficient (NC^1): formulas of depth $O(\log n)$
space efficient (L): Turing Machines with space $O(\log n)$

Known: $\mathrm{NC}^1 \subseteq \mathrm{P}$ and $\mathrm{L} \subseteq \mathrm{P}$

Central goal: show $\mathrm{NC}^1 \subsetneq \mathrm{P}$ and $\mathrm{L} \subsetneq \mathrm{P}$ using composition

Composition and lower bounds

$\text{TreeEval}_{k,d,h} \in \text{P}$: evaluating it bottom-up takes linear time



Composition and lower bounds

Conjecture 1 [KRW'95]: $\mathsf{TreeEval} \notin \mathsf{NC}^1$

Conjecture 2 [CMWBS'12]: $\mathsf{TreeEval} \notin \mathsf{L}$

Composition and depth lower bounds

KRW conjecture [KRW'95]: $depth(f \circ g) \approx depth(f) + depth(g)$

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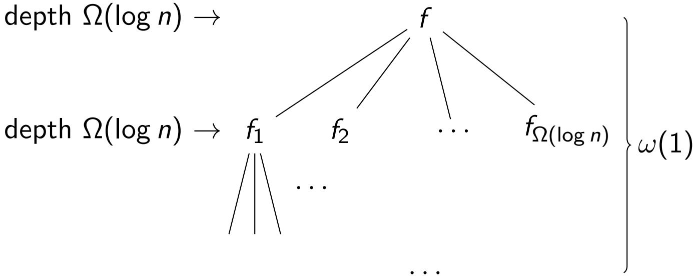

Depth $\omega(1) \cdot \Omega(\log n)$

Composition and depth lower bounds

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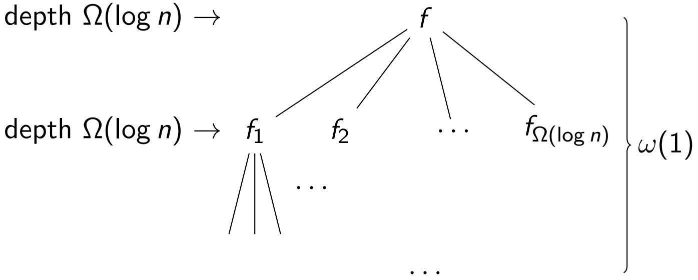

Depth $\omega(1) \cdot \Omega(\log n) \rightarrow \text{TreeEval}_{2,d,h} \notin \text{NC}^1$

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z-f conjecture [CMWBS'12]: computing f while remembering output of g requires space to compute f plus space to remember output of g

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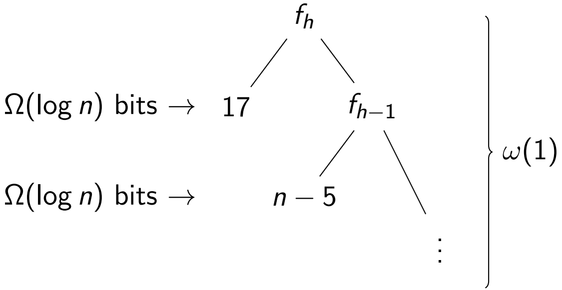

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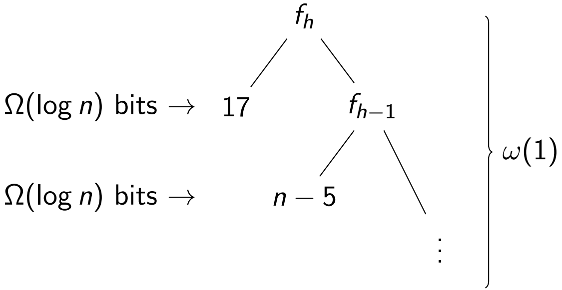

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Part I: getting closer to showing KRW conjecture is true

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Part I: getting closer to showing KRW conjecture is true

Part II: unconditionally showing z-f conjecture is false

Part II

Space

Reusing space

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Proof explicitly refutes z-f conjecture for $f \in \{+, \times\}$:

$$space(z, f) = |z| \ll |z| + space(f)$$

Main contribution: upper bounds on TreeEval

[CM'20,21]: for any k, h, TreeEval$_{k,2,h}$ can be solved in space $O(h \log k / \log h) = o(h \log k)$

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time-space tradeoff to reduce the degree of these polynomials

Application: upper bounds on amortized non-uniform space

[P'17] (informally): every function f can be solved in time $O(n)$ and amortized non-uniform space $O(1)$, as long as we have 2^{2^n-1} copies.

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different sort of space-time tradeoff to reduce degree

Part I

Depth

Communication complexity

Another model of computation: communication complexity

- Alice receives $x \in \mathcal{X}$, Bob receives $y \in \mathcal{Y}$
- goal is to compute $F(x, y)$ together
- allowed to do any amount of computation on their own, charged for every bit exchanged

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We *do* have composition-style results for communication!

Query-to-Communication Lifting

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$$cc(F \circ g) \approx dt(F) \cdot \log m$$

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Main goal: more general g, starting with smaller m

Main contribution: better lifting

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Tools: "structure vs randomness" framework (earlier proofs)

$$+$$

"structure vs randomness" combinatorics to directly handle the random case (the bottleneck for previous proofs)

Application: proof complexity

[AM'20]: given $\tau \in \mathit{UNSAT}$*, it is* NP*-hard to approximate the size of the best Resolution or tree-like Resolution refutation of* τ*.*

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block-width and *graduated* lifting theorems using [LMMPZ'22]

Conclusions

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2. *Broadening and applying our results* (from lifting to KRW, better catalytic computing results)
3. *Structure of our results* (lifting and combinatorics, which classes catalytic techniques inherently lie in)

Works used

1. Shachar Lovett, Raghu Meka, Ian Mertz, Toniann Pitassi, Jiapeng Zhang. *Lifting with Sunflowers*. ITCS 2022.
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Thanks!