Modeling and Analysis of Wireless Local Area Networks*

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1 Introduction

We develop a simple mathematical model to analyze the interaction of TCP Reno rate control and IEEE 802.11 medium access control in local area networks. Our work builds on existing models proposed in the literature to study TCP Reno [3, 7] and IEEE 802.11 [6]. We show that under suitable assumptions combining these two models still allows a formal analysis.

The paper is organized as follows. In Section 2, we model the interaction between TCP Reno and 802.11 as a discrete-time system. In Section 3 we study the existence of a system operating point, characterize the performance at an operating point, and study local stability. In Section 4 we provide a case study to illustrate our results. Due to space constraints proofs are omitted.

2 Models

In this section we develop the model that we use for our analysis. In particular, we define the model for TCP Reno rate control and IEEE 802.11 medium access control. We assume that the reader is familiar with these two protocols and we focus on the features of these protocols that are relevant to our analysis. For an overview and detailed description of TCP Reno and IEEE 802.11 we refer to [5] and [6].

2.1 Network Model

We consider a wireless local area network where nodes belonging to the set $\mathcal{N} = \{1, 2, \cdots, N\}$ access a base station. We assume that each node has one active TCP connection, and use n to refer both to node n as well as the connection associated with node n. We assume that all nodes have finite buffer space and use $\bar{B}_n$ to denote the buffer space available at node n. Let $\bar{B}_0$ denote the buffer space available at the base station. Naturally, we have $\bar{B}_0 > 0$ and $\bar{B}_n > 0$. We assume that the wireless local area network is the bottleneck for all connections, and all packets (data packets and ACKs) are stored either at a wireless

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node or at the base station. Let B_n denote the total backlog of the connection associated with node n, let $b_{0,n}$ denote the backlog of this connection at the base station and let $b_{1,n}$ denote that backlog at node n. Note that

$$B_n = b_{0,n} + b_{1,n}.$$

Let B_0 be the total backlog at the base station.

Finally let $\mathcal{N}_u$ be the set of nodes with upload traffic and $\mathcal{N}_d$ the set of nodes with download traffic. Note that for a connection $n \in \mathcal{N}_u$, $b_{1,n}$ denotes the data packets stored at node n and $b_{0,n}$ the ACKs belonging to connection n currently stored at the base station.

2.2 TCP Reno Model

For our analysis, we adapt the TCP Reno model proposed by Kelly in [3] and used by Low in [7] to the framework that we consider here. That is, we model TCP Reno as a synchronous discrete time system where each slot has unit time and the transmission of all packets (data packets and ACKs) requires L_p time slots, i.e. we have that all packets are of equal and deterministic length. Consider connection $n \in \mathcal{N}$ and let $w_n(t)$ be the window size (in terms of packets) of connection n at the beginning of the tth time slot. The transmission rate $\lambda_n(t)$ (in terms of packets per time slot) of connection n at time t is then given by

$$\lambda_n(t) = w_n(t)/D_n(t), \tag{1}$$

where $D_n(t)$ is the round-trip time (RTT) of connection n, i.e. $D_n(t)$ denotes the number of time slots it takes for connection n to receive a positive ACK for a packet sent at the beginning of slot t.

TCP Reno uses packet loss as a congestion indicator, and halves the window w_n for each packet that is not acknowledged. As in [3], we use a congestion indicator function to characterize the likelihood that a given data packet belonging to connection n is not acknowledged and the window w_n is halved at the end of time slot t. We make the following assumption.

Assumption 1 *The congestion indicator function $P_0(B_0)$ depends on the total backlog B_0 at the base station. Furthermore, we have that $P_0 : [0, \bar{B}_0] \mapsto [0, 1]$ is continuously differentiable and strictly increasing with $P_0(0) = 0$ and $P_0(\bar{B}_0) = 1$.*

The above assumption states that the less buffer space that is available at the base station, the more likely it is that a packet loss occurs and causes the window w_n to be halved. This assumption is based on the observation that in a wireless local area network the backlog of individual connections is concentrated at the base station [8].

Using the above definitions, we characterize the dynamics of TCP Reno as follows (see also [3]). As TCP Reno increases the window size by $\frac{1}{w_n}$ for each positive acknowledged packet and halves the window for each congestion indication, the expected rate of change of the congestion window (i.e. the expected change of the congestion window per time slot) at time t is given by

$$\frac{1}{w_n(t)}\lambda_n(t)\Big(1 - P_0(B_0(t))\Big) - \frac{1}{2}w_n(t)\lambda_n(t)P_0(B_0(t)). \tag{2}$$

Assuming that the the wireless local area network is the bottle-neck and that the propagation delay is negligibly small compared with the queueing delay, we have that all "active" packets (i.e. all data packets and ACKs within to the current window size) of connection n are queued either at node n or at the base station, i.e. we have that

$$w_n(t) = b_{0,n}(t) + b_{1,n}(t) = B_n(t).$$

Combining the above observation with Eq. (1) and (2), the expected rate of change in the transmission rate λ_n at time t is given by

$$\frac{\lambda_n(t)^2}{B_n(t)^2}\Big(1 - P_0(B_0(t))\Big) - \frac{\lambda_n(t)^2}{2} P_0(B_0(t)).$$

2.3 IEEE 802.11 Medium Access Control Model

The medium access control of IEEE 802.11 works roughly as follows. Before each transmission attempt, a node senses the channel to detect whether the channel is free (idle). If the channel is sensed to be free, a node will wait for some random time before starting with the transmission as follows. Each node keeps track of a *back-off* timer and a *retry* counter. The back-off timer interval is selected uniformly from the interval $[1, CW]$, where CW is set equal to a given constant $CW_{\min}$ for the first transmission attempt of a packet. The back-off timer is then decremented by one every time the channel is sensed to be idle. When the back-off timer reaches zero, a transmission starts. If a collision occurs (i.e. two or more nodes make a transmission attempt at the same time), CW is doubled (up to some given constant $CW_{\max}$), the retry counter is incremented and the packet is backlogged waiting for the next transmission. When the retry counter reaches the maximal *retry limit* K, the packet is discarded.

For our analysis, we use a simplified model of IEEE 802.11 which was proposed and analyzed first in [2] and then extended by Kumar et al. in [6]. Under suitable assumptions, it is shown that the 802.11 protocol can be modeled as if each node makes a transmission attempt (after sensing the channel to be idle) with probability $q(N_a)$, where N_a is the number of nodes with a packet to send and $q(N_a)$ is the unique solution to the following fixed-point equation [6]

$$\begin{cases} \vartheta = 1 - (1-q)^{N_a - 1} \\ q = \frac{1+\vartheta^1+\cdots+\vartheta^K}{\omega_0+\vartheta^1\omega_1+\cdots+\vartheta^K\omega_K}, \end{cases} \tag{3}$$

with $\omega_k = \min\{\frac{2^k CW_{\min}+1}{2}, \frac{CW_{\max}+1}{2}\}$, $k = 0, 1, ..., K$. In the above equation, ϑ is the collision probability observed by a node.

Using the above result, we model 802.11 as a slotted CSMA/CD system as follows (see for example [1]). Let L_i be the number of slots a node has to detect the channel to be idle before making a transmission attempt. We will refer to this time as an idle period. If N_a is the number of active nodes at the end of an idle period of L_i slots (i.e. the number of nodes that received a new packet during this idle period or have a backlogged packet to retransmit), then each active node makes a transmission attempt with probability $q(N_a)$. If exactly one node makes a transmission attempt, then the packet is successfully transmitted within L_p slots. It two or more nodes make a transmission attempt, then all transmitted packets are lost and have to be retransmitted. The time to detect a collision is equal to L_c slots. Note that L_p is equal to the transmission delay in the TCP Reno model of the previous subsection.

Note that in the above CSMA/CD model, nodes make a transmission attempt only after an idle period of length L_i. Using this observation, we characterize the system dynamics by considering the times when idle slots start. More precisely, we mark the times when a new idle period starts as follows. Starting the system at time $t = 0$, let t_k be the time at which the kth idle period of length L_i ends. The probability that node n makes a successful transmission attempt between t_k and t_{k+1} is then equal to

$$\frac{[b_{1,n}(k)]^\dagger q(N_a(k))}{1-[b_{1,n}(k)]^\dagger q(N_a(k))}\Big(1-[B_0(k)]^\dagger q(N_a(k))\Big)\prod_{l=1}^{N}\Big(1-[b_{1,l}(k)]^\dagger q(N_a(k))\Big),$$

where $[x]^\dagger = \min\{x, 1\}$, $b_{1,n}(k)$ is the backlog at node n at time t_k, and

$$N_a(k) = [B_0(k)]^\dagger + \sum_{l=1}^{N}[b_{1,l}(k)]^\dagger$$

is the number of active nodes at time t_k. A standard simplification for the analysis for CSMA/CD is to assume that $q(N_a(t))$ is small and to approximate the probability that node n makes a successful transmission attempt between t_k and t_{k+1} by (see for example [1])

$$[b_{1,n}(k)]^\dagger q(N_a(k))e^{-G(k)},$$

where

$$G(k) = [B_0(k)]^\dagger q(N_a(k)) + \sum_{l=1}^{N}[b_{1,l}(k)]^\dagger q(N_a(k)) = N_a(k)q(N_a(k)).$$

We will refer to $G(k)$ as the offered load at time t_k.

In addition, for our analysis we assume that when the base station makes a transmission attempt then the probability that this packet belongs to connection n is equal to $\frac{b_{0,n}}{B_0}$. If $B_0(k) > 0$, then the probability that the base station makes a transmission attempt between time t_k and t_{k+1} of a packet belonging to connection n is given by

$$G_{0,n}(k) = \frac{b_{0,n}(k)}{B_0(k)}[B_0(k)]^\dagger q(N_a(k)),$$

and $G_{0,n}(k) = 0$ if $B_0(k) = 0$. The probability that a packet of connection n is successfully transmitted between t_k and t_{k+1} by the base station is then given by

$$G_{0,n}(k)e^{-G(k)}.$$

Finally, the probability that no node makes a transmission attempt between t_k and t_{k+1} is equal to $e^{-G(k)}$.

2.4 System Model

Using the models of the previous two subsections, we model the system consisting of TCP Reno rate and 802.11 medium access control by a synchronous discrete time system where the system state at time t is given by the vector

$$x(t) = (\lambda(t), b_0(t), b_1(t)), \qquad t \geq 0$$

where $\lambda(t) = (\lambda_1(t), ..., \lambda_N(t))$ indicates the transmission rates, $b_0(t) = (b_{0,1}(t), ..., b_{0,N}(t))$ the backlog at the base station, and $b_1(t) = (b_{1,1}(t), ..., b_{1,N}(t))$ the backlog at the wireless node, of connection $n = 1, ..., N$.

Again, let t_k be the time at which the kth idle period of length L_i ends and let $x(k)$ denote the system state at time t_k. We can then represent expected change of the system state between time t_k and t_{k+1} by a nonlinear equation

$$x(k+1) = \phi(x(k)), \qquad k \geq 1,$$

which is given as follows.

The expected change in the transmission rate of connection m is given by

$$\lambda_n(k+1) = \lambda_n(k) + \lambda_n(k)^2 \frac{1 - P_0(B_0(k))}{B_n(k)^2} L(k) - \lambda_n(k)^2 \frac{P_0(B_0(k))}{2} L(k),$$

where

$$L(k) = L_i + G(k)e^{-G(k)} L_p + (1 - e^{-G(k)} - G(k)e^{-G(k)}) L_c$$

is the expected time between time t_k and t_{k+1}.

For a node $n \in \mathcal{N}_u$, the expected change of the backlog of connection n at node n between time t_k and t_{k+1} is given by

$$b_{1,n}(k+1) = b_{1,n}(k) + \lambda_n(k)L(k) - [b_{1,n}(k)]^\dagger q(N_a(k))e^{-G(k)},$$

where $\lambda_n(k)L(k)$ is the expected number of new packet arrivals at node node n in the interval between t_k and t_{k+1} and $[b_{1,n}(k)]^\dagger q(N_a(k))e^{-G(k)}$ is the probability of a packet departure at node n (i.e. a successful transmission by node n) between t_k and t_{k+1}.

The expected change in the backlog of connection n at the base station between time t_k and t_{k+1} is given by

$$b_{0,n}(k+1) = b_{0,n}(k) - G_{0,n}(k)e^{-G(k)} + [b_{1,n}(k)]^\dagger q(N_a(k))e^{-G(k)},$$

where $G_{0,n}(k)e^{-G(k)}$ is the probability that the base station successfully transmits an ACK belonging to connection n and $[b_{1,n}(k)]^\dagger q(N_a(k))e^{-G(k)}$ is the probability that a new ACK belonging to connection n arrives at the base station. Here we assumed that a successful packet transmission by node n will immediately generate a new ACK at the base station.

Similarly, for a node $n \in \mathcal{N}_d$, the expected change of the backlog of connection n at node n between time t_k and t_{k+1} is given by

$$b_{1,n}(k+1) = b_{1,n}(k) + G_{0,n}(k)e^{-G(k)} - [b_{1,n}(k)]^\dagger q(N_a(k))e^{-G(k)},$$

and the expected change in the backlog of connection n at the base station between time t_k and t_{k+1} is given by

$$b_{0,n}(k+1) = b_{0,n}(k) + \lambda_n(k)L(k) - G_{0,n}(k)e^{-G(k)}.$$

3 Analysis

Having modeled the dynamics, we wish next to study the system performance. In order to do that, we study the system performance at an operating point.

Definition 1 *We call x^* an operating point if we have that*

$$x^* = \phi(x^*).$$

Note that an operating point is a fixed-point of the system dynamics given by ϕ, i.e. the expected change of the state variables is equal to 0 at an operating point. We are interested in the following questions.

1. Does an operating point exist?
2. Does a unique operating point exist?
3. What is the system performance at an operating point?
4. Are the operating points locally stable?

3.1 Existence of an Operating Point

We make the following additional assumptions for our analysis.

Assumption 2 *We have that $P_0(1) \leq 2/3$.*

The above assumption is a very mild assumption which ensures that there is some minimal buffer space available at the base station.

For a given operating point x^*, let $N_a^* = \sum_{n=1}^{N} [b_n^*]^\dagger$ denote the number of active nodes at x^*, and let $G^* = N_a^* q(N_a^*)$. We will refer to G^* as the offered load at x^*.

Using this notation, we have the following result.

Proposition 1 *Under Assumption 1 and 2 there always exists a unique operating point x^*. Furthermore, for x^* we have $N_a^* = 2$ and $G^* = 2q(2)$.*

The above result states that the number of active nodes at the operating point is always equal to 2, independent of the number of active connections N. Similarly, the offered load G^* is always equal to $G^* = 2q(2)$, independent of N. Intuitively, the above results states that on average there are 2 nodes active, the base station and one of the wireless nodes in the set $\mathcal{N}$.

3.2 System Performance

Using the fact that there exists a unique operating point, we can characterize the system performance as follows. For slotted CSMA/CD, it is well known that the system throughput $T(G)$ (in packets per time slot) as a function of the expected number of transmission attempts is given as follows (see for example [1]),

$$T(G) = \frac{Ge^{-G}}{L_i + L_c + \Big(GL_p - (1+G)L_c\Big)e^{-G}}.$$

Therefore, the throughput at the operating point is equal to

$$T(G^*) = T(2q(2)).$$

More precisely, we have the following result.

Proposition 2 *Let Assumption 1 and 2 hold, and let x^* be an operating point. Then the backlog at the base station B_0^* at the operating point is equal to the unique solution to the equation*

$$B_0^* = N\sqrt{\frac{2(1-P_0(B_0^*))}{P_0(B_0^*)}} + 1,$$

and for every connection $n \in \mathcal{N}$ we have

(a) $\lambda_n^* = \frac{T(G^*)}{N}$,

(b) $b_{1,n}^* = \frac{1}{N}$,

(c) $b_{0,n}^* = \frac{B_0^*}{N}$.

3.3 Local Stability

In the previous subsection, we characterized the throughput and backlog at an operating point. In this subsection, we study whether an operating point is locally stable and the system will indeed spend most of the time in the vicinity of the operating point.

Given two system states $x = (\lambda, b_0, b_1)$ and $x' = (\lambda', b_0', b_1')$ let

$$|x - x'| \;=\; \sqrt{\sum_{n\in N}\left(\lambda_n - \lambda_m'\right)^2 + \sum_{n\in N}\left(b_{0,n} - b_{0,n}'\right)^2 + \sum_{n\in N}\left(b_{1,n} - b_{1,n}'\right)^2}.$$

Definition 2 *We call an operating point x^* locally stable if there exists an $\epsilon > 0$ such that for all initial states $x(1)$ with $|x(1) - x^*| \leq \epsilon$ we have*

$$\lim_{k\to\infty} x(k) = x^*$$

where $x(k+1) = \phi(x(k))$, $k \geq 1$.

The following lemma can be used to determine whether an operating point x^* is locally stable.

Lemma 1 *Consider a nonlinear discrete time dynamical system $x(k+1) = \phi(x(k))$, $\phi : \mathbb{R}^n \to \mathbb{R}^n$. Suppose the system has a fixed point x^* and the system can be linearized as $x(k+1) = Bx(k) + D$ around x^*, where B is a $n \times n$ matrix and D is a n-dimensional constant column vector. Suppose $|B| \neq 0$. Let $\{\zeta_1, \cdots, \zeta_n\}$ be the eigenvalues of B and let $\zeta_{\max} = \max_{i=1,\ldots,n} |\zeta_i|$. Then the fixed point x^* is locally stable if and only if $\zeta_{\max} < 1$.*

The above lemma states that an operating point x^* is locally stable if and only if the largest absolute value of all eigenvalues is strictly smaller than 1. We have the following result.

Proposition 3 *Under Assumptions 1 and 2 we have $\zeta_{\max} > 1$ and the operating point x^* is not locally stable.*

To get a better understanding of the local stability, we next characterize the (asymptotic) behavior of the $\zeta_{\max}$ as the number of nodes N becomes large. To do that, we consider a series of systems indexed by N, $N \geq 1$, where N refers to the number of wireless nodes present in the network. Let $P_0^N(B_0)$ be the the congestion indicator function for system N. We have the following assumption.

Assumption 3 *There exists a constant $\epsilon_0 > 0$ such that*

$$P_0^N(N) < 1 - \epsilon_0, \qquad N \geq 1.$$

The above assumption ensures that there is enough buffer space available at the base station to accommodate at least one packet for each TCP connection.

Let $\zeta_{\max}^N$ be the absolute value of the largest eigenvalue for the linearized dynamics around the operating point for system N. Then we have the following result.

Proposition 4 *Under Assumptions 1, 2, and 3, we have that* $\lim_{N\to\infty} \zeta_{\max}^N = 1$.

The above results state that the operating point x^* is not locally stable with $\zeta_{\max}$ being close to 1. This suggests that the dynamics is such that the system will "oscillate" in an area around the operating point. We illustrate this in the next section through a numerical case study.

4 Numerical Results

In this subsection, we illustrate the above results using a numerical case study. For this, we implemented the actual TCP Reno protocol and IEEE 802.11 protocol without making the simplifying assumptions used to formulate the above system model. This explains why the numerical results below do not perfectly match the theoretical predictions obtained in the previous section.

For the simulation, we assumed that all data packets and ACKs are of the same lengths and equal to 1 MSS. The buffer size at each wireless node was set equal to 10 MSS, and the buffer size at the base station was set equal to 100 MSS. For the IEEE 802.11 medium access control we used the following parameter values: we set $CW_{\min} = 32$, $CW_{\max} = 1024$, and $K = 7$. The values for L_i, L_p, and L_c that we considered are given in the table below, along with the corresponding throughput $T(G^*)$ at the operating point predicted by the model in the previous section.

(L_i, L_p, L_c)	$T(G^*)$
$(1, 100, 1)$	0.0091
$(1, 100, 17)$	0.0090
$(1, 100, 100)$	0.0086

We simulated the above system with the total number of nodes N equal to 5, 10, 15, 20, and 30, to investigate how the system performance changes as the number of nodes increases. In the simulation, all nodes were uploading traffic to the base station. Using this setup, we simulated the system for 90000, 150000 180000, 210000, and 240000 time steps for the case of the total number of nodes N equal to 5, 10, 15, 20, and 30, respectively.

4.1 Throughput

Fig. 1 shows the time-average system throughput obtained through simulation for the different scenarios described above, as well as the corresponding theoretical results indicated in the above table. We note that the experimental results indeed confirm the results of the previous section, i.e. the throughput stays (roughly) constant as the number of nodes increases. In addition, we note that the values predicted by the model of the previous section match reasonably well the values obtained in the numerical case study.

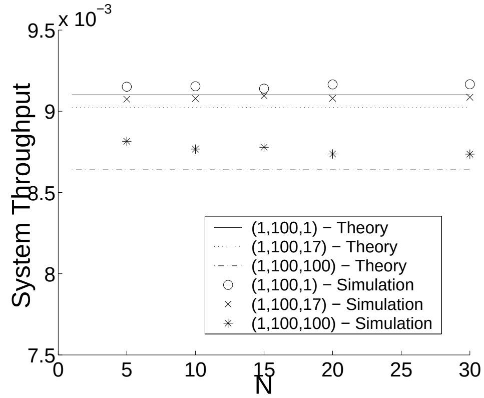

Figure 1: Time-average system throughput for different values of (L_i, L_p, L_c) as a function of the number of wireless nodes N.

4.2 Local Stability

Fig. 2 shows the trajectory of the total system backlog

$$B = B_0 + \sum_n B_n,$$

and the total system throughput

$$\lambda = \sum_n \lambda_n,$$

as well as the histogram of the distribution of the system throughput for the case of $(L_i, L_p, L_c, N) = (1, 100, 17, 15)$. We note that the trajectory "oscillates" around the operating point as predicted by the theoretical results of the previous section.

As we do not know the actual congestion indicator function, we use the following two example functions to investigate local stability. Using the congestion indicator

$$P_0(B) = \frac{e^{5B/\bar{B}_0} - 1}{e^5 - 1}, \qquad B \geq 0,$$

the value of $\zeta_{\max}$ is equal to $|1 \pm 0.0018i|$, i.e. $|\zeta_{\max}|$ is very close to 1. For the congestion indicator

$$P_0(B) = \frac{e^{50B/\bar{B}_0} - 1}{e^{50} - 1}, \qquad B \geq 0,$$

the value of $\zeta_{\max}$ is equal to $|1 \pm 0.0009i|$. This confirms the theoretical result on local stability obtained in the previous Section.

5 Conclusion

We developed a simple model to study the interaction among TCP Reno and IEEE 802.11 in local area networks. Even though we used several simplifying assumptions, the numerical results suggest that the model does capture the important features and does reasonably well predict the actual system behavior and performance.

There are several issues that we did ignore in our analysis. One such issue is the well-known asymmetry between upload and download traffic in wireless local area networks using TCP Reno and IEEE 802.11. This asymmetry is due to the fact that losses of

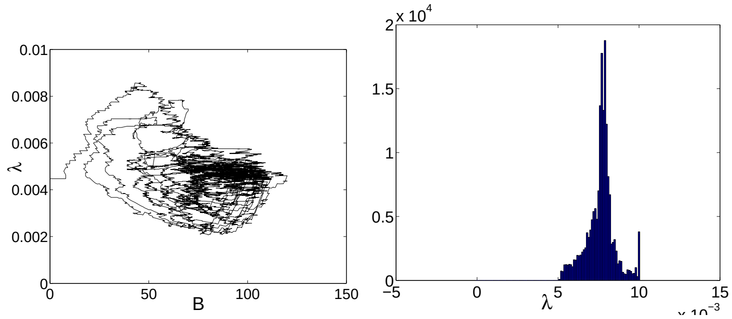

Figure 2: Trajectory of the system backlog and transmission rate (left), and histogram of the system throughput (right), for $(L_i, L_p, L_c, M_A, M_B) = (1, 100, 17, 3, 15)$.

data packets and ACK's are interpreted/handled by TCP Reno in a slightly different manner [8]. This characteristic of TCP Reno could be incorporated into our model by using different congestion indicator functions for upload and download traffic.

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