Rate Control for Random Access Networks: The Finite Node Case

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***Abstract*— We consider rate control for random access networks. In earlier work, we proposed a rate control mechanism which we analyzed using the well-known slotted Aloha model with an infinite set of nodes. Here, we extend this work to the finite node case and analyze two different packet-scheduling schemes: a backlog-dependent scheme where the retransmission probabilities depend on the total number of a backlogged packets at a given node, and a backlog-independent scheme. Using a Markov chain model, we derive conditions under which the rate control stabilizes the network. We also discuss how this mechanism can be used to provide differentiated quality-of-service both in terms of throughput and delay. We use numerical case studies to illustrate our results.**

I. Introduction

We consider rate control for random access networks that support the CSMA family of protocols [1]. These protocols emphasize zero-coordination by allowing nodes to contend for channel access at will ("free-for-all" approach) [1]. However, if two or more nodes transmit a packet at the same time, then the packets interfere ("collide") and are not successfully received by the destination nodes. Such packets become backlogged and must be retransmitted at a later time. An important feature of such random access networks is that the network can become unstable (*i.e.* the number of backlog packets grows without bound) if the packet generation rate exceeds the channel capacity [1]. To avoid this, we consider a rate control mechanism to stabilize the system.

Rate control for random access networks is challenging as the aggregate input rate and the total backlog cannot be observed directly. To allow a distributed implementation, we require that the control mechanism uses only local information available at each node, namely the channel feedback information indicating whether the channel is idle, or whether a successful transmission or a collision between two or more packets has occurred. In the event of a collision we require that nodes decrease their transmission rates to avoid an increase in the backlog and additional collisions in the future. When the channel is idle, we allow nodes to increase their rates.

In earlier work, we analyzed the above rate control scheme assuming that an infinite set of nodes share the network [2]–[4]. For this case, we obtained the following results: (1) the proposed rate control mechanism achieves a stable operation of the network in the face of excessive bandwidth demand; (2) (under appropriate assumptions) there exists a unique operating point which can be used to characterize the average throughput and packet delay; and (3) the operating point can be chosen a priori by suitably designing the rate control mechanism. An important property of this model is that each node has at most one backlogged packet. Here, we extend this work to the finite node case where we allow multiple backlogged packets at individual nodes. For this case, we study two different retransmission strategies for backlogged packets: a backlog-dependent scheme where the retransmission probabilities depend on the total number of a backlogged packets at a given node and a backlog-independent scheme.

For the analysis of the above rate control, we model the random access network using the slotted Aloha channel [1]. Slotted Aloha and its unslotted version (pure Aloha) has been central to the understanding of random access networks. These two protocols have over the years evolved into a rich family of medium access control schemes, most notably CSMA/CD, the Ethernet standard, and CSMA/CA which is the basis of the IEEE 802.11 standard.

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II. Problem Formulation

In this section, we describe the channel model and rate control algorithm that we consider, and present a Markov chain formulation that we use for the stability analysis.

A. Channel Model

Consider the situation where M nodes communicate over a slotted multiaccess broadcast channel [1]. All packets have the same length and each packet requires one time unit for transmission. All nodes are synchronized such that the transmission of each packet starts at an integer time $t \geq 0$. We refer to the slot beginning at time t as slot t. Whenever a node receives one or more new packets, it will transmit in the next time slot and store the additional new packets as backlogged packets in the buffer where we assume that each node has an infinite buffer. If two or more nodes transmit a packet in a given time slot, then there is a collision and the receiver obtains no information about the contents or source of the transmitted packets. Packets that experience a collision become backlogged and are retransmitted at a later time. We consider two different schemes for retransmitting backlogged packets: *backlog-dependent* and *backlog-independent* packet scheduling.

In the backlog-dependent scheduling, nodes retransmit backlogged packets as follows. Let ϵ, $0 < \epsilon < 1$, be a given constant. When node m, $m = 1, ..., M$, has n_m backlogged packets and did not receive a new packet in the previous time slot, then it will retransmit a backlogged packet with probability $q_m(n_m)$ given by

$$q_m(n_m) = \begin{cases} n_m q_m, & n_m q_m \leq 1-\epsilon, \\ 1-\epsilon, & \text{otherwise,} \end{cases}$$

where q_m, $0 < q_m < 1$ is a fixed retransmission factor associated with node m. Note that nodes with a large number of backlogged packets are more likely to retransmit thus reducing their backlog at a higher rate than nodes with fewer backlogged packets.

In backlog-independent scheduling. when a node m has n_m backlogged packets and did not receive a new packet in the previous time slot, then it will retransmit a backlogged packet with probability q_m, $0 < q_m < 1$.

B. Rate Control Scheme

For above packet retransmission schemes, if the rate at which nodes generate new packets is left uncontrolled then the aggregated traffic load can exceed the network capacity and the random access network can get clogged with collisions. This will lead to an unstable system where the total system backlog will increase without bound. To avoid this, we consider a rate control mechanism to stabilize the system where we use a control signal u, $u \geq 0$, to modulate the arrival rate of new packets to the system.

More precisely, we assume that each node m generates new packets according to an independent Poisson process with rate $\lambda_m(u)$. We refer to $\lambda_m(u)$ as the *rate function* of node m. The aggregate rate function is given by

$$\lambda(u) = \sum_{m=1}^{M} \lambda_m(u).$$

We make the following assumption.

Assumption 1: For each node $m = 1, ..., M$, the rate function $\lambda_m(u)$ is bounded, continuous and strictly decreasing such that $\lim_{u \to \infty} \lambda_m(u) = 0$.

Let Z_t be the random variable indicating the total number of transmission attempts in slot t. Note that Z_t includes the transmissions of new packets as well as the retransmissions of backlogged packets. Let $I[\cdot]$ be the indicator function and let $I[Z_t = 0]$, $I[Z_t = 1]$, and $I[Z_t \geq 2]$ indicate an idle slot, a successful transmission, or a collision, respectively, in slot t. Let u_t be the control signal that determines the rate of arrivals of new packets in slot t. We then update the control u_t by setting

$$u_{t+1} = \big[u_t + \alpha I[Z_t = 0] + \beta I[Z_t = 1] + \gamma I[Z_t \geq 2]\big]^+, \tag{1}$$

where $[x]^+$ denotes $\max\{x, 0\}, x \in \mathbb{R}$. The parameters α, β and γ are real-valued control updates corresponding to the observed channel outputs of "idle", "success" and "collision" respectively. As the aggregate rate is a decreasing function, the control signal should be increased in the event of a collision to reduce the arrival rate of new packets and to avoid an increase in the backlog and additional collisions in the future. In the event of an idle slot the control should be lowered to allow nodes to increase their rates to make more efficient use of the channel. In the event of a successful transmission, the control can be left unchanged, or lowered (raised) in the case of a more conservative (aggressive) control scheme. In line with this intuition, we make the following assumption.

Assumption 2: We have that $\alpha < 0$ and $\gamma > 0$; while β can take on any real value.

In the following, we study whether this scheme stabilizes the slotted Aloha channel and how we should choose the control parameters α, β, and γ, to achieve a desired throughput. We study these issues by analyzing the Markov chain representation of the system.

C. Service Differentiation

An important feature of the above scheme is that it allows service differentiation both with respect to throughput and delay. In particular, the throughput differentiation can be achieved by assigning different rate functions to nodes. Delay differentiation can be achieved by assigning different retransmission parameters to nodes. In Section VI, we illustrate through a numerical case study how these two mechanisms can combined to provide multidimensional quality-of-service.

III. Backlog-Dependent Scheduling

In this section, we study the stability properties of backlog-dependent scheduling.

Let $(\underline{n}_t, u_t)$ denote the state of the system at the beginning of slot t, where $\underline{n}_t = (n_{1,t}, ..., n_{M,t})$ is the vector indicating the number of backlogged packets at each node and u_t denotes the control signal which determines the arrival rates in slot t. The dynamics of the backlog at node m is given by

$$n_{m,t+1} = n_t + Y_{t,m} - I[Z_t = 1 \wedge Z_{t_m = 1}],$$

where $Y_{t,m}$ is a random variable indicating the number of newly arrived packets to node m in slot t and Z_{t_m} is the random variable indicating whether node m is attempting to transmit a packet in slot t. Note that the dynamics of the system is then given by a Markov chain $(\underline{n}_t, u_t)$ where u_t is the value of the control at time t, and $\underline{n}_t = (n_{1,t}, ..., n_{M,t})$ is the vector indicating the number of backlogged-packets at individual nodes. Furthermore, without loss of generality, we can assume a countable state space $\mathcal{S}$ for the Markov chain where the control signal u_t takes on values only from the countable set $\mathcal{U} = \{[\alpha a + \beta b + \gamma c]^+ \mid a, b, c \in \mathbb{Z}_+\}$[1]. One can also show that the Markov chain is irreducible and aperiodic. Let $\mathcal{S} = \mathbb{Z}_+^M \times \mathcal{U}$ be the state space of the Markov chain.

A. Stability Analysis

At a state $(\underline{n}, u)$ such that $q_m(n_m) = 1-\epsilon$, $m = 1, ..., M$, the probabilities $P_i(u)$, $P_s(u)$, and $P_c(u)$, of having an idle

[1]The initial signal u_0 in fact may assume a value outside of $\mathcal{U}$, but as far as stability analysis is concerned, only the absorbing sets, in this case $\mathcal{U}$, are of interest.

slot, successful transmission, and collision slot, respectively, are given by

$$
\begin{aligned}
P_i(u) &= e^{-\lambda(u)}\epsilon^M,\\
P_s(u) &= \left(\sum_{m=1}^{M} e^{\lambda_m(u)} - M\epsilon\right) e^{-\lambda(u)}\epsilon^{M-1},\\
P_c(u) &= 1 - P_i(u) - P_s(u).
\end{aligned}
$$

We make the following assumption.

Assumption 3: There exists a constant $\epsilon_0 > 0$ such that $P_i(u)\alpha + P_s(u)\beta + P_c(u)\gamma > \epsilon_0$, for $u \geq 0$.

The above assumption implies that when the system is congested and each node has a large number of backlogged packets, then the control signal u tends to increase. Note that

$$
\lim_{\epsilon \to 0} P_i(u)\alpha + P_s(u)\beta + P_c(u)\gamma = \gamma,
$$

and Assumption 3 can always be satisfied. Furthermore, it can be shown that one can find a constant ϵ to satisfy Assumption 3 independent of the actual rate function $\lambda(u)$, the retransmission parameters, and the number of nodes contending for network access.

We have the following result.

Proposition 1: Under the Assumptions 1, 2, and 3, the Markov chain $(\underline{n}_t, n_t)$ for the backlog-dependent retransmission scheme is positive recurrent.

We provide a proof of Proposition 1 in Appendix A and illustrate the result through a numerical case study in Section VI.

IV. Backlog-Independent Scheduling

Next, we study the stability properties of backlog-independent scheduling. In this case, when each node has at least one backlogged packet then the probabilities $P_i(u)$, $P_s(u)$, and $P_c(u)$, of having an idle slot, successful transmission, and collision slot, respectively, are given by

$$
\begin{aligned}
P_i(u) &= e^{-\lambda(u)} \prod_{m=1}^{M} \big(1 - q_m\big),\\
P_s(u) &= \sum_{m=1}^{M} \left[\frac{e^{\lambda_m(u)}}{1-q_m} - 1\right] e^{-\lambda(u)} \prod_{m=1}^{M} \big(1 - q_m\big),\\
P_c(u) &= 1 - P_i(u) - P_s(u).
\end{aligned}
$$

We make the following assumption to ensure that the control signal u tends to increase when all nodes are backlogged.

Assumption 4: There exists a constant $\epsilon_0 > 0$ such that $P_i(u)\alpha + P_s(u)\beta + P_c(u)\gamma > \epsilon_0$, for $u \geq 0$.

Note that

$$
P_c(u) \geq \min_{m=1,\dots,M} q_m^2, \qquad u \geq 0
$$

and we can always satisfy Assumption 4 independent of the actual rate function $\lambda(u)$ and the number of nodes contending for network access by choosing the retransmission probabilities q_m, $m = 1, ..., M$, large enough.

Again, we can model the rate control with backlog-dependent scheduling scheme as a Markov chain where $(\underline{n}_t, u_t)$ denotes the state of the chain at the beginning of slot t. Using a similar argument as given to prove Proposition 1, we can show that under the Assumptions 1, 2, and 4, the system is stable; we omit a detailed derivation, but illustrate the result through a numerical case study in the next section.

V. Selection of the Control Parameters

In Section III and IV, we showed that the proposed rate control mechanism can be used to stabilize the system as long as the control parameters α, β, and γ, satisfy Assumption 2. A natural question that then arises is whether it is possible to choose α, β, and γ in order to achieve a desired performance objective such as system throughput. In [2], [3], we addressed this question for the case of an infinite set of nodes and showed that in this case it is indeed possible to choose the control parameters to obtain a desired throughput and delay. For the finite node case, choosing the control parameters α, β, and γ to achieve a desired system throughput would require a priori knowledge of the actual number M of active nodes in the network. Unfortunately, this is generally not the case in practice. To avoid this difficulty, we use the procedure derived for the infinite node case as an "approximation" for the case when the number of nodes accessing the network is finite but unknown. One can show that the infinite node case provides a conservative estimate for the channel capacity in the finite node case, and we can expect the system to perform better than predicted by the analysis for the infinite node case. In the next section, we provide a numerical case study to illustrate this. Below, we briefly outline the procedure derived for the infinite node case that we are going to use to choose the control parameters α, β, and γ, in the finite node case.

It is well known that in the infinite node case the highest possible throughput that we can achieve is equal to e^{-1} [1]. In this case, we can choose the control parameter as follows to obtain a desired throughput S (see [2], [3] for a derivation). Let S, $0 \leq S \leq e^{-1}$, be a given throughput and let G, $0 \leq G \leq 1$, be such that $S = Ge^{-G}$. Furthermore, let α and γ be given control parameters which satisfy Assumption 2, and let β be given by

$$
\beta = \frac{\gamma}{G}(G + 1 - e^{G}) - \frac{\alpha}{G}. \tag{2}
$$

Then for the above choice of the control parameters, the throughput is equal to S if $\lambda\big(\max\{-\alpha, -\beta\}\big) \geq e^{-1}$. We refer to [3] for a detailed derivation and the discussion of the case when $\lambda\big(\max\{-\alpha, -\beta\}\big) < e^{-1}$. Note that the the value of β given in Eq. (2) only depends on α, γ, and G, but not on the aggregate rate function $\lambda(u)$ or the retransmission probabilities. This result is important in practice as it implies that one can choose the control parameters α, β, and γ to achieve a certain throughput S, without any prior knowledge of the aggregate rate function or the retransmission probabilities.

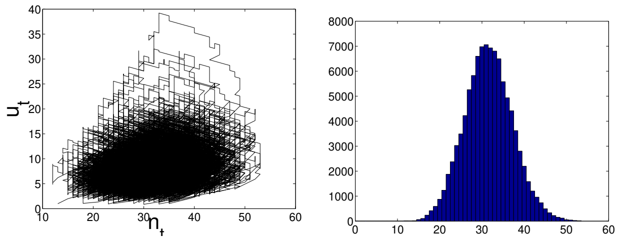

Fig. 1. State trajectory (n_t, u_t) (left) and total system backlog distribution (right) using backlog-dependent scheduling.

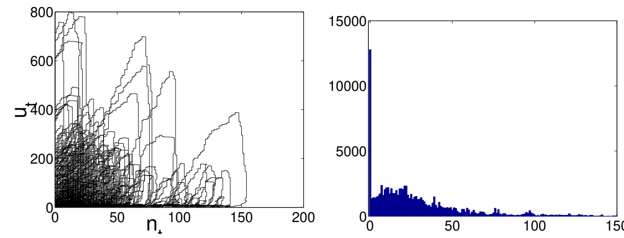

Fig. 2. State trajectory (n_t, u_t) (left) and total system backlog distribution (right) using backlog-independent scheduling.

VI. Numerical Results

In this section, we illustrate the results of Section III and IV and show how the above schemes can be used to provide service differentiation with respect to throughput and delay. For this purpose, we introduce 4 nodes with differing bandwidth and delay requirements. Suppose that the system should provide node 1 and 2 the same throughput and this throughput should be 10 times larger than the throughput of node 3 and 4. In addition, node 2 and 4 should obtain the same delay and this delay should be 10 times smaller than the delay of node 1 and 3. We wish to investigate whether the above schemes support the above multi-dimensional service requirements.

We use the procedure of Section V to choose the control parameters. In particular, throughout the section we set $\alpha = -1$, $\gamma = 1$, and $\beta = 0.2817$ to obtain a system throughput $S = e^{-1}$, which is the highest possible throughput for the infinite node case.

A. Backlog-Dependent Scheduling

We first consider backlog-dependent scheduling and associate with the different nodes a rate function of the form

$$\lambda_m(u) = \frac{k_m}{(1+u)^{1.5}}, \qquad m = 1, ..., 4.$$

The values of k_m and retransmission factors q_m are given in the table below. To satisfy Assumption 3, the parameter ϵ in the scheduling scheme is chosen as 0.01.

Node	k_m	q_m
1	0.5	0.01
2	0.5	0.1
3	5	0.01
4	5	0.1

Note that the parameters k_m and q_m are chosen to achieve the service requirements stated at the beginning of the section.

Fig. 1 shows the system trajectory for a simulation run of 100,000 steps starting with state $(0, 0)$ and the distribution of the backlog. Note that after a transient, the system seems to "hover" around the operating point. The measured average throughput S_m and delay D_m of the individual nodes $m = 1, ..., 4$ is given in the table below. Note that node 1 and 2 indeed obtain about same throughput and this throughput is about 10 times larger than the throughput of node 3 and 4. In addition, node 2 and 4 obtain about the same delay and this delay is roughly 10 times smaller than the delay of node 1 and 3. This illustrates that backlog-dependent can be used to satisfy the multi-dimensional service requirements. The system throughput S is equal 0.460. Note that the system throughput in the finite node case exceeds the predicted throughput of e^{-1} obtained for the infinite node model.

Node m	S_m	D_m
1	0.021	186.7
2	0.021	19.9
3	0.206	116.5
4	0.210	11.8

B. Backlog-Independent Scheduling

Next, we consider backlog-independent scheduling using the same rate function as in the previous subsection. The values of k_m, and the probabilities q_m for retransmitting backlogged packets at node $m = 1, ..., 4$, are given in the table below. The retransmission probabilities are chose such that Assumption 4 is satisfied.

Node	k_m	q_m
1	0.5	0.099
2	0.5	0.99
3	5	0.099
4	5	0.99

Fig. 2 shows the system trajectory for a simulation run of 100,000 steps starting with state $(0, 0)$ and the distribution of the backlog. Compared with backlog-dependent scheduling, the system behavior is more erratic. The measured average throughput S_m and delay D_m of the individual nodes $m = 1, ..., 4$ is given in the table below. Note that node 1 and 2 indeed obtain about the same throughput and this throughput is about 10 times larger than the throughput of node 3 and 4. However, the delay requirements are clearly not met. The system throughput S is equal 0.150, well below the system throughput obtained for backlog-dependent scheduling.

Node m	S_m	D_m
1	0.006	93.0
2	0.006	85.4
3	0.069	417.6
4	0.068	46.0

C. Discussion

We make the following observations for the above numerical results. Backlog-dependent scheduling performs well and allows multi-dimensional quality-of-service provisioning with respect to throughput and delay. Backlog-independent scheduling leads to an erratic system behavior and a system throughput which is significantly worse than the one obtained for backlog-dependent scheduling. The reason for the poor performance of backlog-independent scheduling is that the retransmission probabilities have to be chosen to be large in order to satisfy Assumption 4 leading to periods with frequent collisions.

VII. Related Work

The study of random access networks, and in particular slotted Aloha, has spurred a plethora of research interests [5], [6]. An extensive treatment on random access communication protocols can be found in books by Bertsekas and Gallager [1], and by Rom and Sidi [7]. A general discussion on the topic is also provided in [8]. Most work in this area has focused on retransmission strategies for backlogged packets [9]–[13]. To our knowledge, only Kleinrock and Lam [14] and Mittal and Venetsanopoulos [15] have studied rate (input) control for slotted Aloha. The work in [14] and [15] considers an ON-OFF control where the transmission rate switches between two values, depending on the backlog with respect to a given threshold. In addition, it is assumed that the system backlog is globally known which is an unrealistic assumption in practice. In this paper, we consider a decentralized control policy based on channel feedback information. Our approach also differs from [14] and [15] in the sense that we use a continuous rate function, rather than an ON-OFF control. This enables a "smoother" control of the input rates in the spirit of price-based rate control schemes for point-to-point networks [16].

Price-based schemes to control the traffic load in random access networks have been proposed in [12], [17]. In [17], Battiti *et. al* propose and analyze a mechanism where pricing is used to control the number of hosts that contend for channel access. In [12], Jin and Kesidis consider a price-based scheme to control the retransmission probabilities at individual nodes. The aim of the scheme proposed by Jin and Kesidis is not to control the transmission rate (packet generation rate) in order to avoid that nodes become saturated; rather the aim is to avoid a throughput collapse when the nodes become saturated [12], [17]. The main difference between the rate control considered here and the mechanism proposed by Battiti *et. al* is that the scheme considered in [17] assumes that the rate function $\lambda(u)$ is known, where the scheme considered here does not make this assumption.

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Appendix

A. Proof of Proposition 1

Using Assumption 3, we have the following lemma (we omit here a detailed derivation).

Lemma 1: Under Assumption 3, there exist constants $\kappa > 0$ and $\delta_0 > 0$ such that for $u \geq 0$ we have

$$P_i(u)e^{-\kappa\alpha} + P_s(u)e^{-\kappa\beta} + P_c(u)e^{-\kappa\gamma} < 1 - \delta_0.$$

In the following, whenever we refer to constants κ and δ_0 then we assume that they satisfy the condition in the above lemma.

Let $P_s(\underline{n}, u)$ be the probability of a successful packet transmission at state $(\underline{n}, u) \in \mathcal{S}$. Note this probability is given by

$$P_s(\underline{n}, u) = \sum_{m=1}^{M} \left[\frac{e^{\lambda_m(u)}}{1 - q_m(n_m)} - 1 \right] e^{-\lambda(u)} \prod_{m=1}^{M} \Big(1 - q_m(n_m)\Big).$$

Furthermore, let S_1 be the set of all states at which all nodes have at least one backlogged packet, *i.e.* we have $S_1 = \{(\underline{n}, u) \in \mathcal{S} \mid n_m > 0, m = 1, ..., M\}$.

Using the above definition, let P_s^* and $u_{\max}$ be positive constants such that

$$P_s^* - \lambda(u_{\max}) \frac{2\lambda(0)}{\delta_0} > 0,$$

and for every state $(\underline{n}, u) \in S_1$ with $u \geq u_{\max}$ we have

$$P_s(\underline{n}, u) - \lambda(u) > P_s^*.$$

To see that such constants exist we make the following observations. Note that for every state $(\underline{n}, u) \in S_1$ we have

$$\lim_{u \to \infty} P_s(\underline{n}, u) = \left(\sum_{m=1}^{M} \frac{q_m(n_m)}{1 - q_m(n_m)} \right) \prod_{m=1}^{M} \Big(1 - q_m(n_m)\Big),$$

It follows that $\lim_{u \to \infty} P_s(\underline{n}, u) > 0$, $(\underline{n}, u) \in S_1$, as at every state $(\underline{n}, n)$ the transmission probability at each node m is upper-bounded by $1 - \epsilon$ and lower-bounded by q_m.

Furthermore, let the the positive constants A, B, and C, be such that

$$A \max\{\beta, \gamma\} + B\lambda(0) - \delta_0 C e^{-\kappa u_{\max}} < \frac{1}{2} A\alpha$$

and

$$A \max\{\beta, \gamma\} - BP_s^* + \lambda(u_{\max}) C e^{-\kappa u_{\max}} < \frac{1}{2} A\alpha,$$

and let u_0 be such that for $u \geq u_0$ we have

$$A \left(\alpha e^{-\lambda(u)} + \max\{\beta, \gamma\}(1 - e^{-\lambda(u)}) \right) + ... + B\lambda(u) < \frac{1}{2} A\alpha.$$

Note that such constants exist.

Let S_b be the set of states at which all nodes retransmit backlogged packets with probability $1 - \epsilon$, *i.e.* we set

$$S_b = \{(\underline{n}, u) \mid (n_m - 1)q \geq 1 - \epsilon, m = 1, ..., M\}.$$

Finally, let the set S_λ be given by

$$S_\lambda = \{(\underline{n}, u) \mid u \geq u_{\max} + \gamma + \max(|\, \alpha \,|, |\, \beta \,|)\},$$

and we let S_0 be given by be the set of all states $(\underline{n}, u) \in S_\lambda$ for which the system backlog is equal to zero,

$$S_0 = \{(\underline{n}, u) \in S_\lambda \mid u \geq u_0; n_m = 0, m = 1, ..., M\}.$$

Using the above notation, we construct below a Lyapunov function which satisfies the following *mean-drift* conditions [18].

Proposition 2: The Markov chain $(\underline{n}_t, u_t)$ for the backlog-dependent retransmission scheme is positive recurrent if and only if there exist a non-negative function $V(\underline{n}, u), (\underline{n}, u) \in \mathcal{S}$, positive constants δ and b, and a finite set $\mathcal{F} \subset \mathcal{S}$ such that

$$\mathbf{E}\big(\Delta V_t \mid \underline{n}_t = \underline{n}, u_t = u\big) \leq -\delta, \quad (\underline{n}, u) \notin \mathcal{F},$$

and

$$\mathbf{E}\big(\Delta V_t \mid \underline{n}_t = \underline{n}, u_t = u\big) < b, \qquad (\underline{n}, u) \in \mathcal{F},$$

where we use the notation that

$$\mathbf{E}\big(\Delta V_t \mid \underline{n}_t = \underline{n}, u_t = u\big) = \mathbf{E}\big(V(\underline{n}_{t+1}, u_{t+1}) - V(\underline{n}_t, u_t) \mid \underline{n}_t = \underline{n}, u_t = u\big).$$

The Lyapunov function $V(\underline{n}, u)$ that we consider is given by

$$V(\underline{n}, u) = Au + Bn + Cv(\underline{n}, u),$$

where

$$v(\underline{n}, u) = \begin{cases} e^{-\kappa u}, & (\underline{n}, u) \in S_b, \\ 0, & \text{otherwise} \end{cases}$$

and the positive constants κ, A, B, and C, are chosen as given above. Note that the function V is a non-negative.

Proof [Proposition 1]: Note that by the choice of the constants A, B, and C, for $(\underline{n}, u) \in S_b$ with $u \leq u_{\max}$ we have

$$\begin{aligned} E[\Delta V_t \mid \underline{n}_t = \underline{n}, u_t = u] &< A \max\{\beta, \gamma\} + B\lambda(0) - Ce^{-\kappa u_{\max}} \delta_0 \\ &\leq \frac{1}{2} A\alpha, \end{aligned}$$

and for $(\underline{n}, u) \in S_b$ with $u \geq u_{\max}$ we have

$$E[\Delta V_t \mid \underline{n}_t = \underline{n}, u_t = u] < A \max\{\beta, \gamma\} - BP_s^* \leq \frac{1}{2} A\alpha.$$

Furthermore, for all states $(\underline{n}, n) \in S_\lambda$ with $n \geq 0$ we have

$$\begin{aligned} E[\Delta V_t \mid \underline{n}_t = \underline{n}, u_t = u] &< A \max\{\beta, \gamma\} - BP_s^* + \lambda(u_{\max}) Ce^{-\kappa u_{\max}} \\ &\leq \frac{1}{2} A\alpha, \end{aligned}$$

and for all states $(\underline{n}, u) \in S_0$ we have

$$E[\Delta V_t \mid \underline{n}_t = \underline{n}, u_t = u] \leq \frac{1}{2} A\alpha.$$

Finally, note that for all states $(\underline{n}, u) \in \mathcal{S}$ such that $(\underline{n}, u) \notin S_b$ and $(\underline{n}, u) \notin S_\lambda$, we have

$$\mathbf{E}(\Delta V_t |\, \underline{n}_t = \underline{n}, u_t = u) \leq b,$$

where

$$b = A \max\{\beta, \gamma\} + B\lambda(0) + C \left(\max\left\{ e^{-\kappa\alpha}, e^{-\kappa\beta} \right\} - 1 \right).$$

Then for the finite set

$$\mathcal{F} = \{(\underline{n}, u) \in \mathcal{S} \mid (\underline{n}, u) \notin S_b \cup S_\lambda\}$$

we have

(i) $\mathbf{E}(\Delta V_t \mid \underline{n}_t, u_t) \leq \frac{1}{2} A\alpha$, for all $(\underline{n}_t, u_t) \notin \mathcal{F}$,

(ii) $\mathbf{E}(\Delta V_t \mid \underline{n}_t, u_t) < b$, for all $(\underline{n}_t, u_t) \in \mathcal{F}$.

and Proposition 1 follows. ■